Counting surfaces on Calabi-Yau fourfolds

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Part I. Introduction

1.1 A leading question

- Let \( X \) be a smooth, projective Calabi-Yau fourfold / \( \mathbb{C} \), i.e. \( K_X \equiv 0 \) (includes H.K. abelian fourfolds).
- \( \gamma \in H^2(X, \Omega^2_X) \).

**Goal**: Enumerate surfaces \( S \subset X \) in class \( \gamma \) via sheaf theory.

**Two hurdles**

a/ A (2,2) class \( \gamma \) does not remain (2,2) as \( X \) deforms.

\( \Rightarrow \) Deformation invariant quantities = 0

\[ H^3(X, \Omega^2_X) \cong H^4(X, \Omega^3_X)^* \cong H^4(X, T_X) \]

obstruction space of \( \gamma \) (2,2) class remains (2,2)

\( \uparrow \) The Hodge conjecture is not known for CY fourfolds.

b/ Free roaming points and curves appear in the compactification.

We will resolve a/ in Part II and b/ in Part III.

**Example of CY4**: \( X \subset \mathbb{P}^5 \) degree 6 hypersurface.

\[ h^{1,1} = 1, \quad h^{1,0} = h^{2,0} = h^{1,2} = 0, \quad h^{22} = 1752, \quad h^{13} = 426. \]
I.2 DT4 type virtual class (after Borisov-Joyce, Oh-Thomas)

\[ \nu = (0,0,\gamma,\beta,n) \in \Theta \mathbb{H}^{\psi}(X,\Theta). \]

For simplicity, we consider the Hilbert scheme

\[ I_v(X) = \{ I_2 \subset \mathcal{O}_X : ch_2(I_2) = \nu, ch_3(I_2) = \beta, \chi(I_2) = n \}. \]

The discussion generalizes to other moduli spaces.

- **Deformation theory**: \( \text{Def} = \text{Ext}^1(I,I) \)
  \( \text{Obs} = \text{Ext}^2(I,I) \)
  Higher \( = \text{Ext}^3(I,I) \)

Over the family,

\[ \xymatrix{ \Pi \ar[r] & X \ar[r]^{\pi_X} & X } \]

i) \( \phi : E \to R\text{Hom}_{\pi_X}(I,I) \)

ii) \( \theta : E \to E^v \)

iii) \( \sigma : \mathcal{O}_X \to \det(E) \text{ s.t. } \sigma^2 = \det(\theta) \)

Existence of \( \sigma \) is due to [Cao-Gross-Joyce].

Then [Borisov-Joyce, Oh-Thomas]. There exists a virtual class

\[ [I_v(X)]^{\text{vir}} \in \mathbb{H}_{2vd}(I_v(X),\mathbb{Z}) \]

[BJ]

\[ \uparrow \text{cl, [2-1]} \]

[COT]

\[ [I_v(X)]^{\text{vir}} \in \text{Avd}(I_v(X),\mathbb{Z}[1/2]) \]

[COT]

which depends on the choice of an orientation.
(Different choice of \( \sigma \) changes the sign of each connected component.)

\[ \text{vd} = n - \frac{1}{2} \gamma^2 \quad \left( \text{does not depend on } \text{ch}_3 \right) \]

- For special cases, earlier version [Cao-Leung]
- [BJ] Integral class, [COT] \( \hat{\mathcal{O}}^{\text{vir}} \), torus localization formula.
Local model (after [Brav-Bussi-Joyce], [Pantev-Toën-Vaquié-Vezzosi])

- \( U \): smooth affine scheme, \( \dim U = \text{ext}^4(\mathcal{I}, \mathcal{I}) \).
- \( E \rightarrow U \): vector bundle of rank \( = \text{ext}^2(\mathcal{I}, \mathcal{I}) \).
- \( Q : E \otimes E \rightarrow \Omega_U \): nondegenerate symmetric bilinear form
- \( t \in H^0(E) \): isotropic section \( \Rightarrow Q(t, t) = 0 \).

\[ M := Z(s) \subset U. \]

\[ \Omega_U |_M \xrightarrow{\text{d} \cdot t^*} \Omega_U |_M \xrightarrow{\text{d} \cdot t^*} \Omega_U |_M \]

If \( \exists \Lambda \subset E \) maximally isotropic subbundle, \( t \) factors through \( \Lambda \),

\[ [M]^{\text{vir}} = \pm c(E, \Lambda, t) \cdot [U] \in \text{Avd}(M) \text{ localized top Chern class.} \]

General case: \( \sqrt{c(E, t)} \). by [OT] using [Kiem-Li].

