Verlinde Series on Hirzebruch Surfaces

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Please ask questions!

Background

The Hilbert scheme of n points on a smooth (quasi-)projective surface X (defined over $\mathbb{C}$) is

$$X^{[n]} = \left\{ \begin{matrix} \text{0-dimensional closed subschemes } Z \subseteq X \\ \text{such that } \mathrm{length}(Z) := \sum_p \dim_{\mathbb{C}}(\mathcal{O}_{Z,p}) = n \end{matrix} \right\}.$$

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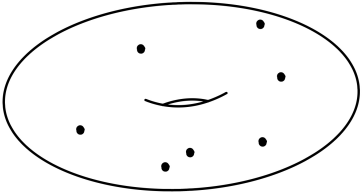

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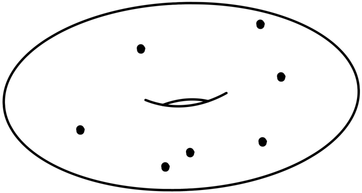
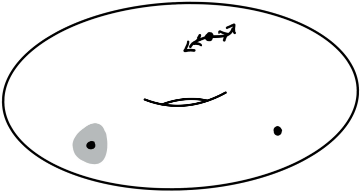


2. not reduced, supported at $< n$ points with some multiplicities



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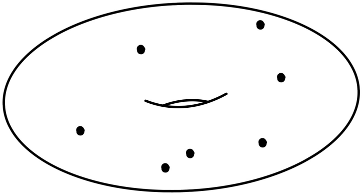

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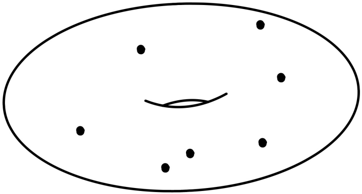

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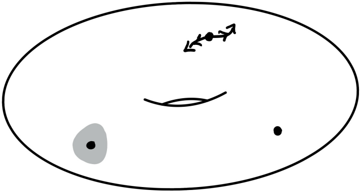

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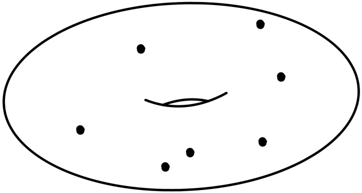

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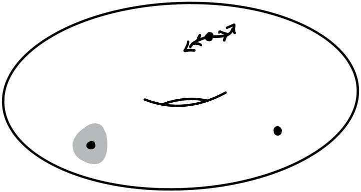

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All of this fails dramatically if $\dim(X) > 2$.

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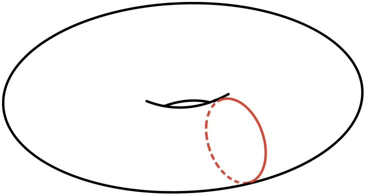

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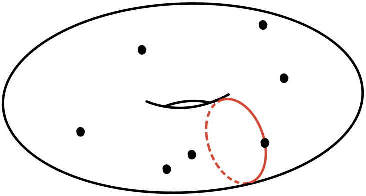

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$\mathcal{O}(D)$ where $D \subseteq X^{[n]}$ is the locus of schemes Z whose support meets C.

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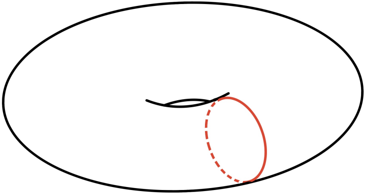

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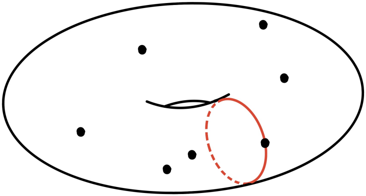

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Theorem (Fogarty ('73))

For $n \geq 2$,
$$Pic(X^{[n]}) \simeq Pic(X)_n \times \mathbb{Z}E$$

where $c_1(E)$ is $-1/2$ times the divisor of nonreduced schemes.

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$$\chi(Y, \mathscr{L}) = \sum_{i=0}^{\dim Y} (-1)^i \dim_{\mathbb{C}} H^i(Y, \mathscr{L})$$

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Segre and Verlinde series are related by a change of variables (J, MOP, GM).

