The decomposition theorem for 2-Calabi-Yau categories

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Mixed Hodge structures and BM homology

For X a complex variety, we define $\mathrm{H}^{\mathrm{BM}}(X,\mathbb{Q}) := s_*\mathbb{D}\mathbb{Q}_X$, where

- $\mathbb{Q}_X$ is the constant sheaf on X
- $\mathbb{D}$ is the Verdier duality functor
- s_* is the derived global sections functor for the structure map $X \xrightarrow{s} \mathrm{pt}$.

1. If X is smooth, this is (up to a shift) the usual cohomology, since $\mathbb{D}\mathbb{Q}_X \cong \mathbb{Q}_X[2\dim_{\mathbb{C}}(X)]$.
2. If $U \subset X$ is open, have long exact sequence

$$\to \mathrm{H}^{\mathrm{BM}}_{-i}(X \setminus U,\mathbb{Q}) \to \mathrm{H}^{\mathrm{BM}}_{-i}(X,\mathbb{Q}) \to \mathrm{H}^{\mathrm{BM}}_{-i}(U,\mathbb{Q}) \to \mathrm{H}^{\mathrm{BM}}_{-(i+1)}(X \setminus U,\mathbb{Q}) \to .$$

   Can *define* the mixed Hodge structure on $\mathrm{H}^{\mathrm{BM}}(X,\mathbb{Q})$ by decreeing that this is a long exact sequence in MHS, and taking standard pure Hodge structure on each $\mathrm{H}^{\mathrm{BM}}(X,\mathbb{Q})$ if X is smooth projective.
3. There is an essentially unique functorial way to extend $\mathrm{H}^{\mathrm{BM}} \colon \mathrm{Var} \to \mathcal{D}(\mathrm{MHS})$ to global quotient stacks.

(2) says that H^{BM} is "motivic", so this cohomology theory connects well with point-counting over $\mathbb{F}_q$ as well as DT theory.

The BBDG decomposition theorem

Theorem (Beilinson–Bernstein–Deligne–Gabber)

Let $p\colon X \to Y$ be a projective morphism of complex algebraic varieties, with X smooth. Then

$$p_*\mathbb{Q}_X \cong \bigoplus_{i\in\mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)\,[-i]$$

and each perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)$ is semisimple.

- If X is singular variety, same theorem remains true, but with $\mathbb{Q}_X$ replaced everywhere by $\mathcal{IC}_X$, the intersection complex, i.e. the intermediate extension of the constant perverse sheaf $\mathbb{Q}[\dim(X)]|_{X^{\mathrm{sm}}}$.
- For X smooth, defining

$$\mathfrak{P}^i\mathrm{H}(X,\mathbb{Q}) := \mathrm{H}(Y, {}^{\mathfrak{p}}\tau^{\leq i}p_*\mathbb{Q}_X) \subset \mathrm{H}(X,\mathbb{Q})$$

gives the perverse filtration of X with respect to q, with subquotients

$$\mathfrak{P}^i\mathrm{H}(X,\mathbb{Q})/\mathfrak{P}^{i-1}\mathrm{H}(X,\mathbb{Q}) \cong \mathrm{H}(Y, {}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)).$$

Saito's version

Given X a complex variety, Saito defines the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X, along with faithful functor $\mathrm{MHM}(X) \to \mathrm{Perv}(X)$;

- So a MHM on X is a perverse sheaf $\mathcal{F}$ on X along with some extra data. Most important part (for us) is the *weight filtration* $W_\bullet \mathcal{F}$.
- A MHM $\mathcal{F}$ is *pure of weight* i if $\mathrm{Gr}_j^W \mathcal{F} = 0$ for $i \neq j$.
- A complex $\mathcal{F} \in D^b(\mathrm{MHM}(X))$ is called *pure* if each $\mathcal{H}^i(\mathcal{F})$ is pure of weight i.
- The perverse sheaves $\mathbb{Q}_X[\dim(X)]$ on a smooth variety X (or $\mathcal{IC}_X$ on a general variety) have lifts to simple pure weight zero MHMs (at least if $\dim(X)$ even).