Counting points and curves on CY4

When \( \gamma = 0 \) (curves or points), the theory is related to \( GW \) invariants

[ Klemm-Pandharipande], [Cao-Maulik-Toda], [Cao-Kool], [Cao-Toda],

GW side

GW side is nice because \( GW_{g, \mu} = 0 \), \( g \geq 2 \).
II.1 Variation of Hodge Structure (VHS)

\[ f : \mathcal{X} \longrightarrow (\mathcal{B}, \mathcal{O}) \] : smooth, projective morphism. \( \mathcal{X}_0 \equiv \mathcal{X} \)

\( \mathcal{B} \) is smooth, connected, quasi-projective.

\( (\mathcal{H}^4, \mathcal{F}^p, \mathcal{H}_{\Theta}, \nabla) \) : VHS on \( \mathcal{B} \).

- \( \mathcal{H}^4 \) : Hodge bundle, \( \mathcal{H}^4 = H^4(\mathcal{X}_b, \mathcal{O}) \), \( b \in \mathcal{B} \)
- \( \mathcal{F}^p \) : filtration, \( \mathcal{F}^p = \bigoplus_{i=0}^{p} \mathcal{H}^{4-i}(\mathcal{X}_b, \Omega_{\mathcal{X}_b}^i) \)
- \( \mathcal{H}_{\Theta} \) : rational structure, \( \mathcal{H}_{\Theta} = H^4(\mathcal{X}_b, \Theta) \)
- \( \nabla : \mathcal{H}^4 \longrightarrow \mathcal{H}^4 \otimes \Omega^1_{\mathcal{B}} \) : Gauss-Manin connection.

Hodge locus (analytic local description)

\( \mathcal{Y} \in H^{2,2}(\mathcal{X}_0, \mathcal{O}) \). Let \( \mathcal{U} \subset \mathcal{B} \) : contractible nbh of 0.

\( f|_{\mathcal{U}} \) : topologically trivial, \( \mathcal{Y} \in \Gamma(\mathcal{U}, \mathcal{H}^4) = H^4(\mathcal{X}_0, \mathcal{O}) \).

Define the Hodge locus

\[ H_{\mathcal{Y}}(\mathcal{U}) = \{ b \in \mathcal{U} \mid \mathcal{Y}(b) \in H^{2,2}(\mathcal{X}_b, \mathcal{O}) \} \subseteq \mathcal{U} \]

For a general choice of \( \mathcal{Y} \), \( \text{codim}(H_{\mathcal{Y}}(\mathcal{U})) > 0 \) (\( \Rightarrow \) virtual class = 0)

Curves on surfaces

This locus is called the Noether-Lefschetz locus. Related reduction in the enumerative geometry is developed by

[Li], [Bryan-Leung], [Maurer-Pandharipande-Thomas],
[Kiem-Li], [Kool-Thomas], ...
II.2 Deformation of sheaves vs Hodge classes

$X$ : smooth, projective CY4, $\omega \in H^0(\Omega^4_X)$ : hol. volume form
Define a symmetric bilinear form

$$B_\gamma : H^4(T_X) \otimes H^4(T_X) \to \mathbb{C}, \exists_1 \otimes \exists_2 \mapsto \int_X 2 \exists_1 \exists_2 \langle \gamma \rangle \cdot \omega$$

where $\gamma : H^4(T_X) \otimes H^4(\Omega_X) \to H^{4+\gamma}(\Omega_X)$ is a contraction
Let $E = \text{rank}(B_\gamma) (\leq h^{4,3}(X))$.

