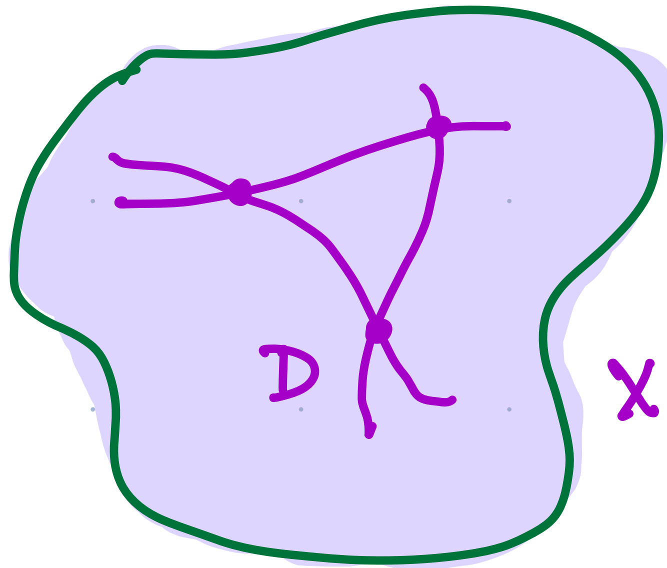
LOGARITHMIC ENUMERATIVE GEOMETRY for CURVES and SHEAVES

with Davesh Maulik

Enumerative geometry of pairs

Fix a pair (X,D): projective manifold X
s.n.c. divisor $D \subseteq X$

<u>Goal</u>: Study enumerative geometry of curves in X



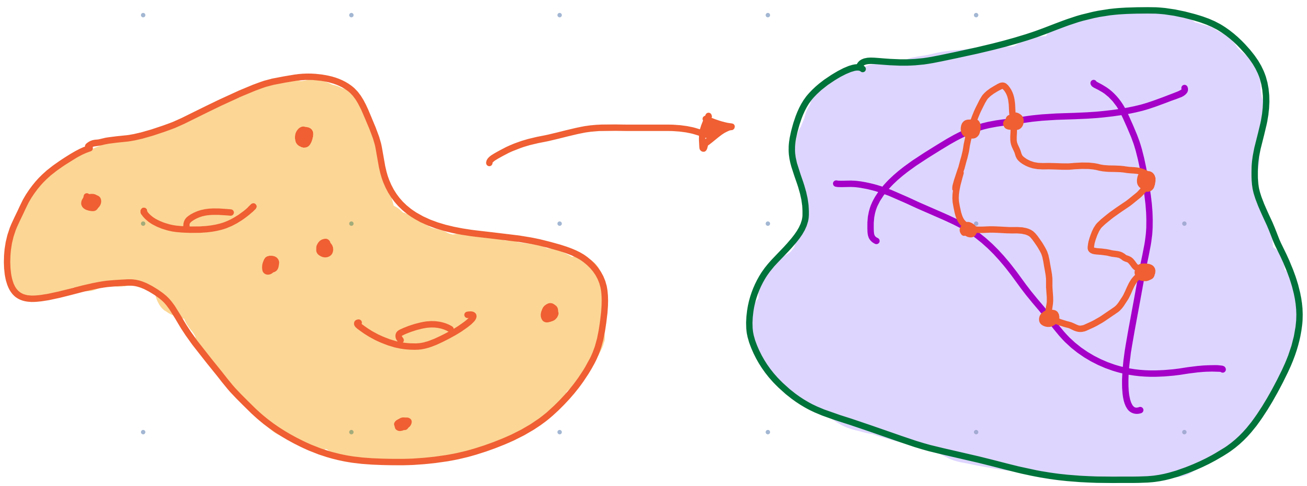
▶ Fix tangency profile along D

...via GW theory

a quick reminder

Intersection theory on $\overline{M}_\Lambda(X,D)$

Non-degenerate locus: Maps from C to (X,D)



