MATH 426 HOMEWORK 2

(1) Let the multiplicative group of non-zero real numbers $\mathbb{R}^1 - \{0\}$ act on $\mathbb{R}^2 - \{(0,0)\}$ by

$$(x, y) \mapsto (\lambda x, \lambda^{-1} y)$$

for $\lambda \in \mathbb{R}^1 - \{0\}$. Let

$$X = \frac{\mathbb{R}^2 - \{(0,0)\}}{\mathbb{R}^1 - \{0\}}$$

be the quotient by the group action.

(a) Show that $f : X \to \mathbb{R}^1$ given by $(x, y) \mapsto xy$ is well-defined and continuous.

(b) Find the cardinality of $f^{-1}(t)$ for each $t \in \mathbb{R}^1$.

(c) Show that X is not Hausdorff.

(d) Consider the equivalence relation on the space

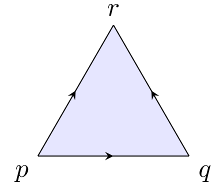
$$Y = \mathbb{R}^1 \times \{0, 1\} \subset \mathbb{R}^2$$

given by $(s, 0) \sim (t, 1)$ if and only if $s = t \neq 0$. Let $Z = Y/\sim$. Show that X is homeomorphic to Z.

(2) Show that the union of the standard torus in $\mathbb{R}^3$ with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle is homotopy equivalent to a 2-sphere. (Hint: Show that both are homotopy equivalent to $\mathbb{R}^3 - \{pt\}$.)

(3) Show that the projective plane $\mathbb{RP}^2$ is homeomorphic to the mapping cone of the map $f : S^1 \to S^1$ given by $z \mapsto z^2$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

(4) Let $T \subset \mathbb{R}^2$ be the convex hull three non-colinear points $p, q, r \in \mathbb{R}^2$ (i.e. T is a triangular region). The "dunce cap" is the quotient of T by an equivalence relation identifying all three edges in a particular way. Namely, we identify the line segment $\overline{pq}$ with the segment $\overline{pr}$ with the segment $\overline{qr}$, with the given orientations. Show that this space is contractible. (Hint: show that the dunce cap can be realized as the mapping cone of a certain map and then study the map. You may use the fact we proved in class: if $f_0 \simeq f_1$ then $C_{f_0} \simeq C_{f_1}$.)



Date: September 23, 2024.