The key diagram (due to Bloch, Buchweitz-Flenner)

Let $I : \text{sheaf (or complex)}$ on $X$ with $ch_1(I) = 0, ch_2(I) = \gamma$. Then

- $At(I) \in \text{Ext}^2(I, I \otimes \Omega_X)$ Atiyah class
- $ob(\exists) = \exists(At(I))$ ← Sheaf theoretic obstruction
- $SR(-) = tr(\cdot \circ At(I))$ Semi-regular map
- $2 \cdot (\gamma) = \text{contraction}(IVHS)$ ← Hodge theoretic obstruction

Special feature of CY4 and (2,2) class

$$tr(ob(\exists_1) \circ ob(\exists_2)) = B_\delta(\exists_1, \exists_2)$$

$\Rightarrow ob$ preserves symmetric bilinear forms, and $SR = ob^\ast$. 
II.3 Reduced obstruction theory

Recall: \( V = (0.0.0.\ldots) \), \( \mathcal{I}_v(X) \) : Hilbert scheme.

\[ \phi: \mathcal{E} = R\text{Hom}_\mathbb{Z}(\mathcal{I}_v, \mathcal{I}_v) \xrightarrow{\text{At}(\mathbb{I})} \mathcal{L}_{\mathcal{I}_v} : \text{obstruction theory}. \]

**Surjective cosections** (removing trivial components from the obs space).

Over the moduli space, the semi-regularity map induces a map

\[ \text{SR}: \mathcal{E}[-1] \to H^2(T_x)^* \otimes \mathcal{O}_{\mathcal{I}_v} \text{, } \text{SR}^2 = B \gamma \otimes 1 \]

Choose a maximal nondegenerate subspace \( V \subseteq H^4(T_x) \).

\[ V \to H^4(T_x) \to H^4(T_x)/\ker(B) := H^4(T_x)_{\gamma} \]

\[ \uparrow \text{Choose orientation} \]

Then

\[ \text{SR}_V: \mathcal{E}[-1] \xrightarrow{\text{SR}} H^4(T_x)^* \otimes \mathcal{O}_{\mathcal{I}_v} \to V^* \otimes \mathcal{O}_{\mathcal{I}_v} \]

\[ \sim \mathcal{E} \simeq (\mathcal{E}^{\text{red}}) \oplus (V \otimes \mathcal{O}[1]) \]

\[ E^{\text{red}} = \text{Cone}(\text{SR}_V[-1]) \]

\[ \chi \]

\[ \gamma \]

\[ \nu \subset \mathcal{B} \]

\[ V = \text{a transverse slice to the Hodge locus} \]

\[ T_{\nu} U \cong H^4(T_x) \]

The main theorem

**Thm (BKP)** There exists a reduced virtual class

\[ [\mathcal{I}_v(X)]^{\text{red}} \in \text{Arvd}(\mathcal{I}_v(X)), \text{rvd} = n + \frac{1}{2}(P_v - \gamma^2) \]

depending on a choice of orientation on \( \mathcal{E} \subset H^4(T_x) \). Moreover, there exists a reduced virtual structure sheaf \( \mathcal{O}^{\text{red}} \in K_0(\mathcal{I}_v(X)) \).

- The class is independent of the choice of \( V \subset H^4(T_x) \).
- Proof uses [Kiem - Li], [Kiem - Park].
- We further checked that \( E^{\text{red}} \) is a reduced obstruction theory adopting the algebraic twistor method of [Kool - Thomas].
II.4 Examples

Ideal geometry

**Def.** A point \([I] \in \mathcal{I}_v(X)\) is **semi-regular** if
\[
\text{SR: } \text{Ext}^2(I, I) \longrightarrow H^3(\Omega_x) \text{ is injective. (i.e. ob is surjective)}
\]

**Thm.** (Blouch, BKP) A point \([I] \) is semi-regular if and only if \(\mathcal{I}_v(X)\) is smooth of \(\text{dim} = \text{rvd at } [I].\)

- If \(\gamma\) is represented by semi-regular sheaf, \(\&x \cdot \gamma^2 = \text{even}.\)
- Reduced theory can be thought of as a tool to handle non semi-regular situations.

Comments on \(\rho\)

a/ If the Kodaira- Spencer map is an isomorphism,
\[\rho_f = \text{codim of } H^3_{dR}, \text{ when } I \text{ is generically reduced.}\]

b/ \(0 \leq \rho_f \leq h^{1,3}.\) Usually \(\rho_f\) is very far from \(h^{1,3}.\)

c/ If \(H^{2,0}(X) = 0\) and \(\gamma = D_1 \cdot D_2, \text{ where } D_1, D_2 \in H^{1,1}(X).\)

Then \(\rho_f = 0\) and we have interesting non-reduced invariants.

