Blowup formulas, strange duality and generating functions for Segre and Verlinde numbers of surfaces

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based on joint work with Martijn Kool
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Let S smooth projective surface
Hilbert scheme of points:

$$S^{[n]} = Hilb^n(S) = \{\text{zero dim. subschemes of degree } n \text{ on } S\}$$

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Universal subscheme:

$$Z_n(S) = \big\{(x, [Z]) \bigm| x \in Z\big\} \subset S \times S^{[n]}$$

$p : Z_n(S) \to S^{[n]}, \quad q : Z_n(S) \to S$ projections
Fibre $p^{-1}([Z]) = Z$.

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Tautological sheaves: V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z)$

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Line bundles on $S^{[n]}$: $\mathrm{Pic}(S^{[n]}) = \mu(\mathrm{Pic}(S)) \oplus \mathbb{Z}E$ with $E = \det(\mathcal{O}_S^{[n]})$.
We have

$$\det(V^{[n]}) = \mu(\det(V)) \otimes E^{\otimes \operatorname{rk}(V)}, \quad V \in K(S)$$

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Want formulas for

$$\chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) \quad \textbf{Verlinde formula}$$

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) = \int_{S^{[n]}} s_{2n}(-V^{[n]}) \quad \textbf{Segre formula}$$

Theorem (Ellingsrud-G-Lehn)

Let $P(x_1, \ldots, x_{2n}, y_1, \ldots, y_n)$ polynomial. Put

$$P[S^{[n]}, L] := \int_{S^{[n]}} P(c_1(S^{[n]}), ..., c_{2n}(S^{[n]}), c_1(L^{[n]}), \ldots, c_n(L^{[n]}))$$

There is a polynomial $\widetilde{P}(x, y, z, w)$, such that for all surfaces S, all line bundles L on S we have

$$P[S^{[n]}, L] = \widetilde{P}(K_S^2, \chi(\mathcal{O}_S), LK_S, K_S^2).$$

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Usually look sequence of polynomials $P_n(x_1, ...x_{2n}, y_1, \ldots, y_n)$, $n \geq 0$, "compatible", then

$$\sum_{n\geq 0} P_n[S^{[n]}, L]x^n = A_1(x)^{L^2} A_2(x)^{LK_S} A_3(x)^{K_S^2} A_4(x)^{\chi(O_S)}$$

for universal power series $A_1, \ldots, A_4 \in \mathbb{Q}[[x]]$

For L a line bundle on S consider the top Segre class

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Conjecture (Lehn 1999)

$$\sum_{n=0}^{\infty} \int_{S^{[n]}} s_{2n}(L^{[n]}) z^n = \frac{(1-w)^a(1-2w)^b}{(1-6w+6w^2)^c},$$

with the change of variable

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3},$$

with $a = LK_S - 2K_S^2$, $b = (L-K_S)^2 + 3\chi(\mathcal{O}_S)$,
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Theorem (Marian-Oprea-Pandharipande, Voisin)

Lehn's conjecture is true.

Marian-Oprea-Pandharipande consider a generalized Segre formula:
a formula for $\sum_{n\geq 0}\int_{S^{[n]}} c_{2n}(\alpha^{[n]})z^n, \quad \alpha\in K(S)$

Theorem (Marian-Oprea-Pandharipande)

For any $s\in\mathbb{Z}$, there exist $V_s, W_s, X_s, Y_s, Z_s\in\mathbb{Q}[[z]]$ s.th. for any $\alpha\in K(S)$ of rank s on S, we have

$$\sum_{n=0}^{\infty} z^n\int_{S^{[n]}} c_{2n}(\alpha^{[n]}) = V_s^{c_2(\alpha)}\, W_s^{c_1(\alpha)^2}\, X_s^{\chi(\mathcal{O}_S)}\, Y_s^{c_1(\alpha)K_S}\, Z_s^{K_S^2}.$$

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With the change of variables $z=t(1+(1-s)t)^{1-s}$, one has

$$V_s(z)=(1+(1-s)t)^{1-s}(1+(2-s)t)^{s},$$
$$W_s(z)=(1+(1-s)t)^{\frac{1}{2}s-1}(1+(2-s)t)^{\frac{1}{2}(1-s)},$$
$$X_s(z)=(1+(1-s)t)^{\frac{1}{2}s^2-s}(1+(2-s)t)^{-\frac{1}{2}s^2+\frac{1}{2}}(1+(2-s)(1-s)t)^{-\frac{1}{2}}.$$

They showed explicit expressions for Y_s, Z_s for $s\in\{-2,-1,0,1,2\}$, and conjecture that Y_s, Z_s are algebraic functions for all $s\in\mathbb{Z}$

Consider the generating series $\sum_{n=0}^{\infty} w^n\, \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r})$.

Theorem (Ellingsrud-G-Lehn)

For any $r \in \mathbb{Z}$, there exist $g_r, f_r, A_r, B_r \in \mathbb{Q}[[w]]$ such that for any $L \in Pic(S)$, we have

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$$g_r(w) = 1 + v, \quad f_r(w) = \frac{(1+v)^{r^2}}{1 + r^2 v}.$$

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Serre duality implies $A_r = B_{-r}/B_r$ for all r. Furthermore, $A_r = B_r = 1$ for $r = 0, \pm 1$. In general the A_r, B_r are unknown.

