MATH 426 HOMEWORK 2

(1) Let the multiplicative group of non-zero real numbers $\mathbb{R}^1 - \{0\}$ act on $\mathbb{R}^2 - \{(0,0)\}$ by

$$(x, y) \mapsto (\lambda x, \lambda^{-1} y)$$

for $\lambda \in \mathbb{R}^1 - \{0\}$. Let

$$X = \frac{\mathbb{R}^2 - \{(0,0)\}}{\mathbb{R}^1 - \{0\}}$$

be the quotient by the group action.

(a) Show that $f : X \to \mathbb{R}^1$ given by $(x, y) \mapsto xy$ is well-defined and continuous.

(b) Find the cardinality of $f^{-1}(t)$ for each $t \in \mathbb{R}^1$.

(c) Show that X is not Hausdorff.

(d) Consider the equivalence relation on the space

$$Y = \mathbb{R}^1 \times \{0, 1\} \subset \mathbb{R}^2$$

given by $(s, 0) \sim (t, 1)$ if and only if $s = t \neq 0$. Let $Z = Y/\sim$. Show that X is homeomorphic to Z.

Solution: (a) Since $f(x, y) = f(\lambda x, \lambda^{-1} y)$, f is well defined as a set mapping. Then since the composition

$$\mathbb{R}^2 - \{0\} \xrightarrow{\pi} X \xrightarrow{f} \mathbb{R}^1$$

is continuous by ordinary calculus, f is continuous by the definition of the quotient topology.

(b) We have

$$f^{-1}(t) = \{(x, y) \in X : \quad xy = t\}$$

and if $t \neq 0$, then $x, y \neq 0$ so $y = \frac{t}{x}$.

Letting $\lambda = y$ we see that the curve $y = \frac{t}{x}$ parameterizes the λ-orbit of $(1, t)$ and so $f^{-1}(t)$ is a single orbit and thus has cardinality 1.

If $t = 0$, then $f^{-1}(0) = \{xy = 0\}$ which has 2 orbits, namely $\lambda \cdot (1, 0)$ and $\lambda \cdot (0, 1)$ and so the cardinality is 2.

$$f^{-1}(t) = \begin{cases} 1 \text{ point} & \text{if } t \neq 0 \\ 2 \text{ points} & \text{if } t = 0 \end{cases}$$

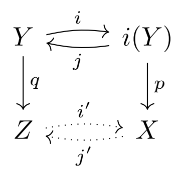
(c) Let $f^{-1}(0) = \{p, p'\}$ where $p = \pi(1, 0)$ and $p' = \pi(0, 1)$. Suppose for the sake of contradiction that X were Hausdorff. Then there are open sets $U, U' \subset X$ with $p \in U$, $p' \in U'$, and $U \cap U' = \emptyset$. Then $V = \pi^{-1}(U)$ and $V' = \pi^{-1}(U')$ are disjoint open sets in $\mathbb{R}^2 - \{0\}$ containing $(1, 0)$ and $(0, 1)$ respectively. Then there exists $\epsilon > 0$ such that $B_\epsilon(1, 0) \subset V$ and $B_\epsilon(0, 1) \subset V'$ so that $(1, \frac{\epsilon}{2}) \in V$ and $(\frac{\epsilon}{2}, 1) \in V'$. But since V and V' are both unions of λ orbits and $(1, \frac{\epsilon}{2})$ is on the same λ-orbit as $(\frac{\epsilon}{2}, 1)$ we see that $V \cap V' \neq \emptyset$, a contradiction.

Date: October 11, 2024.

(d) We define a continuous embedding

$$\begin{aligned} i : Y &\hookrightarrow \mathbb{R}^2 - \{0\} \\ (s,0) &\mapsto (s, e^s) \\ (t,1) &\mapsto (e^t, t) \end{aligned}$$

The map i is clearly a homeomorphism onto its image. Let $j : i(Y) \to Y$ be the inverse map, let $p : i(Y) \to X$ be the restriction of π to $i(Y)$, and let $q : Y \to X$ be the quotient map. We get a diagram:



We claim that the maps i' and j' making the diagram commute exists and provide the desired homeomorphism $Z \approx X$. This follows from the assertion that the homeomorphism $Y \approx i(Y)$ given by i identifies the equivalence relations on Y with the equivalence relation on $i(Y)$ and thus descends to the quotients. Indeed, a point (s, e^s) is in the same λ-orbit as a point (e^t, t) if and only if $s = t \neq 0$. Moreover, the points $(0,1)$ and $(1,0)$ corresponding to $s = 0$ and $t = 0$ respectively lie on distinct λ orbits and are not equivalent.

(2) Show that the union of the standard torus in $\mathbb{R}^3$ with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle is homotopy equivalent to a 2-sphere.

