IC Moduli Zoominar Talk.

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A theory of Gopakumar-Vafa invariants for orbifold Calabi-Yau threefolds

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The story for ordinary CY3's:

Let X be a Calabi-Yau threefold with Gromov-Witten potential:

$$F_X = \sum_{\substack{\beta \neq 0 \\ \beta \in H_2(X)}}' \sum_{g \geq 0}' \langle\, \rangle^X_{g,\beta} Q^\beta \lambda^{2g-2} \qquad \langle\, \rangle^X_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

In 1998, Gopakumar-Vafa defined curve counting invariants $n_g(\beta)$ based on counting BPS states and conjectured

$$(*) \quad F_X = \sum_{\beta \neq 0}' \sum_{g \geq 0}' \sum_{d > 0}' \frac{Q^{d\beta}}{d} \left(2 \sin \frac{d\lambda}{2}\right)^{2g-2} n_g(\beta)$$

where $n_g(\beta) \in \mathbb{Z}$ and for fixed β, $n_g(\beta) = 0$ for all but finitely many $g \geq 0$. $(*)$ can be viewed as:

1. A universal multiple cover / degenerate contributions formula for GW invariants.

2. A structure theorem for GW invariants: the set $\{n_g(\beta)\}$ contains the same information as $\{\langle\, \rangle^X_{g,\beta}\}$ in a smaller, more efficient package.

3. A kind of sheaf $\longleftrightarrow$ map correspondence. In 2018, Maulik-Toda gave a direct sheaf theoretic definition of $n_g(\beta)$.

Examples: • $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ (resolved conifold)

$$n_g(d[\mathbb{P}^1]) = \begin{cases} 1 & \text{if } g=0,\, d=1 \\ 0 & \text{otherwise} \end{cases}$$

• X is a local K3 surface (Yau-Zaslow formula)

$$n_0(\beta) = e(\mathrm{Hilb}^{\beta^2/2+1}(K3))$$

$$\sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(K3))\, q^{n-1} = \frac{1}{\Delta(q)} \qquad \Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

KKV formula generalizes this to $n_g(\beta)$ $g \geq 0$

$1/\chi_{10}$ formula generalizes this to $X = K3 \times E$

GV for orbifolds Let $\mathcal{X}$ be an orbifold CY3 with stacky locus $\mathcal{B} \subset \mathcal{X}$ where $\mathcal{B} \to B$ is a $B\mathbb{Z}_{/N+1}$ gerbe over a smooth curve B (so the singular space X has transverse A_N singularities along B).

$\boxed{\text{Goal: Define GV theory in this setting. Find the analog of } \circledast \text{ and } n_g(\beta)}$

Remark: Our theory works for stacky locus having many components, with transverse type A singularities. Also makes sense for transverse ADE singularities but evidence is much more spotty in DE cases.

Let $H_2(X)^\#$ be the semigroup of effective classes not represented by a curve containing $\mathcal{B}$ as a component.

Let $\gamma_1, \dots, \gamma_N$ be generators for the twisted sector of

$$H^2_{orb}(\mathcal{X}) \cong H^2(X) \bigoplus_{k=1}^{N} H^0(B) .$$

Then the GW potential is:

$$F^\#_{\mathcal{X}}(Q,\lambda,x_1 \cdots x_N) = \sum_{\substack{\beta \in H_2(X)^\# \\ \beta \neq 0}}{}' \; \sum_{g \geq 0}{}' \; \sum_{m_1 \cdots m_N}{}' \langle \gamma_1^{m_1} \cdots \gamma_N^{m_N} \rangle^{\mathcal{X}}_{g,\beta} \, Q^\beta \lambda^{2g-2} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_N^{m_N}}{m_N!}$$

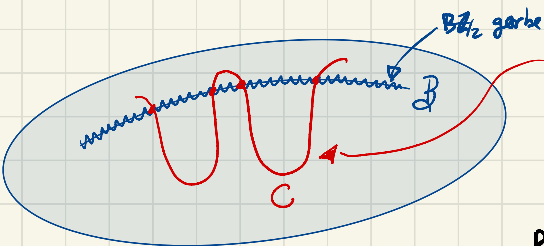
Def'n $\exists$ integers $n_g(\beta)$ finitely many non-zero for fixed $\beta \in H_2(X)^\#$ s.t.