Some values of \(\rho\)

i/ \(X = V(f) \subset \mathbb{P}^5. \) Jacobian ring of \(f\) encodes INVHS of \(X.\)

Values of \(\rho_f\) can be computed via period integrals.

eg. \(X = \text{Fermat sextic. } H^3\text{ is known. } \rho_f\) computed for each \(\gamma.\)

\(\mathbb{P}^2 \subset X\) of type \((1,1,1).\) \(\rho_f = 19, \text{ rvd } = 1 + \frac{1}{2}(19 - 21) = 0\)

(\(\text{rvd } = 1 - 21/2 = -\frac{19}{2}\))

ii/ \(S \subset X: \text{local complete intersection.}\)

\[
\begin{align*}
H^4(T_x) & \xrightarrow{\alpha} H^4(N_{S|X}) \\
\text{ker } \alpha & \xrightarrow{\beta} \text{Ext}^2(I_{S|X}, I_{S|X})
\end{align*}
\]

\(\gamma \leq \dim H^2(N_{S|X})\) equals when \(\alpha:\) surjective.

eg. \(\mathbb{P}^4 \times \mathbb{P}^1 \subset K3 \times K3. \Rightarrow \rho_f = 2, \text{ rvd } = 1 + \frac{1}{2}(2-4) = 0\)
II.4 Deformation invariance

\[ f : X \rightarrow (B, o) : \text{smooth, projective morphism.} \quad \chi_0 = \chi \]

\[ B : \text{smooth, connected, quasi-projective s.t.} \]

\[ \omega_{X/B} \cong \Theta \chi \quad (\text{true Zariski locally on } B). \]

Choose a \( \Theta \) -section \( \tilde{\nu} \in \oplus H^1_{\Theta} (X/B) \) with \( \tilde{\nu}(o) = \nu \).

\[ \mathcal{I}\tilde{\nu}(X/B) \rightarrow B \]

Global invariance cycle \( \text{Thm}(\text{Deligne}) \quad \tilde{\nu}_i(b) \text{ is pure } (i,i) \text{ class.} \)

\[ \sim B \text{ lies in the Hodge locus.} \]

\[ \text{Thm}(\text{BKP}). \text{ Suppose there exists a family of orientations on } \mathcal{I}\tilde{\nu}(X/B)/B. \]

\[ \text{If } \rho(\tilde{\nu}_2(b)) \text{ is constant on } B, \text{ then there exists a reduced class} \]

\[ [\mathcal{I}\tilde{\nu}(X/B)]^{\text{red}} \in \text{Ar}^{\text{red}} + \text{dim}_B (\mathcal{I}\tilde{\nu}(X/B)) \text{ s.t.} \]

\[ \chi_b \rightarrow X \]

\[ \downarrow \quad \downarrow \quad \quad i_b \quad [\mathcal{I}\tilde{\nu}(X/B)]^{\text{red}} = [\mathcal{I}\tilde{\nu}(X_b)]^{\text{red}} \]

\[ b \rightarrow B, \]

Works also for \( \tilde{\nu}^{\text{red}} \).

\[ \bullet \text{ If we choose a coh}/K\text{-theoretic insertion on the family, we get deformation invariance of numbers over the base} \]

Remark. We expect there could be an obstruction of the existence of family orientation on \( \mathcal{I}\tilde{\nu}(X/B) \) in general. At least up to finite étale cover of \( B \), there exists a family orientation.

Remark. Condition \( \bowtie \) is not restrictive.
**Variational Hodge Conjecture (VHC)**

**Conjecture (Grothendieck)** $f: X \to B$: smooth, projective, $B$: smooth, connected, quasi-projective. $v \in H^q(B, R^p f_* \mathbb{Q})$. If $v(o)$ is a cohomology class of an algebraic cycle at some $o \in B(\mathbb{C})$, then the same holds for all $v(b), b \in B(\mathbb{C})$.

**Note.** Hodge Conjecture $\Rightarrow$ VHC

Not much is known for VHC.

**Thm (BKP)** Let $f: X \to B$: family of CY4. Assume $H^2(\Omega_X) = 0$.

If $\left[ \tilde{\iota}_2(b)(X_b) \right]_{red} \neq 0$ in $A_*$ for some $b \in B$, then VHC holds for $\tilde{\iota}_2(b)$.

**Remarks**

a/ We do not assume the existence of family orientation or $P(v)$ constant (maximal).

b/ This recovers the result of [Bloch I], [Buchweitz-Flenner] (semi-regular $\Rightarrow$ VHC).

