MATH 426 HOMEWORK 4 SOLUTIONS

Recall the standing assumption that all covering spaces are path-connected.

(1) The number of degree n covering spaces of $S^1 \times S^1$ up to isomorphism is equal to the sum of the divisors of n, namely

$$\sum_{k|n} k$$

Proof. By the classification of covering spaces, the number of path-connected degree n covering spaces of $S^1 \times S^1$ is equal to the number of conjugacy classes of index-n subgroups of $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$, so we will count these conjugacy classes. Note that since $\mathbb{Z}\times\mathbb{Z}$ is Abelian, conjugacy classes of subgroups of $\mathbb{Z} \times \mathbb{Z}$ are singletons, so we are simply counting index-n subgroups of $\mathbb{Z} \times \mathbb{Z}$. Let us give $\mathbb{Z} \times \mathbb{Z}$ the presentation $\langle x, y | [x, y]\rangle$, or simply $\langle x, y\rangle$; we omit the commutator relation since all of the groups we are working with in this problem are Abelian.

Lemma 1. *Every subgroup $H \subset \mathbb{Z}\times\mathbb{Z}$ is isomorphic to $\langle ax+by, cx+dy\rangle$, for some $a, b, c, d \in \mathbb{Z}$. (Note that any or all of a, b, c, d may be 0).*

Proof. Recall that subgroups of free Abelian groups are free Abelian, so any subgroup $H \subset \mathbb{Z} \times \mathbb{Z}$ is isomorphic to $\mathbb{Z}^r$, for some $r \in \mathbb{N}$. We only need to show that $r \leq 2$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\iota} \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2/H \longrightarrow 0,$$

where H is the image $\iota(\mathbb{Z}^r)$. Using the standard bases for $\mathbb{Z}^r$ and $\mathbb{Z}^2$, the injective homomorphism ι is represented by a $2 \times r$ matrix with integer entries. We may consider it as a matrix with entries in $\mathbb{Q}$. This matrix also represents an injective homomorphism, but this is now a map of vector spaces (check these two statements!)[1], therefore $r \leq 2$. ☐

We call $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ the *presentation matrix* of the group $\langle ax+by, cx+dy\rangle$.

Lemma 2. *Two matrices $P_1, P_2 \in \mathrm{Mat}_2(\mathbb{Z})$ are presentation matrices of the same group $H \subset \mathbb{Z}\times\mathbb{Z}$ if and only if there exists $T \in GL_2(\mathbb{Z})$ such that*

$$P_1 T = P_2.$$

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Written by Mihai Marian.

[1]For those who know these words: we are using the left-exactness of the functor $-\otimes\mathbb{Q}$. It is, in fact, also right-exact.

Proof. That $\langle a_1x + b_1y, c_1x + d_1y\rangle = \langle a_2x + b_2y, c_2x + d_2y\rangle$ means that $a_2x + b_2y$ is a $\mathbb{Z}$-linear combination of $a_1x + b_1y$ and $c_1x + d_1y$, i.e. that there are integers α, β such that

$$a_2x + b_2y = \alpha(a_1x + b_1y) + \beta(c_1x + d_1y).$$

Analogously for $c_2x + d_2y$, we can find integers γ, δ so that

$$\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}$$

Let $T = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$. By expressing the $a_1x + b_1y$ and $c_1x + d_1y$ in terms of the second set of generators, we obtain T^{-1}, showing that $T \in GL_2(\mathbb{Z})$.

Conversely, if there is a matrix $T \in GL_2(\mathbb{Z})$ such that $P_1T = P_2$, then clearly, the generators using the first presentation are transformed into new generators of the same group H. □

Lemma 3. *Let* $P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. *There exists* $T \in GL_2(\mathbb{Z})$ *such that*

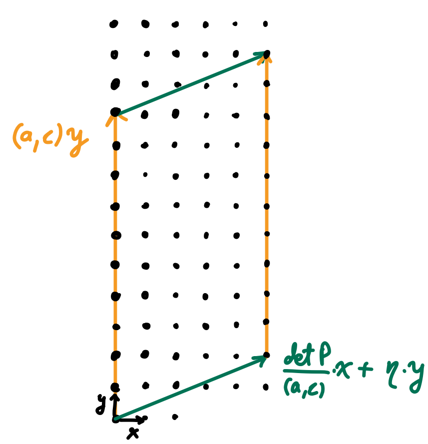
$$PT = \begin{pmatrix} \frac{\det(P)}{(a,c)} & 0 \\ \eta & (a,c) \end{pmatrix}, \qquad (*)$$

where (a, c) *denotes the* gcd *of* a *and* c, *and* $\eta \in \{0, 1, \ldots, (a, c) - 1\}$.