$$F^\#_{\mathcal{X}} = \sum_{d,\beta,g}{}' \frac{Q^{d\beta}}{d} \left(2\sin\frac{d\lambda}{2}\right)^{2g-2} \Theta_{d,g,\beta}(x_1 \cdots x_N)$$

Then

$$n_g(\beta) = \Theta_{1,g,\beta} \Big|_{x_k = \frac{\pi}{N+2} \csc\left(\frac{k\pi}{N+1}\right)}$$

Example familiar to some of you:



genus 0 curve C in a primitive class β meeting $\mathcal{B}$ in P points.

Under idealized conditions, J. Wise proved that the contribution of C to the GW potential is

$$\sum_m{}' \langle \gamma^m \rangle^{\mathcal{X}}_{0,C} \frac{x^m}{m!} = \left(2\sin\frac{x}{2}\right)^P$$

in the specialization $x = \frac{\pi}{N+2} \csc\left(\frac{k\pi}{N+1}\right) = \frac{\pi}{3}\csc\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$ $\quad 2\sin\frac{x}{2} = 2\sin\frac{\pi}{6} = 1$

The invariants $n_g(\beta)$ do not contain all the information of the GW potential. We need more refined GV invariants:

Definition / Conjecture: $\exists$ integers $n_g(\beta; m_1 \cdots m_N)$, finitely many non-zero for fixed $\beta \in H_2(X)^{\#}$, such that if we write

$$F^{\#}_{\mathcal{X}} = \sum_{d,\beta,g}{}' \frac{Q^{d\beta}}{d} \left(2\sin\frac{d\lambda}{2}\right)^{2g-2} \Theta_{d,g,\beta}(x_1 \cdots x_N)$$ then

$$\Theta_{d,g,\beta} = \sum_{m_1 \cdots m_N}{}' n_g(\beta; m_1 \cdots m_N) \prod_{K=1}^{N} \sigma_K^{m_K}(z_1^d \cdots z_{N+1}^d)$$

where σ_K is the Kth elementary symmetric function and

$$z_K = -\omega^{-K+\frac{1}{2}} \exp\left(-\frac{1}{N+1}\sum_{j=1}^{N}{}' \omega^{j(K-1/2)} x_j\right), \quad \omega = \exp\left(\frac{2\pi i}{N+1}\right)$$

$n_g(\beta; m_1 \cdots m_N)$ is a virtual count of genus g curves in the class β meeting $\mathcal{B}$ in m_K points with "weight" $K \in \{1, \cdots, N\}$. The data $\{n_g(\beta; m_1 \cdots m_N)\}$ is equivalent to $\left\{ \langle \gamma_1^{m_1} \cdots \gamma_N^{m_N} \rangle^{\mathcal{X}}_{g,\beta} \right\}$.

Examples ① Local teardrop: $\mathcal{X} = \mathrm{Total}\left(\mathcal{O}(-p_0) \oplus \mathcal{O}(-p_\infty)\right)$, $p_0, p_\infty \in \mathbb{P}^1(N+1, 1)$ so p_0 is a $B\mathbb{Z}/_{N+1}$ point. Johnson-Pandharipande-Tseng compute $F_{\mathcal{X}}$.

$$\rightsquigarrow \quad n_g\left(d[\mathbb{P}^1], m_1 \cdots m_N\right) = \begin{cases} 1 & \text{if } g=0,\ d=1,\ (m_1 \cdots m_N) = (10 \cdots 0) \\ 0 & \text{otherwise} \end{cases}$$

(2) Local orbifold K3 surface $\mathcal{S}$ with a single $B\mathbb{Z}/_{N+1}$ point.

Assume $\mathrm{Pic}(S)$ is generated by curves not meeting the orbifold point (these are easy to find, many examples with $N \leq 17$).

Then orbifold Yau-Zaslow formula is:

<u>Theorem</u> (assuming GW CRC, MNOP)
$$n_0(\beta) = e\left(\mathrm{Hilb}^{\beta^2/2+1}(S)\right)$$

Where S is the singular surface (!)

