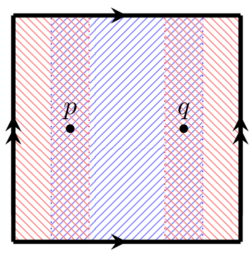
MATH 426 HOMEWORK 3

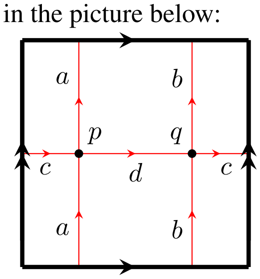
(1) Let $A \subset Y \subset X$ and suppose that $f : X \to Y$ is a strong deformation retract. Show that $f_* : \Pi(X, A) \to \Pi(Y, A)$ is an isomorphism of groupoids.

(2) Consider the torus T^2, with two points $p, q \in T^2$ and the open cover $\{U, V\}$ depicted below:



In the above picture, the sides are identified as usual and the open set U is given by the light blue, northeasterly lined pattern. Similarly the open set V is given by the light red, northwesterly lined pattern.

Consider the paths given in the picture below:



so that

$$a \in \mathrm{Mor}(p,p), \quad b \in \mathrm{Mor}(q,q), \quad c \in \mathrm{Mor}(q,p), \quad d \in \mathrm{Mor}(p,q)$$

in $\Pi(T^2)$.

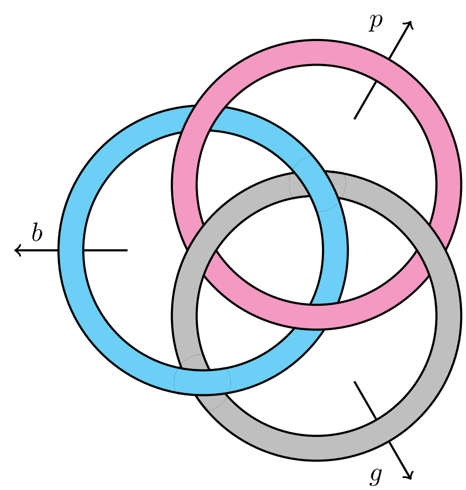
(a) Compute the groupoid $\Pi(U \cap V, \{p, q\})$.
(b) Compute the groupoid $\Pi(U, \{p, q\})$.
(c) Compute the groupoid $\Pi(V, \{p, q\})$.
(d) Use the Van Kampen theorem and the results of the previous parts to compute the groupoid $\Pi(X, \{p, q\})$.
(e) Find an explicit isomorphism of the group $\mathrm{Mor}(p, p)$ with $\mathbb{Z} \times \mathbb{Z}$.

In each of the above (a)–(d), the objects of the groupoid will be the set $\{p, q\}$ so computing the groupoid amounts to giving a set of generators for the morphisms and any non-trivial relations on the morphisms. You may wish to apply the results of problem (1) and/or results we proved in class to justify your computations.

Date: October 6, 2024.

(3) (a) Let $G = \langle x, y \mid xyx = yxy \rangle$ and let $H = \langle a, b \mid a^3 = b^2 \rangle$. Show that $G \cong H$.
(b) Let $G = \langle x, y \mid xy^2 = y^3x, yx^2 = x^3y \rangle$. Prove that G is the trivial group.

(4) Compute the fundamental group of $\mathbb{R}^3 - B$, the complement of the Borromean Rings:



Express your answer as a presentation of a group with generators and relations where your generators should be p, g, b corresponding to loops through the pink, gray, and blue rings respectively. More precisely, consider the base point x_0 to be above the picture and then the loops b, p, g should start at x_0, go to the tail of the indicated arrow, follow along the arrow, and then return to x_0.

Express your relations the form

$$[[\cdot, \cdot], \cdot] = 1$$

where recall that the *commutator bracket* $[\cdot, \cdot]$ is defined by $[x, y] = xyx^{-1}y^{-1}$.