Proof. The matrix on the right-hand side of $(*)$ can be obtained from P by a sequence of column operations. □

Lemma 4. *Let* $H \subset \mathbb{Z} \times \mathbb{Z}$ *be a subgroup and* P *a presentation matrix for* H. *If the index* $[\mathbb{Z} \times \mathbb{Z} : H]$ *is finite, then it is equal to* $|\det(P)|$.

Proof. Let H have a presentation matrix of the form $(*)$. Then, by the previous lemma, we can think of H as being generated by the elements $\frac{\det(P)}{(a,c)}x + \eta y$ and $(a, c)y$. Thinking of $\mathbb{Z} \times \mathbb{Z}$ as a lattice in $\mathbb{R}^2$, we may draw the generators of H as vectors, as in the figure below.



If we assume that $[\mathbb{Z} \times \mathbb{Z} : H]$ is finite, then the two generators of H are linearly independent and the parallelogram that they span has non-zero area. We leave as exercises the following two claims. Claim 1: the cosets in $\mathbb{Z} \times \mathbb{Z}/H$ correspond to the integer lattice points contained inside the parallelogram spanned by the generators of H, together with half of the points contained in the interiors of the sides of the parallelogram, together with the lattice point $(0,0)$. Claim 2: the count of lattice points in claim 1 is precisely $|\det(P)|$. The previous figure should help with proving the claims. □

We can now finish the argument. Every index-n subgroup $H \subset \mathbb{Z} \times \mathbb{Z}$ has a presentation matrix P that looks like the right-hand side of equation $(*)$ and, moreover, this presentation matrix is unique. This proves that the collection of index-n subgroups are in bijective correspondence with the collection of matrices of the above form, so we just need to count these matrices. For every divisor k of n, there are k matrices

$$\begin{pmatrix} \frac{n}{k} & 0 \\ * & k \end{pmatrix}$$

with $* \in \{0, 1, \ldots, k-1\}$. Thus there are $\sum_{k|n} k$ subgroups of index n in $\mathbb{Z} \times \mathbb{Z}$. □

(2) List all of the path-connected degree 2 and degree 3 covering spaces of $X = S^1 \vee S^1$ and compute their groups of deck transformations.

Solution. Here is a lemma we will use over and over again. For its proof, see Hatcher's discussion of covering spaces of X in §1.3.

Lemma 5. *Let Γ be an oriented graph with edges labelled a and b. If Γ has 4 edge ends incident to every vertex*[2] *and at every vertex precisely one of the edge ends carries each different label and orientation, then there is a covering map $\Gamma \to X$.*

Note that a degree n covering space of X is a 4-valent graph (as in the above lemma) with precisely n vertices. Listing these is not something one wants to do by hand for n much larger than 2. The following are all of the degree 2 (path-connected) covering spaces of X. All three covers have group of deck transformations isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The case of degree 3 covers is more complicated. We wish to list 4-valent graphs with 3 vertices. Taking a hint from the degree 2 case, we can start by listing the possibly disconnected degree 3 covers of one of the two circles in the wedge X, for example the red one. There are, up to isomorphism, 3 such covers, listed below.

There is a similar set of possibly-disconnected covers of the other S^1 that forms the wedge X (the blue circle). By the lemma mentionned at the beginning of this solution, we can form a degree 3 covering space by taking a degree 3 covering space of the red S^1 and degree 3 covering space of the blue S^1, and identifying their vertices pairwise. We claim that figure 3 contains the complete list of the isomorphism classes of degree 3 covering spaces of X.

Let us sketch a proof of the claim and then indicate what the group of deck transformation is for each $X_i \to X$. Let $p\colon \widehat{X} \to X$ be a connected

[2] I call this a *4-valent* graph; it is also known as a *4-regular* graph.