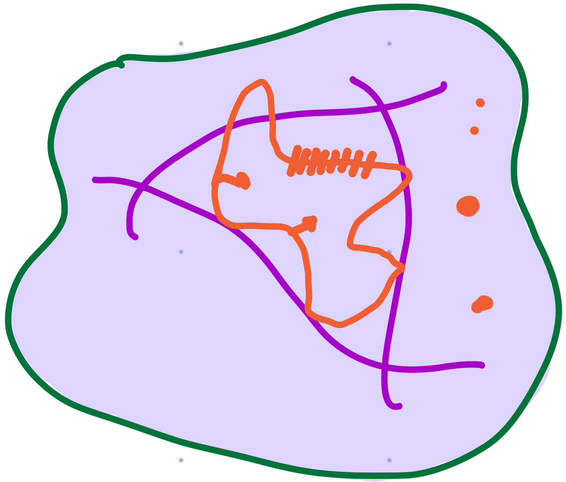
Λ: • Numerical data fixing genus g & class β

• Tangency data c_{ij} for p_i along D_j.

... via DT theory

Intersection theory on $\mathrm{Hilb}_\Lambda(X,D)$ [subschemes of dim 1]

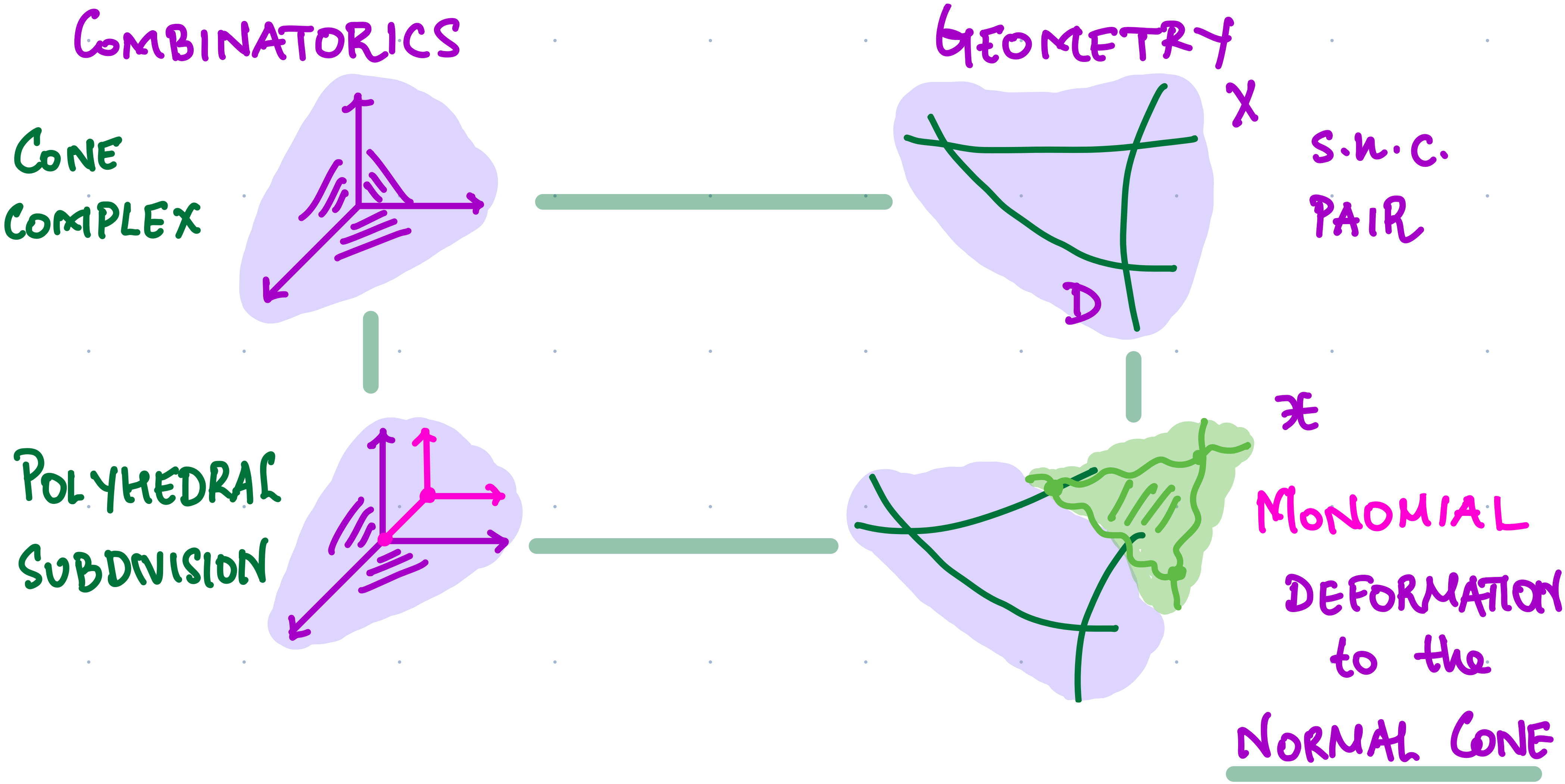
Non-degenerate locus: transverse subschemes of (X,D)