Theorem (Saito)

The category of pure weight n MHMs on a variety is semisimple. If $f\colon X \to Y$ is projective, $f_*\colon D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ preserves pure objects. If $\mathcal{F} \in D^b(\mathrm{MHM}(Y))$ is pure, then $\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(\mathcal{F})[-i]$. $\therefore$ If $\mathcal{G} \in D^b(\mathrm{MHM}(X))$ is pure, then $f_*\mathcal{G} \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(f_*\mathcal{G})[-i]$.

Coherent sheaves on K3 surfaces

Let (S, H) be a polarized K3 surface, or a pair of a smooth complex algebraic surface S with trivial ω_S, along with an ample class on $\overline{S}$.

We denote by $\mathfrak{M}_\alpha^H(S)$ the moduli stack of H-Geiseker semistable coherent sheaves on S with projective support and with Chern character α, and by $\mathcal{M}_\alpha^H(S)$ the coarse moduli space.

- Via the standard quot scheme construction, $\mathfrak{M}_\alpha^H(S)$ is a global quotient stack.
- The canonical morphism $p\colon \mathfrak{M}_\alpha^H(S) \to \mathcal{M}_\alpha^H(S)$ is a good moduli space in the sense of Alper.

One of the motivations of this project was to prove

Conjecture (Halpern–Leistner)

Assume that S is a K3 surface, then the mixed Hodge structure on $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\alpha^H(S), \mathbb{Q})$ is pure.

(We'll prove the conjecture under *weaker* assumptions on S.)

Formal and étale neighbourhoods (after Alper–Hall–Rydh)

Formal neighbourhoods

Fix a finite length category Abelian subcategory $\mathcal{C}$ of $\mathcal{H}(\mathcal{D})$ for $\mathcal{D}$ a "nice" dg category, with a good moduli space $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$.
Formal completion of p at x representing the semisimple object $\mathcal{F} = \bigoplus_{i\in I} \mathcal{F}_i^{\oplus \mathrm{d}_i}$ is determined by the dg algebra $\mathrm{RHom}_{\mathcal{D}}(\mathcal{F},\mathcal{F})$, which is determined by $\{\mathcal{F}_i\}_{i\in I}$, the full dg subcategory of $\mathcal{D}$ containing the $\mathcal{F}_i$.

Let $\mathcal{F}'_i$ be objects of a finite length Abelian subcategory $\mathcal{C}' \subset \mathcal{H}(\mathcal{D}')$ such that there is a dg equivalence $\{\mathcal{F}'_i\} \simeq \{\mathcal{F}_i\}$, and there is a good moduli space $\mathfrak{M}_{\mathcal{C}'} \to \mathcal{M}_{\mathcal{C}'}$ of objects in $\mathcal{C}'$. Let x' represent $\bigoplus_{i\in I} \mathcal{F}_i'^{\oplus \mathrm{d}_i}$. Then there is a commutative diagram with Cartesian squares and étale horizontal morphisms

$$\begin{array}{ccccc}
(\mathfrak{M}_{\mathcal{C}'}, x') & \hookleftarrow & (\mathfrak{U}, u) & \hookrightarrow & (\mathfrak{M}_{\mathcal{C}}, x) \\
\downarrow p' & & \downarrow q & & \downarrow p \\
(\mathcal{M}_{\mathcal{C}'}, x') & \hookleftarrow & (\mathcal{U}, q(u)) & \hookrightarrow & (\mathcal{M}_{\mathcal{C}}, x)
\end{array}$$

Formality

Let $\mathcal{C} \in \mathrm{H}(\mathcal{D})$ be a finite length Abelian category with $\mathcal{D}$ 2CY, e.g.

- Semistable coherent sheaves of fixed slope on smooth S satisfying $\mathcal{O}_S \cong \omega_S$
- Semistable Higgs bundles of fixed slope on a smooth projective curve.
- Representations of a (usual/deformed/multiplicative) preprojective algebra Π_Q.
- Bridgeland semistable objects of fixed slope in Kuznetsov component.
- Representations of $\pi_1(\Sigma_g)$ for Σ_g a compact Riemann surface.