Solution:
Let $X = T^2 \cup D_1 \cup D_2 \subset \mathbb{R}^3$ where T^2 is the "standard" 2-torus in $\mathbb{R}^3$ and D_1 and D_2 are disks spanning the meridian and longitudinal circles. To be explicit, we can write T^2, D_1, and D_2 in cylindrical coordinates as

$$\begin{aligned} T^2 &= \{(r,\theta,z) \in \mathbb{R}^3 : \quad (r-2)^2 + z^2 = 1\} \\ D_1 &= \{(r,\theta,z) \in \mathbb{R}^3 : \quad \theta = 0, (r-2)^2 + z^2 \leq 1\} \\ D_2 &= \{(r,\theta,z) \in \mathbb{R}^3 : \quad z = 0, r \leq 1\} \end{aligned}$$

Claim: Let p be the point with $(r,\theta,z) = (2,\pi,0)$. There is a strong deformation retract

$$r : \mathbb{R}^3 - \{p\} \to X.$$

Assuming the claim, we proceed as follows. $\mathbb{R}^3 - \{p\}$ is clearly homeomorphic to $\mathbb{R}^3 - \{0\}$ which we showed in class admits a strong deformation retract onto S^2, the unit sphere centered at 0. Since we also showed that strong deformation retracts are homotopy equivalences, we get

$$X \approx \mathbb{R}^3 - \{p\} \approx S^2$$

as desired.

Proof of claim: The complement of X in $\mathbb{R}^3$ has two components: the inside of T^2 excluding D_1:

$$I = \{(r-2)^2 + z^2 < 1, \quad 0 < \theta < 2\pi\},$$

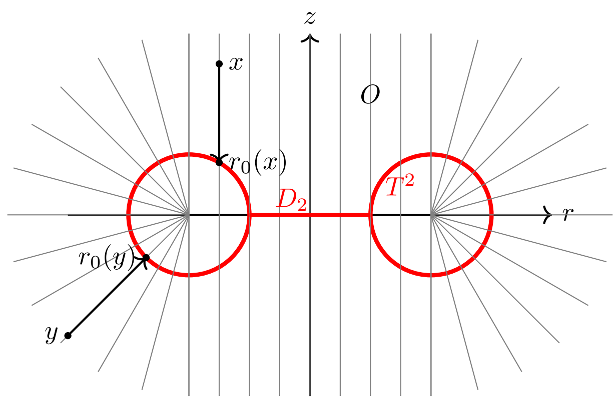
and the outside of T^2 excluding D_2:

$$O = \{(r-2)^2 + z^2 > 1, \text{ and } z \neq 0 \text{ if } r \leq 1\}.$$

We first define a strong deformation retract

$$r_0 : O \cup T^2 \cup D_2 \to T^2 \cup D_2$$

as follows. Let $\mathcal{L}$ be the collection of lines and half lines in $O \cup T^2 \cup D_2$ given by the vertical lines with $r \leq 2$ and the half lines contained in the $\theta = \theta_0$ planes, passing through the $\{r = 2, z = 0\}$ circle, and lying in the region $r \geq 2$. These lines are depicted in the below diagram lying in any $\theta = \theta_0$ plane.



Each point $x \in O \cup T^2 \cup D_2$ lies on a unique line $L_x \in \mathcal{L}$ and we define $r_0(x)$ to be $L_x \cap (T^2 \cup D_2)$ (taking the nearest point if there are two points in the intersection). The map r_0 is a strong deformation retract by the homotopy:

$$H(x,t) = tr_0(x) + (1-t)x.$$

We note that $H(x, 0) = x$ and $H(x, 1) = r_0(x)$ and that $H(x,t) \in O \cup T^2 \cup D_2$ since by construction, the whole line segment between x and $r_0(x)$ is in $O \cup T^2 \cup D_2$.

We next define a strong deformation retract

$$r_1 : I \cup T^2 \cup D_1 - \{p\} \to T^2 \cup D_1$$

as follows. The closed cylinder

$$Cyl = \{(u, v, w) \in \mathbb{R}^3 : \quad u^2 + v^2 \leq 1, \quad -\pi \leq w \leq \pi\}$$

maps to $I \cup T^2 \cup D_1$ by

$$r = u + 2, \quad \theta = w + \pi, \quad z = v$$

and under this map, $(0, 0, 0)$ goes to $p = (2, \pi, 0)$ and the boundary of Cyl goes to $T^2 \cup D_1$. Since Cyl is a star shaped region, $Cyl - (0, 0, 0)$ deformation retracts onto its boundary and we define r_1 to be the composition of this deformation retract with the map $Cyl - \{(0, 0, 0)\} \to I \cup T^2 \cup D_1 - \{p\}$. The composition is also a deformation retract since we can compose the homotopy for the retract of $Cyl - \{(0, 0, 0)\}$ with the map to $I \cup T^2 \cup D_1 - \{p\}$.