**Examples**

i/ $X \subset \mathbb{P}^5, S \subset X$: complete intersection.

(Steenbrink) $S$ is semi-regular. They are rigid when

$(1.1.2) \quad (1.1.3) \quad (1.2.2) \quad (2.2.3) \quad (1.1.4)$

ii/ any "low degree" (Fano) surface $S \subset$ Complete Intersection CY4.
III. Moduli of pairs

- **X**: smooth projective variety, \( \dim X \geq k \).

Hilbert schemes are too big because of free roaming points, curves. We consider alternative compactifications following [Pandharipande - Thomas]

\[
\text{Pair}(X, v) = \{ (x, F) : \text{ch}(F) = v \} \quad \text{moduli stack of pairs, } \dim F = 2.
\]

\( T_0(F) < T_1(F) < F \) : torsion filtration. \( Q := \text{coker}(s) \).

**Def**
- (F,s) is DT (= PT-1) stable if \( s \) is surjective \( (F \cong \mathbb{O}_2) \)
- (F,s) is PT \( T_0 \) stable if \( T_0(F) = 0 \), \( \dim Q \leq 0 \).
- (F,s) is PT \( T_1 \) stable if \( T_1(F) = 0 \), \( \dim Q \leq 1 \). \( (F : \text{pure 2-dim}) \)

\[
\mathbb{P}^q_v(X) = \{ (F,s) : \text{PT}q \text{ stable, ch}(F) = v \} \subseteq \text{Pair}(X, v).
\]

**Construction of moduli spaces**

\( \text{Thm} (BKP) \) There exists a projective scheme \( N \), reductive group \( G \subset N \), \( G \)-linearization \( L_1, \ldots, L_q \), such that there exists no strictly semistable and

\[
N \sslash L_q G \cong \mathbb{P}^q_v(X) \quad q = -1, 0, 1.
\]

Hence \( \mathbb{P}^q_v(X) \) are projective and connected by GIT wall-crossing.

**Remark**
- Existence of DT (Grothendieck), PT \( T_1 \) (Le Potier) is known. Moduli spaces of non-pure sheaves is not much studied. The construction of PT seems new.

- (c) Our construction is motivated by [Stopa - Thomas].
Pointwise description

a/ PT₁ pair. If \( S = \text{Supp}(F) \) is smooth, \( F = L_\text{cl}(C) \) where
\[
\mathbb{Z} \to C \to S
\]
\( 0 \text{-dim.} \uparrow \quad \text{pure } 1 \text{-dim.} \downarrow \)

b/ PT₀ pair. If \( S = \text{Supp}(F) \) pure is Cohen-Macaulay,

\[
0 \to I_{S \setminus X} \to \mathcal{O}_X \to \mathcal{O}_S \to 0
\]

\[
0 \to T_i(F) \to F \to F/T_i(F) \to 0
\]

\( (F, s) \equiv (I_{S \setminus X}, \bar{s}) \).

c/ PT₀ = PT₁ pair. If \( S \) is not CM, then there exist \( \text{PT}_0(\neq \text{PT}_1) \) but not DT.

e.g. \( S_1, S_2 \subset X \) smooth surfaces. \( S_2 \cap S_2 = p, F = \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \)

\[
\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \to \mathcal{O}_p \to 0
\]
III.2 Pairs as complexes

Powerful idea of [Thomas] [PTJ]:
$P^{(q)}(X)$ is a fine moduli space. There exists universal sheaf with section

$$I^* := [\mathcal{O}_{X \times P} \xrightarrow{s} \mathcal{F}] \quad \text{on} \quad X \times P^{(q)}(X).$$

We consider (F.s) as an element in $D^b(X)$:

$$P^{(q)}(X) \xrightarrow{I^*} \text{Perf}(X)_{\text{spl}} \xleftarrow{\text{C}^2\text{-shifted str.}} \text{moduli stack of perfect complex } E \quad \det(E) = \mathcal{O}_X, \quad \text{Ext}^0 = 0, \quad \text{Ext}^* = \mathbb{C}. $$

Thm(BKp) The above morphism is an open immersion.

$\implies \exists$ 3 term symmetric obstruction theory on $P^{(q)}(X)$ and hence

$\exists$ reduced cycle

Remark PTq stability conditions can be realized by Bayer’s polynomial stab.

Remark When $q = 1$, the statement is independently proven by [Gholampour–Jiang–Lo].

