MATH 426 HOMEWORK 3

(1) Let $A \subset Y \subset X$ and suppose that $f : X \to Y$ is a strong deformation retract. Show that $f_* : \Pi(X, A) \to \Pi(Y, A)$ is an isomorphism of groupoids.

By definition of strong deformation retract, $f \circ i = Id_Y$ and $i \circ f \simeq_A Id_X$, so there exists

$$H : I \times X \to X$$

such that

$$\begin{aligned} H(0, x) &= i(f(x)), \\ H(1, x) &= x \\ H(t, a) &= a \text{ if } a \in A. \end{aligned}$$

To show that $f_* : \Pi(X, A) \to \Pi(Y, A)$ is an isomorphism, we show that $i_* : \Pi(Y, A) \to \Pi(X, A)$ is its inverse. $f_* \times i_* = Id_{\Pi(Y,A)}$ follows immediately from $f \circ i = Id_Y$ and functoriality. Thus we need only to show that $f_* \circ i_* = Id_{\Pi(X,A)}$. The objects of $\Pi(X, A)$ are the elements of A and since $i \circ f$ restricted to A is the identity on A (by definition of strong deformation retract), $f_* \circ i_*$ is the identity on objects. So we need only show that for

$$[\gamma] \in \mathrm{Mor}(a_0, a_1),$$

$$[\gamma] = i_* \circ f_*[\gamma] = [i \circ f \circ \gamma],$$

i.e. we need to show that

$$\gamma \simeq_{\{0,1\}} i \circ f \circ \gamma.$$

Let $G : I \times I \to X$ be given by

$$G(s, t) = H(s, \gamma(t)).$$

Then

$$\begin{aligned} G(0, t) &= H(0, \gamma(t)) = i(f(\gamma(t))) \\ G(1, t) &= H(1, \gamma(t)) = \gamma(t) \\ G(s, i) &= H(s, \gamma(i)) = a_i \text{ for } i = 0, 1, \end{aligned}$$

so G is the required homotopy.

(2) Consider the torus T^2, with two points $p, q \in T^2$ and the open cover $\{U, V\}$ depicted below:

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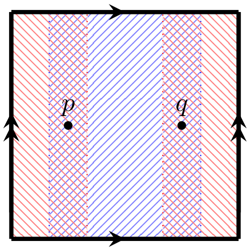
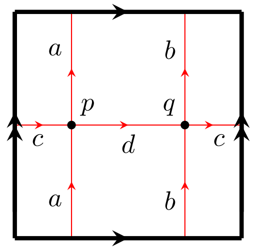


In the above picture, the sides are identified as usual and the open set U is given by the light blue, northeasterly lined pattern. Similarly the open set V is given by the light red, northwesterly lined pattern.

Consider the paths given in the picture below:



so that

$$a \in \mathrm{Mor}(p,p), \quad b \in \mathrm{Mor}(q,q), \quad c \in \mathrm{Mor}(q,p), \quad d \in \mathrm{Mor}(p,q)$$

in $\Pi(T^2)$.

(a) Compute the groupoid $\Pi(U \cap V, \{p,q\})$.
(b) Compute the groupoid $\Pi(U, \{p,q\})$.
(c) Compute the groupoid $\Pi(V, \{p,q\})$.
(d) Use the Van Kampen theorem and the results of the previous parts to compute the groupoid $\Pi(X, \{p,q\})$.
(e) Find an explicit isomorphism of the group $\mathrm{Mor}(p,p)$ with $\mathbb{Z} \times \mathbb{Z}$.

In each of the above (a)–(d), the objects of the groupoid will be the set $\{p,q\}$ so computing the groupoid amounts to giving a set of generators for the morphisms and any non-trivial relations on the morphisms. You may wish to apply the results of problem (1) and/or results we proved in class to justify your computations.

We will use our knowledge of the fundamental groupoid of the circle S^1 from class. Namely, if $x_1, x_2 \in S^1$ are distinct points, then $\Pi(S^1, \{x_1, x_2\})$ is the groupoid with objects $\{x_1, x_2\}$ and generated by the two morphisms $\phi, \psi \in \mathrm{Mor}(x_1, x_2)$ given by the paths from x_1 to x_2 going on opposite sides of the circle. As a corollary, $\Pi(S^1, \{x_1\})$ is the groupoid with one object $\{x_1\}$ and generated by the morphism $\alpha \in \mathrm{Mor}(x_1, x_1)$ which we may take to be $\psi^{-1} \circ \phi$.