Theorem

The dg category $\{\mathcal{F}_i\}_{i\in I}$ is formal for any collection of simple objects in $\mathcal{C}$. I.e. there is a quasi-equivalence of A_∞-algebras

$$\mathrm{RHom}^\bullet_{\mathcal{D}}(\mathcal{F},\mathcal{F}) \simeq \mathrm{Ext}^\bullet_{\mathcal{D}}(\mathcal{F},\mathcal{F}).$$

$\therefore$ The étale neighbourhood of $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ around x representing $\bigoplus_i \mathcal{F}_i^{\oplus \mathrm{d}_i}$ is entirely determined by the dimensions of the $\mathrm{Ext}^1(\mathcal{F}_i,\mathcal{F}_j)$.

Preprojective algebras

- Let Q be a quiver with vertices Q_0 and arrows Q_1.
- Define $\overline{Q}$ to be the doubled quiver formed by adjoining an arrow a^* with opposite orientation to a for each arrow $a \in Q_1$.
- Define $\Pi_Q = \mathbb{C}\overline{Q}/\langle\sum_{a\in Q_1}[a,a^*]\rangle$

E.g. if Q is the *Jordan quiver* with one vertex and one loop, $\Pi_Q \cong \mathbb{C}[x,y]$.

- The category $\mathcal{C}$ of finite-dimensional Π_Q-modules is an Abelian subcategory of a 2CY dg category $\mathcal{D}$ (outside of ADE type $\mathcal{D}$ is just the dg category of Π_Q-modules).
- Let S_i be the one-dimensional nilpotent Π_Q-module with dimension vector 1_i. Then

$$\mathrm{Ext}^1(S_i,S_j) = \begin{cases} a_{ij} & \text{if } i \neq j \\ 2a_{ii} & \text{if } i = j. \end{cases}$$

where (a_{ij}) is the adjacency matrix of the underlying graph of Q.

Local structure of stacks of objects in 2CY categories

Theorem

let $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ *be a good moduli space of objects in* $\mathcal{C}$*, a finite length Abelian subcategory of* $\mathcal{H}(\mathcal{D})$*, for* $\mathcal{D}$ *a 2CY dg category. Let* $x \in \mathcal{M}_{\mathcal{C}}$ *represent the semisimple object* $\mathcal{F} = \bigoplus_{i\in I} \mathcal{F}_i^{\oplus \mathrm{d}_i}$*, and pick* Q *so that the Ext quiver of* $\{\mathcal{F}_i\}_{i\in I}$ *is* $\overline{Q}$*. Then there is a commutative diagram*

$$\begin{array}{ccccc}
(\mathfrak{M}_{\mathrm{d}}(\Pi_Q), 0_{\mathrm{d}}) & \hookleftarrow & (\mathfrak{U}, u) & \hookrightarrow & (\mathfrak{M}, x) \\
\downarrow \mathtt{JH} & & \downarrow q & & \downarrow p \\
(\mathcal{M}_{\mathrm{d}}(\Pi_Q), 0_{\mathrm{d}}) & \hookleftarrow & (\mathcal{U}, q(u)) & \hookrightarrow & (\mathcal{M}, x)
\end{array}$$

with Cartesian squares, and étale horizontal maps, where $\mathtt{JH}$ *is the morphism from the stack of* d*-dimensional* Π_Q*-modules to the coarse moduli space.*

Purity for $p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}}$ around x then follows from purity for $\mathtt{JH}_*\,\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathrm{d}}(\Pi_Q)}$, for which we turn to cohomological DT theory.