Questions

(A) Correspondence between DT/PT0 invariants?

(B) Correspondence between PT0/PT1 invariants?

(C) Structure of PT1 invariants?
IV.1 DT/PT0 correspondence

Conjectural DT/PT0 correspondence

Let $y$ be a formal variable. The Nekrasov genus is given by

$$\langle L \rangle_{X,v}^{\text{top}} = \chi(P_v^{(a)}(X), \hat{\text{red/vir}} \otimes \hat{\wedge}(L_{v}^{\bullet} \otimes y^{-1})$$

where

$$\hat{\wedge}(E) := \sum_i (-1)^i \frac{\Lambda^i E}{\text{det} E}, \quad E \in K^0(P_v^{(0)}(X)).$$

Conjectural DT/PT0 correspondence

**Conjecture (BKP)** Fix $\gamma, \beta$. Then there exists a choice of orientations such that

$$\frac{\sum_n \langle L \rangle_{X,\gamma,\beta,n}^{\text{DT}}}{} = \frac{\sum_n \langle L \rangle_{X,\gamma,\beta,n}^{\text{PT}}}{}$$

**Remark**

a/ Taking $y \to 1^+$, we get cohomological DT/PT0.

b/ When $\gamma = 0$, the conjecture appears in [Gao-Kool-Monavari].

c/ The cohomological version of the denominator, $L = O(D)$. [Park]
d/ Assuming [Joyce], the closed formula of the denominator is given by the plethystic exponential [Bojkov].
e/ Other insertions (ie rk(codim 2)) do not work.

This conjecture is motivated from toric computations (non-reduced).
IV.2 Toric Calabi-Yau fourfold

\[ X : \text{smooth toric CY4, } T = (\mathbb{C}^*)^3 \sim X \text{ Calabi-Yau torus} \]
\[ \{ t, t_2, t_4 = 1 \} \]

**Vertex formalism**

On each toric chart \( X \cong \mathbb{C}^4 \), we fix \( Z \subset X : 2\text{-dim, } T\text{-inv, no embedded points.} \)  
(Asymptotic behavior)

\( \mathcal{O}_Z|_{(\mathbb{C}^*)^2 \times \mathbb{C}^2} \cong \mathcal{O}_A[x_1^\pm, x_2^\pm] \), \( A \subset \mathbb{C}^2 : 0\text{-dim } \sim \{ \lambda, \mu \} \text{ s.t. } \lambda, \mu \text{ are } 2D \text{ part.} \)

\( \mathcal{O}_Z|_{\mathbb{C}^2 \times \mathbb{C}^3} \cong \mathcal{O}_B[x_1^\pm] \), \( B \subset \mathbb{C}^3 : 1\text{-dim } \sim \{ \mu, \nu \} \text{ is } 3D \text{ part.} \)

**Thm (BKP)** Fix \( \lambda, \mu \text{ s.t. there is no moduli on PT side (ie T-fixed are 0-dim, reduced).} \) Then there exist topological vertices (K-theoretic)

\[ V^{DT}_{\lambda, \mu}(q, t, y), \ V^{PT}_{\lambda, \mu}(q, t, y) \]

which depend on the choice of signs.

**Equivariant correspondence**

**Conjecture** Under the same assumption, there exists a choice of signs \( \lambda, \mu \)

\[ \frac{V^{DT}_{\lambda, \mu}(q, t, y)}{V^{DT}_{\phi, \phi}(q, t, y)} = V^{PT}_{\lambda, \mu}(q, t, y). \]

Up to sign issues, **Conjecture** \( \Rightarrow \) equiv. version of Conjecture by localization \([OT]\).

**Remark**

a/ Finding a nice formula on T-fixed locus is a real challenge.

b/ T-fixed on DT is always reduced point.

c/ T-fixed on PT can be singular.

c/ The denominator has a closed formula \([Nekrasov-Rozalskaya]\) proven by \([Kool-Rennemo]\).
Examples

• Crossed instantons $\leftarrow$ due to [Nekrasov, gauge origami]

This consider the case when $P_{T_0} = P_{T_1}$ is a general, $\mu = \mu_{\min}$.

Lemma $S \subset X$; pure 2-dim'l subscheme.