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Theorem (Ellingsrud, Göttsche, Lehn ('99))

There exist universal power series $A_r, B_r, C_r, D_r \in \mathbb{Q}[[z]]$ for each $r \in \mathbb{Z}$ such that

$$\mathbf{V}_{X,L,r}(z) = A_r(z)^{\chi(L)} \cdot B_r(z)^{\chi(\mathcal{O}_X)} \cdot C_r(z)^{c_1(L)\cdot K_X - \frac{1}{2}K_X^2} \cdot D_r(z)^{K_X^2},$$

for all X and L.

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There are no such simple formulas for $r > 1$.

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This determines $A_r(z)$ and $B_r(z)$ for all r.

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Corollary

$$A_r(t(1+t)^{r^2-1}) = 1+t$$

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i.e. $A_r(z)$ and $B_r(z)$ are expressed in terms of the inverse power series to $z = t(1+t)^{r^2-1}$ near $t = 0$:

$$t = z - (r^2-1)z^2 + \left(\frac{3}{2}r^4 - \frac{5}{2}r^2 + 1\right) z^3 + \cdots$$

Key takeaway:
This determines the Verlinde series for all surfaces X with $K_X = 0$.

$$\mathbf{V}_{X,L,r}(z) = A_r(z)^{\chi(L)} \cdot B_r(z)^{\chi(\mathcal{O}_X)} \cdot \cancel{C_r(z)^{c_1(L)\cdot K_X - \frac{1}{2}K_X^2} \cdot D_r(z)^{K_X^2}}$$

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Enumerative interpretation

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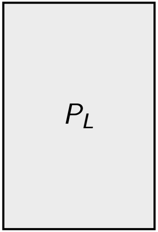

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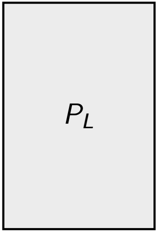

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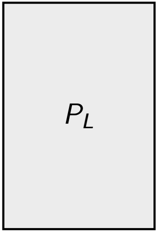

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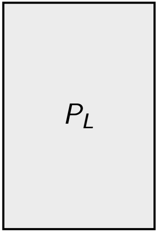

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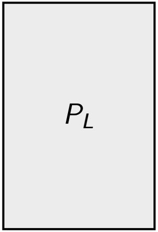

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$\chi(X^{[n]}, L_n \otimes E^r) \leftrightarrow \#$**r-separated** n-tuples of integer points in P_L

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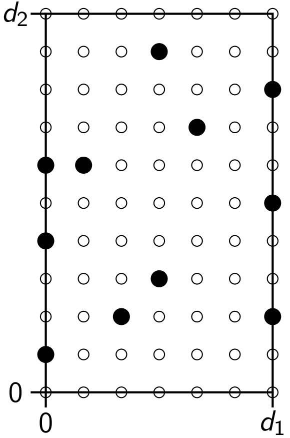

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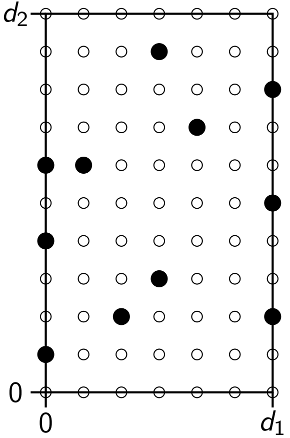

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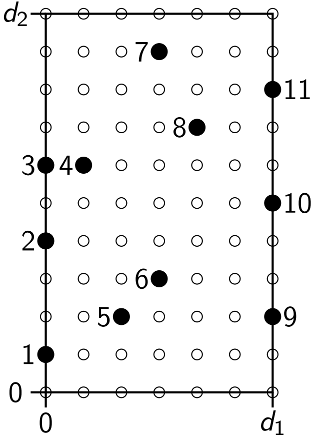