(a) The open set $U \cup V$ is homeomorphic to a disjoint union of two copies of $(-\epsilon, \epsilon) \times S^1$ where p and q are in distinct copies. Each copy admits a strong deformation retract onto $\{0\} \times S^1$ and so by problem 1, we may identify $\Pi(U \cup V, \{p,q\})$ as the union of two copies of the groupoid of a circle with one point, namely

$$\Pi(U \cup V, \{p, q\}) \cong \left\{ \begin{array}{ccc} \overset{a}{\circlearrowright} & & \overset{b}{\circlearrowright} \\ p & & q \end{array} \right\}$$

(b) The set U is clearly homeomorphic to $(-\epsilon, \epsilon) \times S^1$ where under the appropriate identifications, p and q are located at $\{-\frac{\epsilon}{2}\} \times x_1$ and $\{\frac{\epsilon}{2}\} \times x_1$ respectively. Let p' be the point $\{-\frac{\epsilon}{2}\} \times x_2$ where $x_1, x_2 \in S^1$ are distinct points and let γ be a path from q to p'. We then get the following isomorphisms of groupoids:

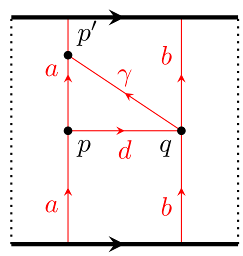
$$\Pi(U, \{p, q\}) \cong \Pi(U, \{p, p'\}) \cong \Pi(S^1, \{x_1, x_2\})$$

where the second isomorphism is induced by the projection $U \to \{-\frac{\epsilon}{2}\} \times S^1$ which is a deformation retract and the first isomorphism is given on objects by $p \mapsto p$, $q \mapsto p'$ and on morphisms by composing with γ.

Since

$$\Pi(S^1, \{x_1, x_2\}) \cong \left\{ x_1 \underset{\phi}{\overset{\psi}{\rightrightarrows}} x_2 \right\}$$

we need to see how the morphisms a, b, and d get written in terms of ϕ and ψ under the above isomorphisms. Consulting the diagram below



we see that under the above groupoid isomorphisms we have

$$\begin{aligned} d &\mapsto \gamma \circ d &&\mapsto \phi \\ a &\mapsto a &&\mapsto \psi^{-1} \circ \phi \\ b &\mapsto \gamma \circ b \circ \gamma^{-1} &&\mapsto \phi \circ a \circ \phi^{-1} \\ & && = \phi \circ \psi^{-1} \circ \phi \circ \phi^{-1} \\ & && = \phi \circ \psi^{-1}. \end{aligned}$$

It follows that d, a, and b generate $\Pi(U, \{p, q\})$ with one relation:

$$d \circ a = \phi \circ \psi^{-1} \circ \phi = b \circ d.$$

So we conclude:

$$\Pi(U, \{p, q\}) \cong \left\{ \begin{array}{ccc} \overset{a}{\circlearrowright} & & \overset{b}{\circlearrowright} \\ p & \xrightarrow{d} & q \end{array} , \quad d \circ a = b \circ d \right\}$$

(c) By an argument parallel to the one for (b), we see that

$$\Pi(V, \{p, q\}) \cong \left\{ \begin{array}{ccc} \overset{a}{\circlearrowright} & & \overset{b}{\circlearrowright} \\ p & \xleftarrow{\ c\ } & q \end{array} \quad , \quad a \circ c = c \circ b \right\}$$

(d) The groupoid

$$\left\{ \begin{array}{ccc} \overset{a}{\circlearrowright} & & \overset{b}{\circlearrowright} \\ p & \underset{d}{\overset{c}{\leftrightarrows}} & q \end{array} \quad , \quad a \circ c = c \circ b, \quad d \circ a = b \circ d \right\}$$

then satisfies the universal property for the diagram

$$\begin{array}{ccc} \Pi(U \cap V, \{p, q\}) & \longrightarrow & \Pi(U, \{p, q\}) \\ \downarrow & & \downarrow \\ \Pi(V, \{p, q\}) & \longrightarrow & \Pi(X, \{p, q\}) \end{array}$$

and hence must be isomorphic to $\Pi(X, \{p, q\}$.