1. $S$ is CM $\iff$ $\text{Ext}^3_x (\mathcal{O}_S, \mathcal{O}_X) = 0$

2. IF $(F_s)$ is supported on $S$ is both $P_{T_0} = P_{T_1}$, then

$$l(\text{coker}(s)) \leq l(\text{Ext}^3_x (\mathcal{O}_S, \mathcal{O}_X))$$

Eq. Take $S = \mathbb{C}^2 \cup \mathbb{C}^2$.

Then the (normalized) topological vertex is

$$V_{(a), \phi, \phi, \phi}^{P_{T_0}, \mathcal{O}_S, \mathcal{O}_X} = 1 + \frac{[\text{cy}]}{[\text{cy}]} = q$$

$[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.

Final remarks

a) For special geometry, $L = \mathcal{O}_X (D)$. $D \subset X$; smooth divisor, $V$ pushed from $D$, we checked Conjecture on compact geometry.

b) (work in progress B - Bojko - Lim). Assuming Joyce's wall-crossing formula, Conjecture holds for $X = \text{strict CY4}$, non-reduced.
IV.3 PT0/PT1 correspondence

We expect PT₀ and PT₁ invariants are related by curve counting invariants.

We don't know the correspondence in full generality. We focus on special geometry.

Weierstrass Calabi-Yau fourfolds

\[ B : \text{smooth, projective Fano 3fold. (or CY3).} \]

Weierstrass CY4:

\[
\begin{array}{ccc}
X & \hookrightarrow & \mathbb{P}(O \oplus L \oplus L^3) \\
\downarrow & & \downarrow \\
\Delta & \hookrightarrow & B
\end{array}
\]

- General fiber of \( p \): smooth elliptic curve
- Singular fiber: nodal, cuspidal

Example \( B = \mathbb{P}^3 \).

\( h^{1,1} = 2, \ h^{1,0} = h^{2,0} = h^{1,2} = 0, \ h^{2,2} = 15,564, \ h^{1,3} = 3878. \)

Let \( f \in H^6(X, \mathbb{Z}) \) be a fiber class.

**Conjecture (BKP).** For \( \beta \in H_2(B, \mathbb{Z}) \), \( \gamma = p^* \beta, \ n = \frac{p \cdot \Delta}{72} \). Then there exists a choice of orientations such that

\[
\frac{\sum Q^d \int [p^i_{y, df, n}] \text{vir } \prod T_{\mathbb{C}}(\sigma_i)}{\sum Q^d \int [p_{o, df, 0}] \text{vir } 1} = \sum Q^d \int [p^i_{y, df, n}(x)] \text{vir } \prod T_{\mathbb{C}}(\sigma_i).
\]

for all \( \sigma_i \in H^4(X) \) with \( p^*(\sigma_i) \in H^{2,2}(B) \).

\( T_{\mathbb{C}}(\sigma_i) = k \)-th descendant insertion

**Thm (BKP)** Conjecture is true when all pure CC B, \( [C] = \beta \) is irreducible Gorenstein and

1/ B: toric Fano 3fold: via stationary DT/PT on \( B \) [Oblomkov-Okounkov-Pandharipande]

2/ B: CY3: DT/PT on \( B \) [Bridgeland, Ito, Toda].

**Question** What about a K-theoretic correspondence?
Moving one irreducible curve

\[ V = (0, 0, 0, \beta, n) \in \bigoplus H^*(X, \Theta). \]

Assume

i) A pure surface \( S \subset X \) with \([S] = \gamma\) are CH with the constant \( \text{ch}(\Theta_S) = V(\gamma) \)

ii) \( \beta - \nu_3(\gamma) \) is irreducible, effective.

iii) A pure 1-dimensional sheaf \( G \) on \( X \) with \( \text{ch}(G) = V - \nu_3 \).

\[ \text{Ext}^2(\mathcal{I}_S/X, G) = 0, \forall [\mathcal{I}_S/X] \in \text{IV}_{\nu_3}(X). \]

Consider a morphism

\[ \Phi: P^{(\nu)}(X) \to M_{\nu-\nu_3}(X) \times \text{IV}_{\nu_3}(X) \]

\[ (F, s) \mapsto (T_1(F), \text{Isupp}(F)/s). \]

Thm (BKP) Under the above assumption, \( \Phi \) is virtually smooth and

\[ [P^{(\nu)}(X)]_{\text{red}} = \Phi^*([M_{\nu-\nu_3}(X)]^{\text{vir}} \times [\text{IV}_{\nu_3}(X)]_{\text{red}}) \]

is virtual pullback [Manolache]