Dimensional reduction I

Theorem

Let X be a smooth variety, $\overline{X} = X \times \mathbb{A}^m$. Let $f \in \Gamma(X \times \mathbb{A}^m)$ be a weight one function for the scaling action of $\mathbb{A}^m$, i.e. we can write $f = \sum_{i \leq m} x_i f_i$ for x_i coordinates on $\mathbb{A}^m$ and f_i functions on X. Let $Z = V(f_1, \ldots, f_m) \subset X$. Then the natural morphism

$$\pi_! \phi_f \mathbb{Q}_{\overline{X}} \to \pi_! \mathbb{Q}_{Z \times \mathbb{A}^m}$$

is an isomorphism. Taking Verdier duals, and derived global sections, yields

$$\mathrm{H}^{\mathrm{BM}}(Z \times \mathbb{A}^m, \mathbb{Q}) \xrightarrow{\cong} \mathrm{H}(\overline{X}, \phi_f \mathbb{Q}).$$

Example

Take function $\mathrm{Tr}(W) = \mathrm{Tr}([A, B]C)$ on $\overline{X} = \mathrm{Mat}_{n \times n}(\mathbb{C})^{\times 2} \times \mathrm{Mat}_{n \times n}(\mathbb{C})$. Then $\pi_* \phi_{\mathrm{Tr}(W)} \mathbb{Q}_{\overline{X}} \cong \mathbb{D}\mathbb{Q}_{\mathcal{C}_m} \otimes \mathbb{L}^?$, where $\mathcal{C}_m$ is the variety of pairs of commuting matrices, and $\mathbb{L} = \mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$ is the pure Tate twist.

Dimensional reduction II

Fix a quiver Q, and define $\tilde{Q}$ by adding a loop ω_i to $\overline{Q}$ at each vertex Q_0. Define

$$\tilde{W} = \sum_{a\in Q_1} [a, a^*] \sum_{i\in Q_0} \omega_i$$

Define $\pi\colon \mathfrak{M}(\tilde{Q}) \to \mathfrak{M}(\overline{Q})$ to be the forgetful morphism. Then there is an isomorphism

$$\pi_* \phi_{\mathrm{Tr}(\tilde{W})} \mathbb{Q} \cong \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)} \otimes \mathbb{L}^?.$$

Proof strategy

In the diagram

$$\begin{array}{ccc} \mathfrak{M}(\tilde{Q}) & \xrightarrow{\pi} & \mathfrak{M}(\overline{Q}) \\ \downarrow{\scriptstyle \tilde{\mathtt{JH}}} & & \downarrow{\scriptstyle \mathtt{JH}} \\ \mathcal{M}(\tilde{Q}) & \xrightarrow{\pi'} & \mathcal{M}(\overline{Q}) \end{array}$$

prove purity of $\mathtt{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}$ via purity of (isomorphic) $\pi'_* \tilde{\mathtt{JH}}_* \phi_{\mathrm{Tr}(\tilde{W})} \mathbb{Q}$.

Integrality and BPS sheaves for $\tilde{Q}, \tilde{W}$

Tensor structure

There is a finite morphism $\mu\colon \mathcal{M}(\tilde{Q}) \times \mathcal{M}(\tilde{Q}) \to \mathcal{M}(\tilde{Q})$ taking a pair of modules to their direct sum. This induces a symmetric tensor structure on $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}(\tilde{Q})))$:

$$\mathcal{F} \otimes_{\oplus} \mathcal{G} := \mu_*(\mathcal{F} \boxtimes \mathcal{G}).$$

$\pi'_*\colon \mathcal{D}^+(\mathrm{MHM}(\mathcal{M}(\tilde{Q}))) \to \mathcal{D}^+(\mathrm{MHM}(\mathcal{M}(\overline{Q})))$ is a tensor functor.

For $\mathfrak{M}$ a stack with virtual dimension d, we set $\mathbb{Q}_{\mathfrak{M},\mathrm{vir}} := \mathbb{Q}_{\mathfrak{M}} \otimes \mathbb{L}^{-d/2}$.