In DT theory Λ DOES NOT fix tangency data!

the stack of expansions

On the boundary we study curves in expansions



the stack of expansions

Main construction in Maulik-R '20 is moduli of expansions

Zero-dimensional Artin stack

standard notice about choices

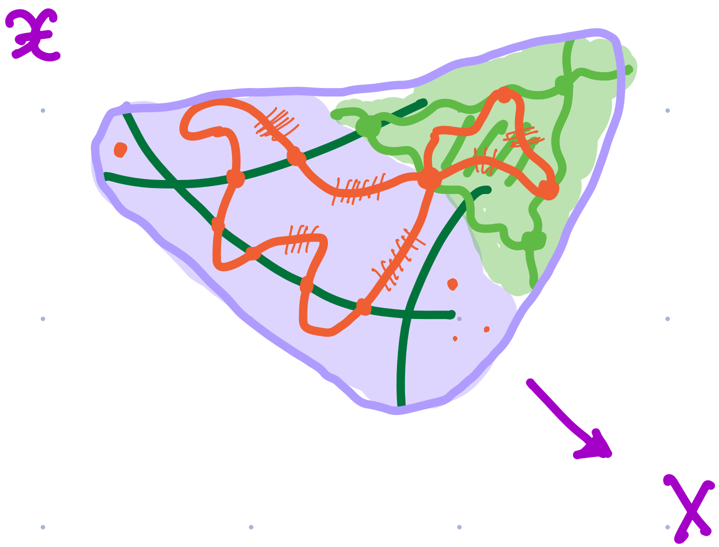
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & X \\ \mathrm{Exp}(X|D) & \end{tikzcd}$$

Example: If D is smooth then $\mathrm{Exp}(X,D) \subseteq \mathcal{M}^{ss}_{0,3}$

For (X,D) is a toric surface, work of Kennedy-Hunt via GKZ theory

... Logarithmic GW/DT theory

On the boundary we study curves in expansions

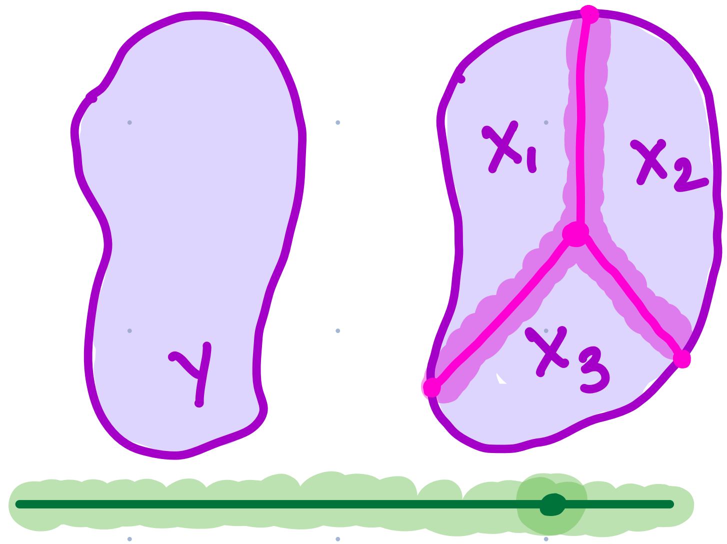


- curves should be transverse
- Require "no useless components"
- Carries VIRTUAL class