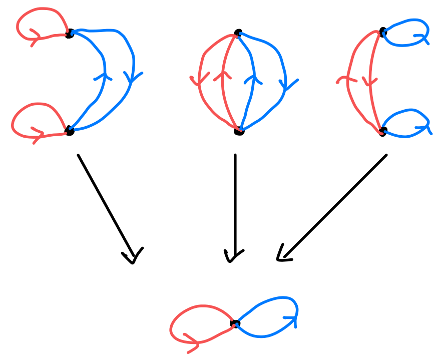


FIGURE 1. The degree 2 covers of X.

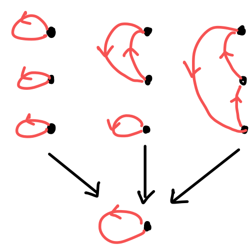


FIGURE 2. The degree 3 not-necessarily-connected covers of S^1

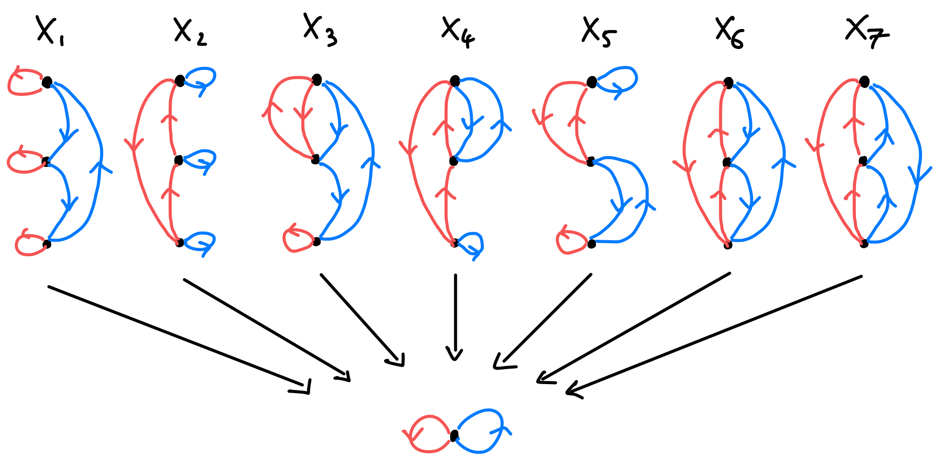


FIGURE 3. The degree 3 covers of X.

covering space and let S^1_r be the red circle in X, and S^1_b, the blue one. We will argue by cases.

(a) Suppose first that $p^{-1}(S^1_r)$ has three connected components. Then, since $\widehat{X}$ is connected, it must be that $p^{-1}(S^1_b)$ is connected. Therefore, up to orientation, $\widehat{X} = X_1$. There are no choices of orientation on the red edges and there are two choices of orientation on the blue edges (remember that each vertex of $\widehat{X}$ needs to have one incoming and one outgoing blue edge). It is not hard to see that the two choices yield isomorphic covering spaces.

(b) Suppose next that $p^{-1}(S^1_r)$ has two connected components. Then $p^{-1}(S^1_b)$ can have either one or two connected components. If $p^{-1}(S^1_b)$ is connected, then, up to isomorphism, $\widehat{X} = X^3$. If instead $p^{-1}(S^1_b)$ has two components, then $\widehat{X} = X_5$, up to isomorphism. This may be a little surprising, but you can convince yourself that the following isomorphisms of covering spaces can be defined.

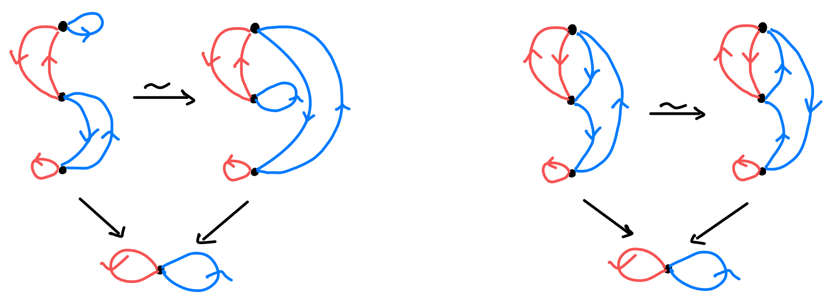


FIGURE 4. Two isomorphisms of covering spaces.