Theorem (-,Meinhardt)

There is a canonical isomorphism

$$\tilde{\mathrm{JH}}_* \phi_{\mathrm{Tr}(\tilde{W})} \mathbb{Q}_{\mathrm{vir}} \cong \mathrm{Sym}\left(\bigoplus_{\mathrm{d} \neq 0} \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q}_{\mathrm{vir}}) \right)$$

where $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}} = \phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathcal{M}_{\mathrm{d}}(\tilde{Q})} \in \mathrm{MHM}(\mathcal{M}_{\mathrm{d}}(\tilde{Q}))$ *is Verdier self dual.*

Finishing the proof

$$\mathrm{JH}_* \, \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q),\mathrm{vir}} \cong (\mathcal{M}(\tilde{Q}) \xrightarrow{\pi'} \mathcal{M}(\overline{Q}))_* \tilde{\mathrm{JH}}_* \phi_{\mathrm{Tr}(\tilde{W})} \mathbb{Q}_{\mathfrak{M}(\tilde{Q}),\mathrm{vir}}$$
$$\cong \mathrm{Sym}\left(\bigoplus_{\mathrm{d} \in \mathbb{N}^{Q_0}} \pi'_* \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}} \otimes \mathrm{H}(\mathrm{pt}\,/\mathbb{C}^*, \mathbb{Q}_{\mathrm{vir}}) \right)$$

So we need to prove purity of the direct image $\pi'_* \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}}$ of the BPS sheaf. This follows from

1. $\mathrm{JH}_* \, \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q),\mathrm{vir}}$ can only be "impure above":
$$W_i(\mathcal{H}^{i+1}(\mathrm{JH}_* \, \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q),\mathrm{vir}})) = 0.$$
So same is true of summand $\pi'_* \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}}$
2. Verdier self-duality of $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}}$implies that it is impure above iff it is impure below.
3. The "support lemma": $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathrm{d}}$ is entirely supported on locus where all of the eigenvalues of the loops ω_i are the same.

The decomposition theorem

Theorem

Let $\mathcal{C} \subset \mathcal{H}(\mathcal{D})$ be finite length Abelian category, with $\mathcal{D}$ a 2CY dg category. Let $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ be a good moduli space of objects in $\mathcal{C}$, with $\mathfrak{M}_{\mathcal{C}}$ a global quotient stack. Then $p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}}$ is pure, and

$$p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}} \cong \bigoplus_{i\in 2\cdot\mathbb{Z}_{\geq 0}} \mathcal{H}^i(p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}})[-i].$$

Corollary

For S a K3 or Abelian surface, H a polarization, the mixed Hodge structure $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^H_\alpha(S),\mathbb{Q}) \cong \mathrm{H}(\mathcal{M}^H_\alpha(S), p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}^H_\alpha(S)})$ is pure.

Corollary

If there is a stable object in $\mathfrak{M}^H_\alpha(S)$, there is a canonical inclusion $\mathrm{IC}_{\mathcal{M}^H_\alpha(S)} \subset \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^H_\alpha(S),\mathbb{Q}_{\mathrm{vir}})$ as a cuspidal summand of the zeroth piece of the perverse filtration with respect to p.

Corollaries in nonabelian Hodge theory

Corollary

For C a smooth projective curve, and $\mathtt{h}\colon \mathcal{M}^{\mathrm{Dol}}_{r,d}(C) \to \Lambda_r$ the Hitchin map from the moduli space of semistable Higgs bundles, the complex $(\mathtt{h} \circ p)_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}}$ is pure. So the Borel–Moore homology of $\mathbb{D}\mathbb{Q}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}}$ carries a perverse filtration, generalising the case $(r,d) = 1$.

Corollary

For C a smooth complex projective curve, the mixed Hodge structure $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C), \mathbb{Q}) \cong \mathrm{H}(\Lambda, (\mathtt{h} \circ p)_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)})$ is pure.

Finally: can formulate stacky $P = W$ conjecture, that implies $IP = IW$ conjecture, and is conjecturally equivalent to it, along with $P = W$ conjecture (via χ-independence conjectures for BPS cohomology).

Theorem

SP=SW true in genus 0,1.