(e) Any morphism in the groupoid $\Pi(X, \{p, q\}$ must be a word in a, b, c, d. Using the relation

$$b = c^{-1} \circ a \circ c$$

we may reduce any word to a word in a, c, and d. Then if the morphism is in $\mathrm{Mor}(p, p)$ and we have eliminated b using the relation, the resulting word must contain c and d only in the combinations $c \circ d$ or $d^{-1} \circ c^{-1} = (c \circ d)^{-1}$. Writing $x = c \circ d$ we see that all morphisms in $\mathrm{Mor}(p, p)$ are words in a and x. But

$$a \circ c = c \circ b = c \circ (d \circ a \circ d^{-1})$$

and so

$$a \circ c \circ d = c \circ d \circ a$$

which is $a \circ x = x \circ a$ and so by commuting, any word in a and x can be uniquely written in the form $a^k \circ x^l$ for $(k, l) \in \mathbb{Z} \times \mathbb{Z}$. We conclude that

$$(k, l) \mapsto a^k \circ x^l$$

is an isomorphism of the group $\mathbb{Z} \times \mathbb{Z}$ with $\mathrm{Mor}(p, p)$.

(3) (a) Let $G = \langle x, y \mid xyx = yxy \rangle$ and let $H = \langle a, b \mid a^3 = b^2 \rangle$. Show that $G \cong H$.

We define a homomorphism $f : H \to G$ by

$$f(a) = xy, \quad f(b) = xyx = yxy.$$

This is well defined since

$$f(a^3) = (xy)^3 = xyxyxy = (xyx)(yxy) = (xyx)^2 = f(b^2).$$

We define $g : G \to H$ by

$$g(x) = a^2 b^{-1}, \quad g(y) = ba^{-1}.$$

This is well defined since

$$g(xyx) = (a^2b^{-1})ba^{-1}(a^2b^{-1}) = a^3b^{-1} = b^2b^{-1} = b$$
$$g(yxy) = (ba^{-1})a^2b^{-1}(ba^{-1}) = b.$$

Moreover g and f are inverses of each other:

$$g(f(a)) = g(xy) = a^2b^{-1}ba^{-1} = a$$
$$g(f(b)) = g(xyx) = b$$

(b) Let $G = \langle x, y \,|\, xy^2 = y^3x, yx^2 = x^3y\rangle$. Prove that G is the trivial group.

We note that the first relation can be rewritten as $xy^2x^{-1} = y^3$ which by raising both sides to the power k implies

$$xy^{2k}x^{-1} = y^{3k}, \quad y^{2k} = x^{-1}y^{3k}x$$

for any $z \in \mathbb{Z}$. Similarly, the second relation is $yx^2y^{-1} = x^3$ which implies

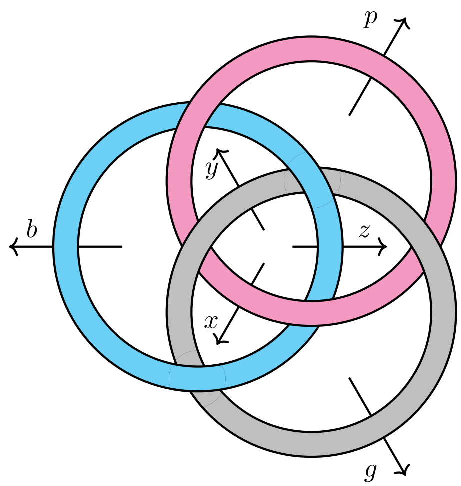
$$yx^{2k}y^{-1} = x^{3k}, \quad x^{2k} = y^{-1}x^{3k}y.$$