In GW theory, this essentially recovers Abramovich–Chen
Gross–Siebert

Main results

preview

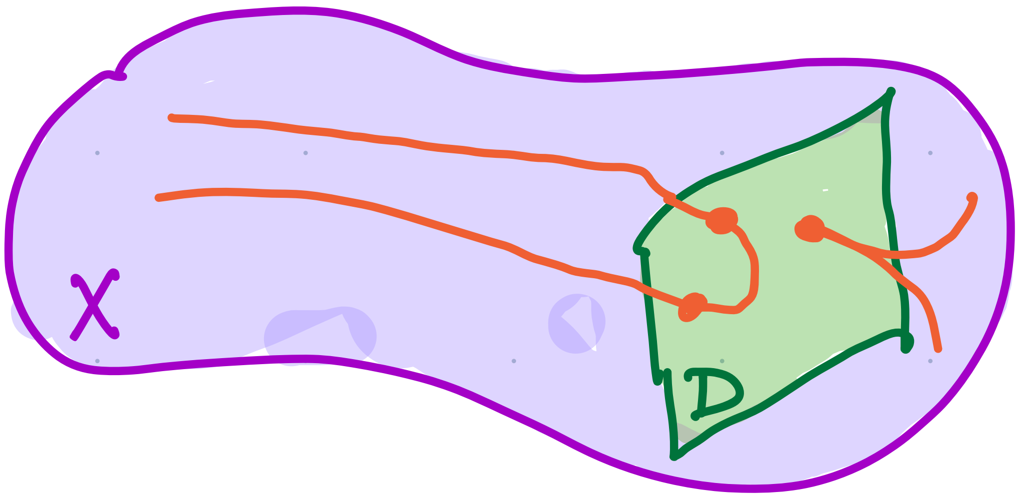


- **THM**: Logarithmic degeneration formula in DT theory
- A logarithmic GW/DT conjecture for pairs
- **THM**: $(GW/DT)^{\log}$ is compatible with degeneration

Hilbert schemes of points

smooth divisor

If D is a smooth surface then a key role in GW/DT played by $\mathrm{Hilb}^n(D)$



$$\mathrm{Hilb}_{\mathrm{curves}}(X,D) \xrightarrow{\cap D} \mathrm{Hilb}_{\mathrm{pts}}(D)$$

Notice: No logarithmic anything!

Hilbert schemes of points

smooth divisor

By Nakajima–Grojnowski

$H^*(\mathrm{Hilb}^n(D);\mathbb{Q})$ has a basis

via $H^*(D)$-weighted partitions

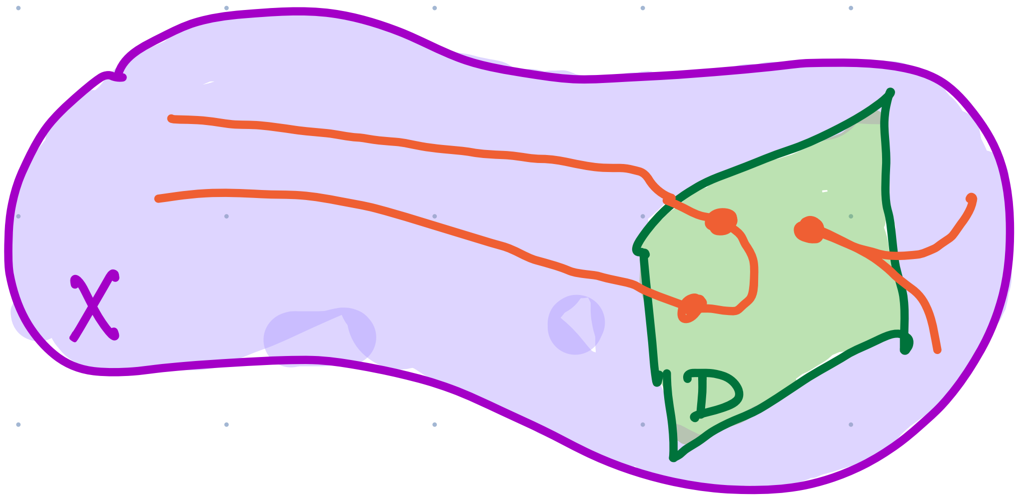
Very Roughly:

Number of entries in μ $\longleftrightarrow$ Number of distinct points

Single entry of μ $\longleftrightarrow$ Length at each point

$H^*(D)$-weights $\longleftrightarrow$ Loci swept out by the supports.

Tangency conditions in GW/DT theory



DT $\left\{ \right.$ $\mathrm{Hilb}_{\text{curves}}(X,D) \xrightarrow{\cap D} \mathrm{Hilb}_{\text{pts}}(D)$

GW $\overline{M}_\lambda(X,D) \longrightarrow D \times \cdots \times D$

Nakajima basis gives tangency

Many different spaces for different tangency.