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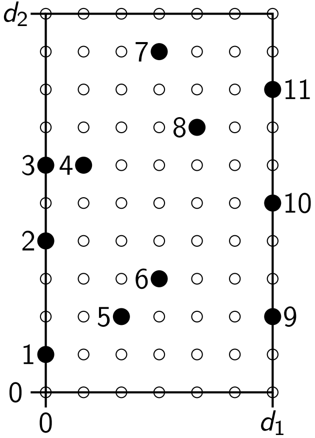

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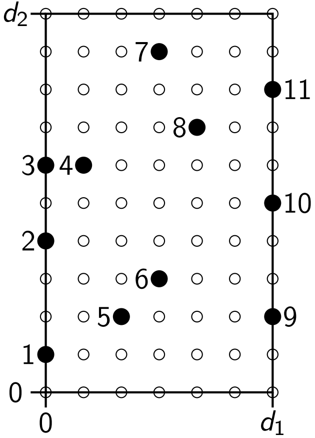

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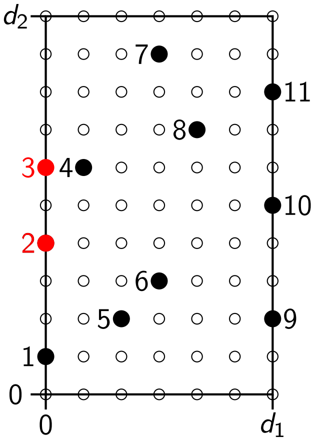

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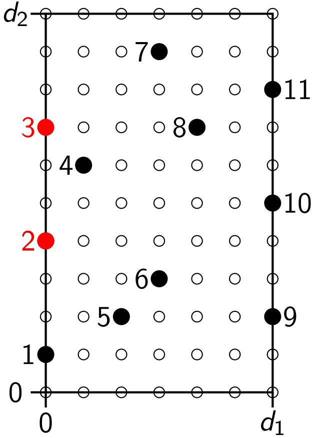

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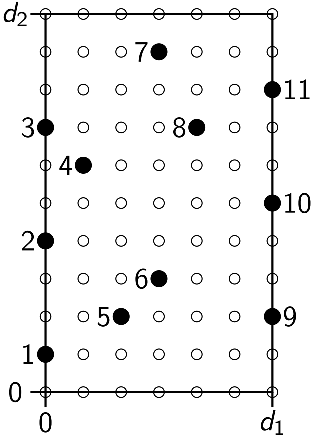

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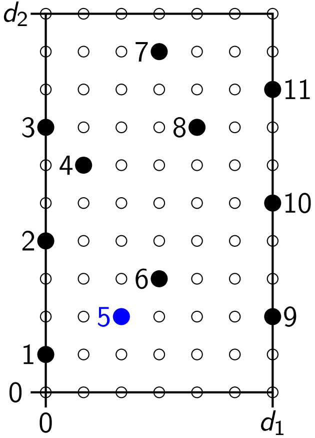

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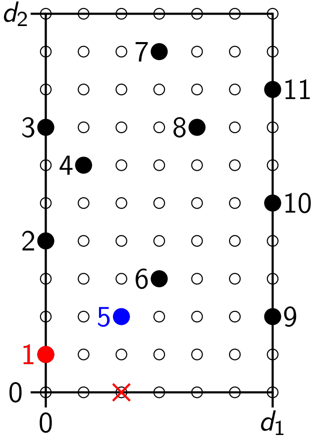

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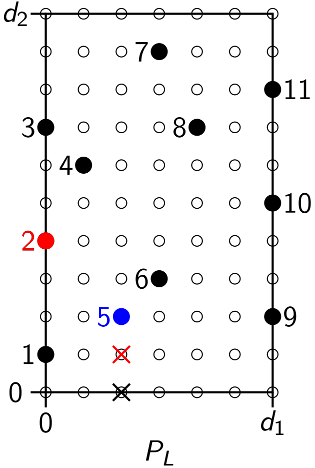

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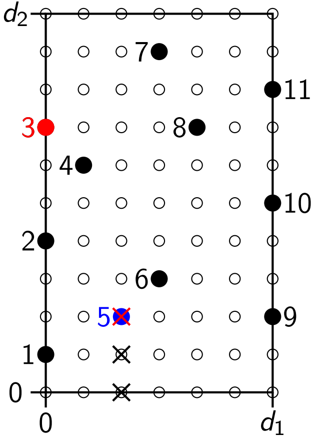