(c) Finally, if $p^{-1}(S^1_r)$ is connected, $p^{-1}(S^1_b)$ can have any number of connected components, so we have to consider subcases.
   (i) If $p^{-1}(S^1_b)$ has three components, then we are in the same case as (a), but with colours reversed, so $\widehat{X} = X_2$.
   (ii) If $p^{-1}(S^1_b)$ has two components, then we are in the same case as in (b), so $\widehat{X} = X_3$ or $\widehat{X} = X_4$.
   (iii) If $p^{-1}(S^1_b)$ is connected, then $\widehat{X}$ is either X_6 or X_7. Surprisingly, there is no isomorphism $X_6 \simeq X_7$. Indeed, the closed path that is the concatenation of both loops in X lifts to a closed path in X_6, but not in X_7.

Finally, we need to compute the groups of deck transformations. Let D_i denote the group of deck transformations of $X_i \to X$. We have
   (i) $D_1 \simeq D_2 \simeq \mathbb{Z}/3\mathbb{Z}$. Proof: the generator of the group of deck transformations of $p^{-1}(S^1_b)$ (rotation by $\pi/3$) also generates D_1, rotate the connected cover of S^1_b and carry the red petals around. Similarly for D_2.
   (ii) $D_3 \simeq D_4 \simeq D_5 \simeq 1$. Proof: since there is at most one self loop of a given colour, any deck transformation must fix a vertex that has a self-loop, hence must fix the whole covering space.

(iii) $D_6 \simeq D_7 \simeq \mathbb{Z}/3\mathbb{Z}$. Proof: similar to the one for D_1. Visually, one may redraw the covering space symmetrically, to see the deck transformations as rotations in $\mathbb{R}^2$.

□

(3) Let G be a path-connected, locally path-connected topological group with basepoint the identity element $e \in G$, multiplication $m\colon G \times G \to G$ and inverse $i\colon G \to G$, and let $\widetilde{G} \xrightarrow{p} G$ be a covering space, with a chosen basepoint $\widetilde{e} \in p^{-1}(e)$. There is a unique group structure on $\widetilde{G}$ with identity element $\widetilde{e}$ such that p is a group homomorphism.

Proof. The niceties of topological groups are contained in the following lemma. Note that we use the isomorphism

$$\pi_1(X \times Y, x \times y) \simeq \pi_1(X, x) \times \pi_1(Y),$$

where (X, x) and (Y, y) are arbitrary pointed topological spaces.

Lemma 6. *The map*

$$m_*\colon \pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$$

coincides with the group operation in $\pi_1(G, e)$. Moreover, $\pi_1(G, e)$ is an Abelian group.

Proof. This is known as the Eckmann-Hilton argument. Let us denote $m_*([\gamma], [\delta])$ by $[\gamma] \cdot [\delta]$, and let us use $*$ to denote the group operation in $\pi_1(G, e)$ (induced from path concatenation). The first observation is that the homotopy class of the constant path at e is the identity element with respect to both $*$ and $\cdot$. Let us abuse notation and call this constant path e. The second observation is the following distributivity identity, which holds for all loops $\alpha, \beta, \gamma, \delta$ in G based at e:

$$(\alpha * \beta) \cdot (\gamma * \delta) = (\alpha \cdot \beta) * (\gamma \cdot \delta).$$

This is immediate from the definitions of the two operations: both the left- and the right-hand sides of the above equation are the path given by

$$t \mapsto \begin{cases} \alpha(2t) \cdot \gamma(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t-1) \cdot \delta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

Now we can run the Eckmann-Hilton argument. Let $[\alpha], [\beta] \in \pi_1(G, e)$. We have

$$[\alpha] \cdot [\beta] = ([\alpha] * [e]) \cdot ([e] * [\beta]) = ([\alpha] \cdot [e]) * ([e] \cdot [\beta]) = [\alpha] * [\beta],$$

which shows that the two group operations are equal. We also have

$$[\alpha] \cdot [\beta] = ([e] * [\alpha]) \cdot ([\beta] * [e]) = [\beta] * [\alpha],$$

which shows that the operation is commutative and completes the proof of the lemma. □

