MATH 426 HOMEWORK 4

(1) Prove that the number of degree d covering spaces (up to isomorphism) of the 2-torus $S^1 \times S^1$ is given by

$$\sigma(d) = \sum_{k|d} k,$$

the sum of divisors of d.

(2) Let $X = S^1 \vee S^1$. In this problem, I will ask you to draw several things and you are allowed to include a pdf picture produced by hand or on a tablet instead of using tikz or some other native LaTeX to make the pictures.

   (a) Draw a picture of all degree two covers of X and determine the group of deck transformations of each cover. How many are there?

   (b) Draw a picture of all degree three covers of X and determine the group of deck transformations of each cover. How many are there?

(3) We say that G is a *topological group* if G is a topological space and a group and the maps $m : G \times G \to G$ and $i : G \to G$ given by multiplication $m(g,h) = gh$ and inverse $i(g) = g^{-1}$ are continuous. Let $e \in G$ be the identity element and suppose that $p : \widetilde{G} \to G$ is a covering space. Let $\tilde{e} \in \widetilde{G}$ be a point with $p(\tilde{e}) = e$. Prove that there exists a unique group structure on $\widetilde{G}$ with identity element $\tilde{e}$ making $\widetilde{G}$ into a topological group and making $p : \widetilde{G} \to G$ a group homomorphism.

(4) Let $X = S^1 \vee S^1$ as in problem (1) and let $x_0 \in X$ be the point where the two circles are attached. As we know from class $\pi_1(X, x_0)$ is the free group on two generators. Construct the covering space $p : \widetilde{X} \to X$ corresponding to the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0)$ and describe the group of deck transformations of $\widetilde{X}$.

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