Therefore we have

$$\begin{aligned}
x^6 &= yx^4y^{-1} \\
\Rightarrow yx^6y^{-1} &= y^2x^4y^{-2} \\
x^9 &= (x^{-1}y^3x)x^4(x^{-1}y^{-3}x) \\
x^9 &= x^{-1}y^3x^4y^{-3}x \\
\Rightarrow x^9 &= y^3x^4y^{-3} \\
x^9 &= y^2(yx^4y^{-1})y^{-2} \\
x^9 &= y^2x^6y^{-2} \\
x^9 &= y(yx^6y^{-1})y^{-1} \\
x^9 &= yx^9y^{-1} \\
\Rightarrow y^{-1}x^9y &= x^9 \\
x^6 &= x^9 \\
\Rightarrow x^3 &= 1.
\end{aligned}$$

But then the relation $yx^2y^{-1} = x^3 = 1$ implies that $x^2 = 1$ and then $x^2 = x^3$ which implies $x = 1$. Finally $x = 1$ and the relation $xy^2 = y^3x$ implies $y^2 = y^3$ which implies $y = 1$. Since we've proved that $x = y = 1$ and they generate G, G is trivial.

(4) Compute the fundamental group of $\mathbb{R}^3 - B$, the complement of the Borromean Rings:



Express your answer as a presentation of a group with generators and relations where your generators should be p, g, b corresponding to loops through the pink, gray, and blue rings respectively. More precisely, consider the base point x_0 to be above the picture and then the loops b, p, g should start at x_0, go to the tail of the indicated arrow, follow along the arrow, and then return to x_0.

Express your relations the form

$$[[\cdot,\cdot],\cdot] = 1$$

where recall that the *commutator bracket* $[\cdot,\cdot]$ is defined by $[x,y] = xyx^{-1}y^{-1}$.

We label the loops through the remaining overpasses by x, y, and z as in the diagram. We then get Wirtinger relations for each of the six crossings, starting with the pink-grey crossing on the right and proceeding counter-clockwise:

$$\begin{aligned} xg = gp &\quad\Rightarrow\quad x = gpg^{-1} \\ yz = zg &\quad\Rightarrow\quad y = zgz^{-1} \\ zp = pb &\quad\Rightarrow\quad z = pbp^{-1} \\ xy = yp &\quad\Rightarrow\quad x = ypy^{-1} \\ yb = bg &\quad\Rightarrow\quad y = bgb^{-1} \\ zx = xb &\quad\Rightarrow\quad z = xbx^{-1} \end{aligned}$$

We use the first, third, and fifth of the equations above to write x, y, and z in terms of p, b, and g. The remaining equations then become:

$$\begin{aligned} y &= zgz^{-1} \\ \Rightarrow\quad bgb^{-1} &= (pbp^{-1})g(pb^{-1}p^{-1}) \\ \Rightarrow\quad (pb^{-1}p^{-1})bg &= g(pb^{-1}p^{-1})b \\ \Rightarrow\quad [p,b^{-1}]g &= g[p,b^{-1}] \\ \Rightarrow\quad [[p,b^{-1}],g] &= 1, \end{aligned}$$

$$\begin{aligned} & x = ypy^{-1} \\ \Rightarrow \quad & gpg^{-1} = (bgb^{-1})p(bg^{-1}b^{-1}) \\ \Rightarrow \quad & (bg^{-1}b^{-1})gp = p(bg^{-1}b^{-1})g \\ \Rightarrow \quad & [b, g^{-1}]p = p[b, g^{-1}] \\ \Rightarrow \quad & [[b, g^{-1}], p] = 1, \end{aligned}$$

$$\begin{aligned} & z = xbx^{-1} \\ \Rightarrow \quad & pbp^{-1} = (gpg^{-1})b(gp^{-1}g^{-1}) \\ \Rightarrow \quad & (gp^{-1}g^{-1})pb = b(gp^{-1}g^{-1})p \\ \Rightarrow \quad & [g, p^{-1}]b = b[g, p^{-1}] \\ \Rightarrow \quad & [[g, p^{-1}], b] = 1. \end{aligned}$$

Therefore $\pi_1(\mathbb{R}^3 - B)$ is the group with presentation:

$$\langle p, b, g \quad | \quad [[g, p^{-1}], b] = [[b, g^{-1}], p] = [[p, b^{-1}], g] = 1 \rangle$$

We remark that only two of the three relations are necessary: one can derive any one relation from the other two.