Remark

a) Assumption i) implies \( \nu_3 \) is "minimal". This is the case when \( \text{DT} = \text{PT}_0 = \text{PT}_1 \)

b) The proof involves functorial property of \( \text{Oh-Thomas} \) class developed by [Park]

c) \( \Phi \) is a virtual projective bundle and \( \exists \) pushforward formula along \( \Phi \).
IV.4 Structure of PT1 invariants

Rigid surface inside CY4

\[ i : S \hookrightarrow X \text{ smooth projective surface. } N := N_{S|X} \Rightarrow \det(N) \cong KS. \]

\begin{align*}
  \text{Assume } &\text{i/ } [S] = \gamma \text{ is irreducible} \\
  &\text{ii/ } H^0(S, N) = 0 \text{ (i.e. } S \subseteq X \text{ cannot move)} \\
  &\text{iii/ } N_{S|X} \text{ is semiregular. }
\end{align*}

For \( \beta \in H^2(S, \mathbb{Z}) \) effective, consider the relative Hilbert scheme

\[ S^m_{\beta} = \{ z \in C \subseteq S : [C] = \beta, \chi(C_2) = n \} \hookrightarrow S_{\beta} \times S^m \]

It has a natural pair obstruction theory by [Kaul]. [Kaul-Thomas]

**Theorem (BKP)** Under the above assumption, the pushforward \( i_* \) induces an open immersion

\[ \Phi : S^m_{\beta} \longrightarrow P^{(\gamma)}_v(X), \quad v = \text{ch}(i_*, I_{S|X}(C)). \]

Moreover, \( \Phi^* \left[ P^{(\gamma)}_v(X) \right]_{\text{red}} = \pm \am \quad \text{where} \quad \am \Omega \left( 1-q^m \right) \chi(N) - e(S) = \sum \am q^m. \quad pt \in H_0(S^m_{\beta}). \]

The additional obstruction bundle can be computed by [Carlson-Okounkov].

**Pair/Sheaf correspondence**

\[ \text{Assume } \gamma \text{ is irreducible. } v = (0, 0, 1, \ast, \ast, \ast) \in \Theta H^*(X). \]

Consider the moduli space of two dimensional torsion stable sheaves.

\[ M_v(X) = \{ F : F \text{ stable, } \text{ch}(F) = v \gamma. \}

(Independent of the choice of polarization because \( \gamma = \text{irred.} \))

\[ [M_v(X)]_{\text{red}} \in A_{1+\frac{1}{2}(p_v^* - v^2)}(M_v(X)) \]

Let

\[ \Phi : P^{(\gamma)}_v(X) \longrightarrow M_v(X), \quad (F, s) \mapsto F \]

\[ \text{Assume } \text{ i/ } H^2(X, F) = 0 \forall [F] \in M_v(X) \quad \leftarrow \text{crucial.} \]

(Existence of universal sheaf \( G \) on \( X \times M_v(X) \))

**Theorem (BKP)** Under the above assumption,

\[ \Phi^! \left[ M_v(X) \right]_{\text{red}} = \left[ P^{(\gamma)}_v(X) \right]_{\text{red}}. \]
Generating series (?)

Full picture is not clear.

**Lemma** For fixed $\gamma, \beta$, $\exists N > 0$ s.t $\mathbb{P}_{\gamma, \beta}^{(n)}(X) = \emptyset$, $n \geq N$. Moreover if $\chi(F)$ is maximal, $F$ is a reflexive sheaf.

E.g. If $S = \text{Supp} F$ is smooth, $F = \mathbb{I}_{2/k}(C)$. $\chi(F)$ is maximal when $l(O_2) = 0$. In that case $F = O_S(C)$.

Case 1. Fix $\gamma, \beta$ and sum over $n$: finitely many terms

Case 2. Fix $\gamma$ and sum over $\beta$: This seems an interesting direction (e.g. Weierstrass model). We should not fix $n$, but rather we should take $n = n_{\text{max}}(\gamma, \beta)$ (reflexive case)

- Natural cohomological insertion? or K-theoretic inv?

Case 3. Sum over $\gamma$: Most complicated question.

The $\mathbb{P}(r)$ is not additive nor constant. rvd jumps around

Many other questions remain ... Relation to string theory? ...