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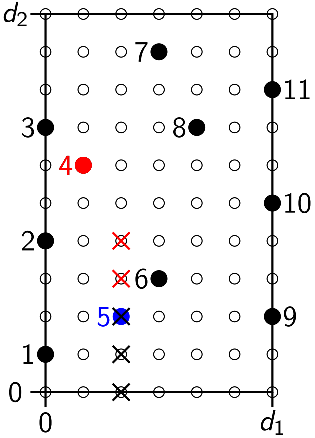

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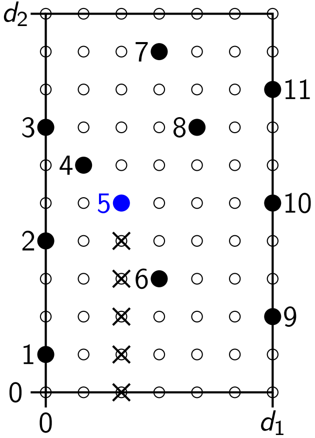

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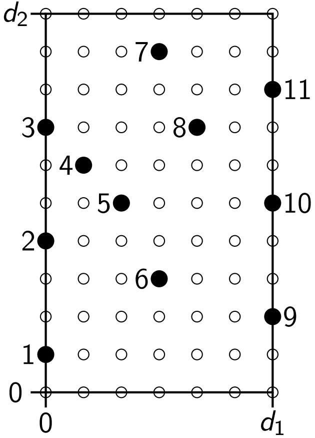

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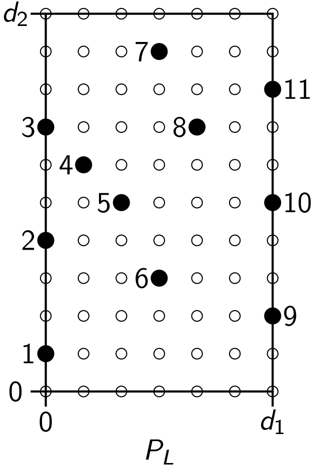

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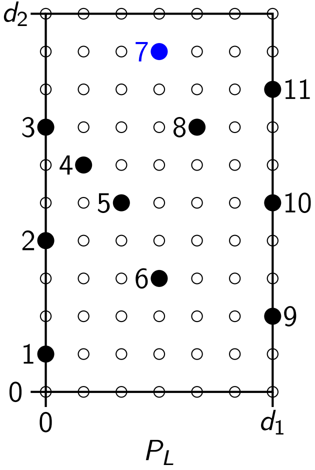

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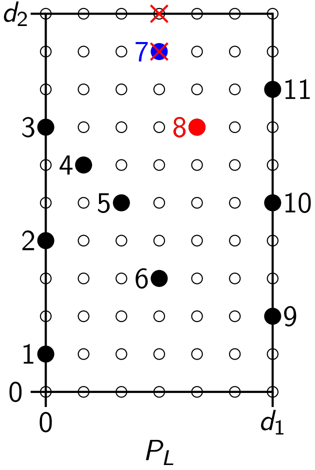

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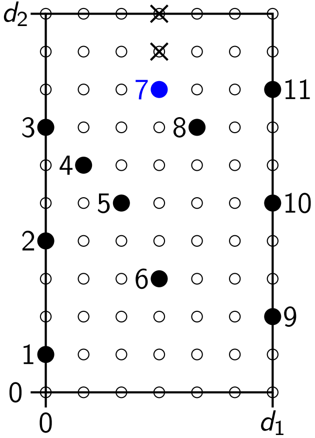

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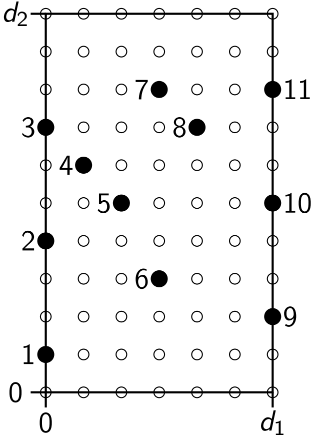

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- The ample cone of $X^{[n]}$ is known: e.g. for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $L = \mathcal{O}(d_1, d_2)$, $L_n \otimes E^r$ is ample iff $r > 0$ and $d_1, d_2 > r(n-1)$.