By a result of Gyenge-Nemethi-Szendroi

$$\sum_{n=0}^{\infty}{}' \, e(\mathrm{Hilb}^n(S))\, q^{n-1} = \frac{\Theta^{GNS}_{A_N}(q)}{\Delta(q)}$$

generalizes in the obvious way to orbi KKV and $\mathcal{S}\times E$

For any ADE root system R with root lattice Λ_R

$$\Theta^{GNS}_{R}(q) = \sum_{v\in\Lambda_R}{}' \, q^{\frac{1}{2}\langle v,v\rangle}\, w_1^{v_1}\cdots w_N^{v_N} \qquad w_k = \exp\left(\frac{2\pi i}{h^\vee+1}\right)$$

$v = v_1e_1+\cdots+v_ne_n$

e_i simple roots

Where does this all come from? On sheaf side, $\mathcal{X}$ has extra derived symmetries. This is partially responsible for the structure.

Sheaf Side: X ordinary CY3

$$Z_X^{PT} = \sum_{\beta,n}{}' PT_{n,\beta}(X)\, Q^\beta y^n$$

numerical K-theory of sheaves supported in dim ≤ 1

$$N_{\leq 1}(X) = \underset{n}{N_0(X)} \oplus \underset{\beta}{N_1(X)}$$

$\chi = 0$

Then the GV formula ⊛ $\iff$

$$\log\left(Z_X^{PT}\right) = \sum_{d,g,\beta}{}' \frac{Q^{d\beta}}{d}\, \psi^{g-1}_{-(-y)^d}\, n_g(\beta) \qquad \psi_y = (y^{1/2}+y^{-1/2})^2 = 2+y+y^{-1}$$

The fact that Z_X^{PT} is invariant under $y \longleftrightarrow y^{-1}$ (and hence can be written in terms of ψ) comes from the derived symmetry of $D(Coh(X))$ given by $F \mapsto RHom(F, \mathcal{O}_X)$.

For orbiCY3 $\mathcal{X}$ with transverse A_N orbi locus

$$N_{\leq 1}(\mathcal{X}) = \text{saturation}\Big(\overbrace{\underset{n}{\mathbb{Z}} \oplus \underset{v}{\Lambda_{A_N}}}^{N_0(\mathcal{X})} \oplus \underset{\beta}{N_1(\mathcal{X})}\Big) \subset \mathbb{Z} \oplus \Lambda^\vee_{A_N} \oplus N_1(\mathcal{X}) \otimes \mathbb{Q}$$

$$Z_{\mathcal{X}}^{PT\#} = \sum_{\substack{\beta \in H_2^\# \\ n \in \mathbb{Z}}}{}' \ \sum_{v \in \Lambda^\vee_{A_N}}{}' PT_{n,\beta,v}(\mathcal{X})\, Q^\beta y^n w^v$$

Let W be the Weyl group of the A_N root lattice. (Bryan–Moriera) There is an action of W on $D(Coh(\mathcal{X}))$ which leads to a corresponding symmetry of $Z_{\mathcal{X}}^{PT\#}$. Namely

$$\operatorname{Coef}_{Q^\beta y^n}\left(Z_{\mathcal{X}}^{PT\#}\right) \in \mathbb{Z}[\Lambda^\vee_{A_N}]^W$$

A fundamental theorem in representation theory (any root lattice)

$$\mathbb{Z}[\Lambda^\vee_{A_N}]^W \cong \mathbb{Z}[\Phi_1, \dots, \Phi_N] \qquad \Phi_k = \mathrm{char}(\omega_k)$$

fundamental weight.

Definition / Conjecture

$$\log\left(Z^{PT\,\#}_{\mathcal{X}}\right) = \sum_{\beta, d, g}' \frac{Q^{d\beta}}{d} \sum_{m_1 \cdots m_N}' n_g(\beta; m_1 \cdots m_N)\, \psi^{g-1}_{-(-y)^d} \cdot \prod_{\ell=1}^{N} \Phi_\ell^{m_\ell}(w^d)$$

The GW version comes from this using DT CRC (BCR), MNOP, and GW CRC

MNOP: lot's of cases by Pandharipande-Pixton

GW CRC: known in toric case

The above theorem/conjecture relies on some guess work checked by examples

local orbi K3s and local orbi curves and sporadic other evidence give high confidence in A_N case, especially A_1. DE cases only real evidence is local K3. Less confident above conjecture is correct.