Logarithmic Hilbert schemes of points

(S,E) is a pair. The relevant stack of expansions is very concrete.

- Form cone complex $\Sigma(S,E)$
- Build symmetric product $\mathrm{Sym}^n(\Sigma(S,E))$
- Turn this cone complex into an Artin stack.

$$\begin{tikzcd} \mathrm{Hilb}^n(S,E) \arrow[d] \arrow[dr] & \\ \mathrm{Sym}^n(S,E) \arrow[r] & \mathrm{Exp} \end{tikzcd}$$

Hilbert–Chow type map.

Exotic Invariants

The degeneration formula requires New types of invariants:

$$\mathrm{Hilb}(X,D) \xrightarrow{ev} \prod_i \mathrm{Hilb}_{pts}(D_i, D_i \cap D)$$

Allow blowups of this product

We need classes from

$$H^*_{log}\Big(\prod_i \mathrm{Hilb}_{pts}(D_i, D_i \cap D); \mathbb{Q}\Big)$$

Not a tensor product over the divisors!

Tangency conditions in GW/DT theory

logarithmic case

$D = D_1 + \cdots + D_r$ $\quad\quad$ $\underline{n} = (n_1, \ldots, n_r)$

We have **EVALUATION SPACES**

DT SIDE $\quad\quad$ GW SIDE

$$\begin{tikzcd}
\mathrm{Hilb}^{\underline{n}}(D, \partial D) \arrow[dr] & & \mathrm{Ev}^{\underline{n}}(D, \partial D) \arrow[dl] \\
& \mathrm{Sym}^{\underline{n}}(D, \partial D) \arrow[d] & \\
& \mathrm{Exp} &
\end{tikzcd}$$

Tangency conditions in GW/DT theory

$$\begin{tikzcd}
\mathrm{Hilb}^{\underline{n}}(D,\partial D) \arrow[dr] & & \mathrm{Ev}^{\underline{n}}(D,\partial D) \arrow[dl] \\
& \mathrm{Sym}^{\underline{n}}(D,\partial D) \arrow[d] & \\
& \mathrm{Exp} &
\end{tikzcd}$$

$\underline{\mu}$ a vector of partitions of $\underline{n}$.

GW Evaluations: $H^{*}_{\log}(\mathrm{Ev}^{\underline{\mu}}(D))^{\mathrm{Aut}}$

DT Evaluations: $H^{*}_{\log}(\mathrm{Hilb}^{\underline{n}}(D,\partial D))$

NEED to MATCH

Tangency conditions in GW/DT theory

THM: There is an isomorphism:

$$\boxed{\bigoplus_{\mu} H^{\star}_{\log}(Ev^{\mu}(D))^{Aut} \longrightarrow H^{\star}_{\log}(\mathrm{Hilb}^{n}(D, \partial D))}$$

This lifts the Grojnowski-Nakajima basis.

Key: The support map $\mathrm{Hilb}^{n} \to \mathrm{Sym}^{n}$ is semismall. [de Cataldo-Migliorini]

The GW/DT correspondence

Fix tangency vector $\underline{\mu}$, one for each D_i

Choose: $H^*(Ev^{\underline{\mu}}(D,\partial D))^{\mathrm{Aut}} \longleftrightarrow H^*(\mathrm{Hilb}^{\underline{n}}(D,\partial D))$

and "bulk" insertions $\underline{\gamma}$

$$\boxed{\begin{gathered} Z'_{GW}(X/D;u,\underline{\gamma})_\beta \text{“}=\text{”} Z'_{DT}(X/D;q,\underline{\gamma})_\beta \\ e^{iu}=-q \end{gathered}}$$

some remarks

Rationality of DT series is conjectured [PRIMARY THEORY]

There are PT versions

DT/PT conjecturally related after removing the degree zero series.

The DT degree 0 series $M(-q)^{\int c_3(T_X^{\log} \otimes K_X^{\log})}$

MacMahon function

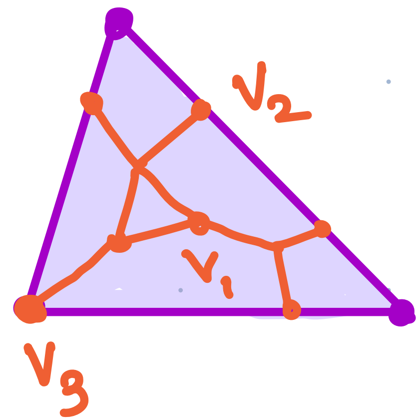
Logarithmic degeneration formula

[there is a GW theory version]

- Start with: DT^Y_β

- Degenerate Y to $\cup X_i$. Form $\Sigma(Y)$.