Corollary 6.1. *The induced map $i_*\colon \pi_1(G, e) \to \pi_1(G, e)$ is given by $[\gamma] \mapsto [\gamma]^{-1}$, where we mean the inverse with respect to the concatenation group operation on $\pi_1(G, e)$.*

Let us now construct the the multiplication and inversion maps $\widetilde{m}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ and $\widetilde{i}\colon \widetilde{G} \to \widetilde{G}$ that provide $\widetilde{G}$ with its group structure. Consider the following diagram:

$$\begin{CD} \widetilde{G} @>{\widetilde{i}}>> \widetilde{G} \\ @V{p}VV @VV{p}V \\ G @>{i}>> G \end{CD}$$

Propositions 1.33 and 1.34 in Hatcher tell us that there exists a *unique* lift $\widetilde{i}\colon \widetilde{G} \to \widetilde{G}$ of i that fixes $\widetilde{e}$ and makes the above diagram commute if and only if

$$(ip)_*(\pi_1(\widetilde{G}, \widetilde{e}) \subset p_*(\pi_1(\widetilde{G}, \widetilde{e}). \tag{i}$$

It follows from the previous corollary that the inclusion (i) holds: the isomorphism i_* is simply the map that takes every group element to its inverse, so it preserves subgroups of $\pi_1(G, e)$; in particular, it preserves $p_*(\pi_1(\widetilde{G}, \widetilde{e})$. This shows the existence and uniqueness of the map $\widetilde{i}\colon \widetilde{G} \to \widetilde{G}$.

To define the multiplication $\widetilde{m}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, consider the diagram

$$\begin{CD} \widetilde{G} \times \widetilde{G} @>{\widetilde{m}}>> \widetilde{G} \\ @V{p \times p}VV @VV{p}V \\ G \times G @>{m}>> G \end{CD}$$

To obtain $\widetilde{m}$ as a lift, we need to argue the inclusion

$$(m \circ p \times p)_* \pi_1(\widetilde{G} \times \widetilde{G}, \widetilde{e} \times \widetilde{e}) \subset p_* \pi_1(\widetilde{G}, \widetilde{e}). \tag{ii}$$

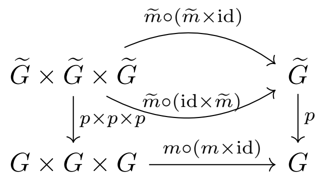
Arguing as for (i), $(p \times p)_*(\pi_1(\widetilde{G} \times \widetilde{G}, \widetilde{e} \times \widetilde{e})$ is a subgroup of $\pi_1(G \times G, e \times e) \simeq \pi_1(G, e) \times \pi_1(G, e)$, so it is preserved by multiplication. This proves the inclusion (ii) and the existence of the *unique* lift $\widetilde{m}$.

Thus we have produced a continuous binary operation $\widetilde{m}$ and a continuous function $\widetilde{i}$ on $\widetilde{G}$ that should give $\widetilde{G}$ the structure of a topological group with identity $\widetilde{e}$. We need to check that the group axioms are satisfied for the tuple $(\widetilde{(G)}, \widetilde{m}, \widetilde{i}, \widetilde{e})$, in other words we need to check

(a) (Identity) For all $a \in \widetilde{G}$, we have
$$\widetilde{m}(\widetilde{e}, a) = \widetilde{m}(a, \widetilde{e}) = a.$$
(b) (Inversion) For all $a \in \widetilde{G}$, we have
$$\widetilde{m}(a, \widetilde{i}(a)) = \widetilde{m}(\widetilde{i}(a), a) = \widetilde{e}.$$
(c) (Associativity) For all $a, b, c \in \widetilde{G}$, we have
$$\widetilde{m}(\widetilde{m}(a, b), c) = \widetilde{m}(a, \widetilde{m}(b, c)).$$

Assuming that $\widetilde{G}$ is indeed a group, the commutative diagrams defining $\widetilde{m}$ and $\widetilde{i}$ immediately prove that p is a group homomorphism (check this). Thus, all we have left is to check the axioms (a), (b), (c). The only tool we need is proposition 1.34 from Hatcher, which says that lifts to covering spaces that agree at a point are equal at all points. We will show how the argument works for proving (c), and sketch the proofs of (a) and (b).