Since

$$r_0|_{T^2} = Id_{T^2} = r_1|_{T^2}$$

and

$$\big(O \cup T^2 \cup D_2\big) \cap \big(I \cup T^2 \cup D_1 - \{p\}\big) = T^2$$

we see that r_0 and r_1 glue together to give a continuous map

$$r : \mathbb{R}^3 - \{p\} = O \cup T^2 \cup D_1 \cup D_2 \cup I - \{p\} \to T^2 \cup D_1 \cup D_2 = X$$

which by construction is a strong deformation retract whose existence was asserted by the claim.

(3) Show that the projective plane $\mathbb{RP}^2$ is homeomorphic to the mapping cone of the map $f : S^1 \to S^1$ given by $z \mapsto z^2$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Solution:

The preimage of $z \in S^1$ under f is $f^{-1}(z) = \pm\sqrt{z}$. Therefore we may identify $f : S^1 \to S^1$ with the quotient map

$$S^1 \to S^1/\sim$$

where the equivalence relation is $z \sim -z$.

The mapping cone is then by definition

$$C_f = I \times S^1/\sim$$

where now the equivalence relation is $(1, z) \sim *, (0, z) \sim (0, -z)$.

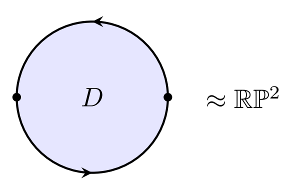
We may identify the quotient of $I \times S^1$ by the equivalence relation $(1, z) \sim *$ as $D = \{z \in \mathbb{C}, |z| \leq 1\}$ by the map

$$(r, z) \mapsto (1 - r)z.$$

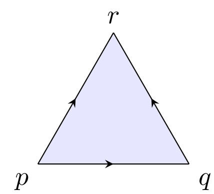
We thus get a homeomorphism

$$C_f \approx \{z \in \mathbb{C}, |z| \leq 1\}/\sim$$

where $z \sim -z$ if $|z| = 1$. This is the unit disk with points on the boundary glued to their opposite point (depicted below). In class we showed this is homeomorphic to $\mathbb{RP}^2$.

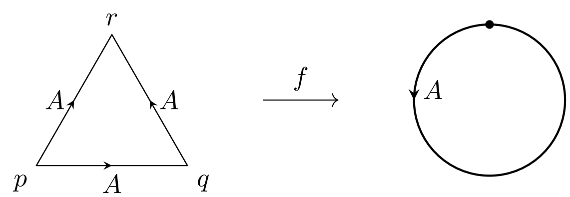


(4) Let $T \subset \mathbb{R}^2$ be the convex hull three non-colinear points $p, q, r \in \mathbb{R}^2$ (i.e. T is a triangular region). The "dunce cap" is the quotient of T by an equivalence relation identifying all three edges in a particular way. Namely, we identify the line segment $\overline{pq}$ with the segment $\overline{pr}$ with the segment $\overline{qr}$, with the given orientations. Show that this space is contractible. (Hint: show that the dunce cap can be realized as the mapping cone of a certain map and then study the map. You may use the fact we proved in class: if $f_0 \simeq f_1$ then $C_{f_0} \simeq C_{f_1}$.)



Solution:

We identify the boundary of the triangle T with S^1 and define $f : S^1 \to S^1$ to be the quotient map depicted below:



We can make f more explicit by identifying $S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$ so $\theta \in S^1$ is the equivalence classes of real numbers θ under $\theta \sim \theta + 2\pi$ and we parameterize the boundary of the triangle so that p is at $\theta = 0$, q is at $\theta = \frac{2\pi}{3}$, and r is at $\theta = \frac{4\pi}{3}$. Then f is given by:

$$f(\theta) = \begin{cases} 3\theta & 0 \le \theta \le \frac{4\pi}{3}, \\ 8\pi - 3\theta & \frac{4\pi}{3} \le \theta \le 2\pi. \end{cases}$$

Here we have written f as a function from $[0, 2\pi]$ to $\mathbb{R}^1$ which we are implicitly composing with the map $\mathbb{R}^1 \to \mathbb{R}^1/2\pi\mathbb{Z}$. By inspection we see that f is well-defined and agrees with the quotient map depicted above.

By the same argument as in the previous problem, the Dunce Cap (which we denote by X) is homeomorphic to the mapping cone of f:

$$X \approx C_f.$$

We claim that $f \simeq Id_{S^1}$. We know from class that homotopic maps give rise to homotopic mapping cones and so assuming the claim we get

$$X \approx C_f \simeq C_{Id_{S^1}} \approx D^2 \simeq *$$

since the mapping cone of Id_{S^1} is

$$I \times S^1/\{1\} \times S^1 \approx D^2$$

and the 2-disk D^2 is contractible.

To prove the claim, we use the following homotopy:

$$H(\theta, t) = \begin{cases} 3\theta t + (1-t)\theta & 0 \le \theta \le \frac{4\pi}{3}, \\ (8\pi - 3\theta)t + (1-t)\theta & \frac{4\pi}{3} \le \theta \le 2\pi. \end{cases}$$