- Find relevant tropical curves γ.

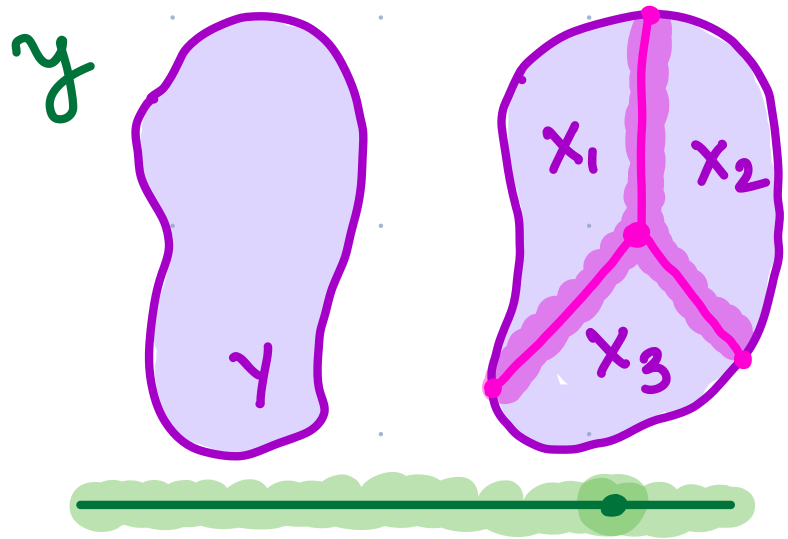


Write: $$DT^Y_\beta = \sum_\gamma DT_\gamma$$

- Express $$DT_\gamma = \mathop{\Large\ast}_{v \in \gamma} DT_v$$

Exotic invariants are crucial here!

Degeneration compatibility

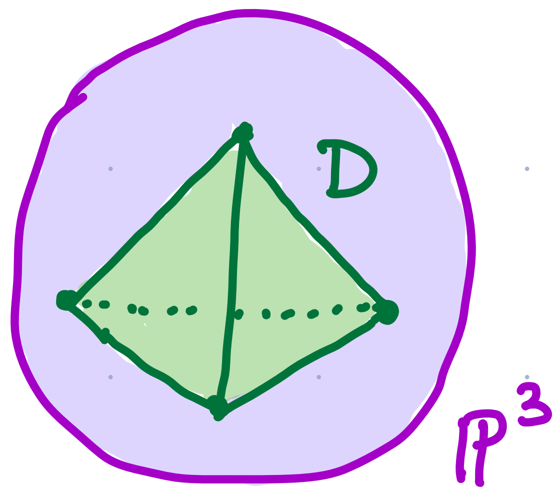


<u>THM</u>: For classes coming from $\mathcal{Y}$,

$(GW/DT)^{\log}$ on special fiber components $\Longrightarrow$ GW/DT on Y.

The most basic log CY3

Take $(X, D) = (\mathbb{P}^3, \text{4 PLANES})$



What we know:

First : Degree 0 series $= 1$

follows from degeneration & birational invariance

For curve class $\beta =$ line or 2[line]

Second: Log GW/DT holds for these classes

The most basic log CY3

We believe that GW/DT for $(\mathbb{P}^3, 4H)$ is close

Strategy: $\mathrm{Hilb}_\lambda(\mathbb{P}^3, 4) \longrightarrow \mathrm{Hilb}_{pts}(4H, \partial 4H)$

Lots of relations on the target; heavily constrains the $DT^{\log}$-invariants

Very closely related:
- Bousseau's tropical refined curve counting
- Parker's 3D tropical paper

Final Remarks

- Degree 0 conjectures appear very provable; localization here is easier than general case.

- So far only "ancient" DT theory has a log upgrade, don't know how to do moduli of sheaves

- Kennedy-Hunt (shortly forthcoming) builds a full logarithmic Quot scheme [in particular, full Hilb]

- Fun to be had on surfaces

HAPPY DIWALI!

and thanks!