(c) Consider the following diagram



We can check that the two maps at the top of the diagram are both lifts of the map $(p \times p \times p) \circ m \circ (m \times \mathrm{id})\colon \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$: let $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3 \in \widetilde{G}$ and let $g_i = p(\widetilde{g}_i)$, for $i = 1, 2, 3$. Then we have

$$\begin{aligned} p(\widetilde{m}(\widetilde{g}_1, \widetilde{m}(\widetilde{g}_2, \widetilde{g}_3)) &= m(p(\widetilde{g}_1), p(\widetilde{m}(\widetilde{g}_2, \widetilde{g}_3))) \\ &= m(g_1, m(p(\widetilde{g}_2), p(\widetilde{g}_3)) \\ &= m(g_1, m(g_2, g_3)). \end{aligned}$$

Similarly,

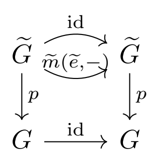
$$p(\widetilde{m}(\widetilde{m}(\widetilde{g}_1, \widetilde{g}_2), \widetilde{g}_3)) = m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3)).$$

Moreover, the two maps at the top of the diagram agree at $\widetilde{e}$, since $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$. Therefore, by the uniqueness of lifts, the two maps are equal, so we have

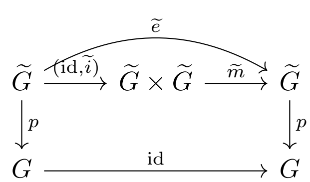
$$\widetilde{m}(a, \widetilde{m}(b, c)) = \widetilde{m}(\widetilde{m}(a, b), c),$$

for all $a, b, c \in \widetilde{G}$, as was to be proved.

(a) To prove (a), consider the diagram



(b) To prove (b), consider the diagram



where the topmost map is the constant map at $\widetilde{e}$, and argue that the composite $\widetilde{m} \circ (\mathrm{id} \times \widetilde{i})$ agrees with $\widetilde{e}$.

□

(4) Construct the covering space of $X = S^1 \vee S^1$ that corresponds to the commutator subgroup $[\pi_1(X), \pi_1(X)] \subset \pi_1(X)$.

Proof. Let S^1 be the unit complex numbers, with basepoint 1 and consider $X \subset S^1 \times S^1$ to be the set

$$\{(x, y) \in S^1 \times S^1 : x = 1 \text{ or } y = 1\}.$$

We claim that the covering space corresponding to the commutator subgroup is the space

$$\widehat{X} := \{(x, n) : x \in \mathbb{R}, n \in \mathbb{Z}\} \cup \{(m, y) : m \in \mathbb{Z}, y \in \mathbb{R}\} \subset \mathbb{R}^2,$$

which we call the integer lattice grid in $\mathbb{R}^2$. A covering space is not a covering space without a covering map, so let us define $p\colon \widehat{X} \to X$ as the restriction of the universal covering map $\mathbb{R}^2 \to S^1 \times S^1$, which is given by

$$(x, y) \mapsto (e^{i2\pi x}, e^{i2\pi y}).$$

Note that the above map restricted to $\widehat{X}$ does indeed land in $X \subset S^1 \times S^1$, since one of the two coordinates in (x, y) is an integer, so one of $e^{i2\pi x}$ or $e^{i2\pi y}$ is equal to 1.

The restriction of a covering map is a covering map, so $p\colon \widehat{X} \to X$ is a covering space, so all we need in order to prove the claim is to show that $\widehat{X}$ is indeed the covering space that corresponds to the commutator subgroup. This is equivalent to showing that the group of deck transformations is $F_2/[F_2, F_2] = \mathbb{Z} \times \mathbb{Z}$. This is clear, since deck transformations are determined by the image of a single point, and the group $\mathbb{Z} \times \mathbb{Z}$ acting by translations on $\mathbb{R}^2$ preserves the integer lattice grid and commutes with the projection, and, moreover, the orbit of the point $(0, 0)$ is $\mathbb{Z} \times \mathbb{Z}$, so we can map $(0, 0)$ to any other vertex in the integer lattice grid. This completes the proof that the group of deck transformations of $\widehat{X} \to X$ is indeed $\mathbb{Z} \times \mathbb{Z}$. □