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- Same formula gives the T-equivariant refinement of Euler characteristic.

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Corollary

For $X = \mathbb{P}^1 \times \mathbb{P}^1$ or any Hirzebruch surface and any **ample** *line bundle $L_n \otimes E^r$ on $X^{[n]}$, there is a section $s \in H^0(X^{[n]}, L_n \otimes E^r)$ with trailing term $x_1^{a_1} y_1^{a_1} \cdots x_n^{a_n} y_n^{b_n}$ iff $(a_1, b_1), \ldots, (a_n, b_n)$ is an r-separated n-tuple in P_L.*

Proof Outline

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- $(\mathbb{C}^*)^2 \simeq T$-action on X induces T-action on $X^{[n]}$, so we have a refined invariant

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Proof of this result is direct.

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This is an "area/bounce" formula for $\chi^T((\mathbb{C}^2)^{[n]}, E^r)$ (picture at the end).

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For $d_1, d_2 \gg n, r$,

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$X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}(1, 2)$, $r = 5$,

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$$\begin{aligned}
\mathbf{V}_{X,L,r}(z) &= A_r(z)^{\chi(L)} \cdot B_r(z)^{\chi(\mathcal{O}_X)} \cdot C_r(z)^{c_1(L)\cdot K_X - \frac{1}{2}K_X^2} \cdot D_r(z)^{K_X^2} \\
&= A_r(z)^{(d_1+1)(d_2+1)} \cdot B_r(z) \cdot C_r(z)^{-2d_1-2d_2-4} \cdot D_r(z)^8 \\
&= 1 + \sum_{n\geq 1} z^n \sum_{\delta\in\{0,\dots,r\}^{n-1}} \binom{d_1 + \cdots}{\dots} \prod_{\cdots} \binom{d_2 + \cdots}{\dots}.
\end{aligned}$$

This formula determines D_r, and therefore the Verlinde series for all X, L, r.

e.g. take $d_1 = d_2 = -1$ to get $B_r(z) \cdot D_r(z)^8$, solve for $D_r(z)$.

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Thank you!

For each $\delta \in \{0, 1, \ldots, r\}^{n-1}$ there is a "minimal" such choice of r-separated points and an r-bounce path. The sum of horizontal positions of points is the bounce statistic of the path, and the sum of the vertical positions is the area.

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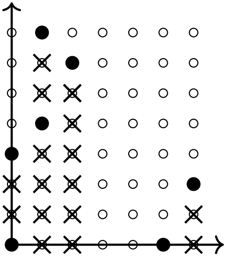
E.g. $r = 3$, $\delta = (0, 1, 0, 1, 3, 1)$



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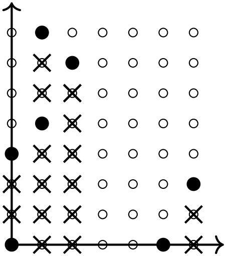

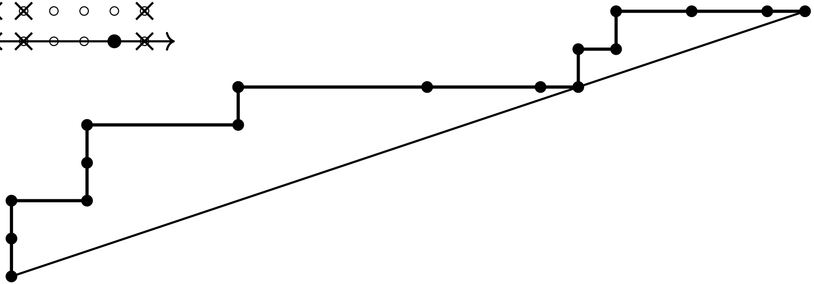

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