We have seen

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c_{2n}(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}, \quad s = \mathrm{rk}(\alpha)$$

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with $V_s, W_s, X_s \in \mathbb{Q}[[z]]$, $f_r, g_r \in \mathbb{Q}[[w]]$ known algebraic functions, and $Y_s, Z_s \in \mathbb{Q}[[z]]$, $A_r, B_r \in \mathbb{Q}[[w]]$ unknown

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with $V_s,\ W_s,\ X_s \in \mathbb{Q}[[z]]$, $f_r,\ g_r \in \mathbb{Q}[[w]]$ known algebraic functions, and $Y_s,\ Z_s \in \mathbb{Q}[[z]]$, $A_r,\ B_r \in \mathbb{Q}[[w]]$ unknown
Based on strange duality there is a conjectural relation between these two generating functions

Conjecture (Johnson, Marian-Oprea-Pandharipande)

For any $r \in \mathbb{Z}$, we have

$$A_r(w) = W_s(z)\, Y_s(z), \quad B_r(w) = Z_s(z),$$

where $s = 1 + r$ and $w = v(1+v)^{r^2-1}$, $z = t(1+(1-s)t)^{1-s}$, and $v = t(1-rt)^{-1}$.

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Theorem

$$A_r(w) = (1+v)^{-\frac{r}{2}} \exp\Big(\sum_{i>0} \frac{(-1)^{i+1} v^i}{2i} \mathrm{Coeff}_{x^0}\Big[\Big(\frac{x^r - x^{-r}}{x - x^{-1}}\Big)^{2i}\Big]\Big)$$

equivalently if $A_{i,r}(w)^{\frac{1}{2}}$ are the $r-1$ solutions of $\frac{y^{-1}+(-1)^r y}{y^{-r}-y^r} = v^{\frac{1}{2}}$, then $A_r(w) = \frac{1}{v^{\frac{1}{2}}(1+v)^{\frac{r}{2}} \prod_{i=1}^{r-1} A_{i,r}^{\frac{1}{2}}}$

We can conjecturally also determine the B-series.

Conjecture

$$B_r(w)^8 = \left(\frac{\prod_{i=1}^{r-1} A_{i,r}}{v}\right)^{4r+2} (1+v)^{r^2+2r}(1+r^2v)^3 \cdot \prod_{i,j=1}^{r-1}(1-A_{i,r}A_{j,r})^2 \prod_{\substack{i,j=1\\ i\neq j}}^{r-1}(1-A_{i,r}^r A_{j,r}^r)^2$$

$$\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L)\otimes E^{\otimes r}) = g_r^{\chi(L)} f_r^{\frac{1}{2}\chi(\mathcal{O}_S)} A_r^{LK_S} B_r^{K_S^2}$$

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c_{2n}(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}.$$

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Theorem

The Verlinde-Segre correspondence is true:
$A_r(w) = W_{r+1}(z)\, Y_{r+1}(z), \quad B_r(w) = Z_{r+1}(z)$
with $w = v(1+v)^{r^2-1}$, $z = t(1+(1-s)t)^{1-s}$, *and* $v = t(1-rt)^{-1}$

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Let (S, H) polarized surface.
Assume in the following that $p_g(S) > 0$, $b_1(S) = 0$
For $\rho \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, and $c_2 \in H^4(S, \mathbb{Z})$, let
$M := M_S^H(\rho, c_1, c_2)$ moduli space of rank ρ H-semistable sheaves on S with Chern classes c_1, c_2

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Note: via $Z \mapsto I_Z$, we have $S^{[n]} = M_S^H(1, 0, n)$.

Assume M contains no strictly semistable sheaves
For simplicity also assume there exists a universal sheaf $\mathcal{E}$ on $S \times M$, (i.e. $\mathcal{E}|_{S\times\{[E]\}} = E$)

$M = M_S^H(\rho, c_1, c_2)$ has a perfect obstruction theory of expected dimension

$$\mathrm{vd}(M) := 2\rho c_2 - (\rho - 1)c_1^2 - (\rho^2 - 1)\chi(\mathcal{O}_S)$$

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In particular

- it carries a virtual class $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}(M)}(M)$
- has a virtual Tangent bundle $T_M^{\mathrm{vir}} \in K^0(M)$
- has a virtual structure sheaf $\mathcal{O}_M^{\mathrm{vir}} \in K_0(S)$
  For any $V \in K^0(M)$ the virtual holomorphic Euler characteristic of V is $\chi^{\mathrm{vir}}(M, V) := \chi(M, V \otimes \mathcal{O}_M^{\mathrm{vir}})$

Determinant bundles: Let $c \in K(S)$ be the class of $E \in M = M_S^H(\rho, c_1, c_2)$ and $K_c := \{v \in K(S) \,:\, \chi(S, c \otimes v) = 0\}$
For $\alpha \in K_c$ put with $\pi_S : S \times M \to S$, $\pi_M : S \times M \to M$ projections

$$\lambda(\alpha) := \det\left(\pi_{M!}\left(\pi_S^*\alpha \cdot [\mathcal{E}]\right)\right)^{-1} \in \mathrm{Pic}(M)$$

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take $v \in K_c$ such that $\mathrm{rk}(v) = r$ and $c_1(v) = \mathcal{L}$, put

$$\mu(L) \otimes E^{\otimes r} := \lambda(v) \in \mathrm{Pic}(M).$$

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Denote by $\mathcal{O}_M^{\mathrm{vir}}$ the virtual structure sheaf of M
The *virtual Verlinde numbers* of S are the virtual holomorphic Euler characteristics

$$\chi^{\mathrm{vir}}(M, \mu(L) \otimes E^{\otimes r}) := \chi(M, \mu(L) \otimes E^{\otimes r} \otimes \mathcal{O}_M^{\mathrm{vir}})$$

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and S has a smooth connected canonical divisor
Write $\varepsilon_\rho := \exp(2\pi i/\rho)$ and $[n] := \{1, \ldots, n\}$. For any $J \subset [n]$, write $|J|$ for its cardinality and $\|J\| := \sum_{j \in J} j$

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Conjecture (GK)

Let $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$. There exist $A_{J,r} = A_{J,r}^{(\rho)}$, $B_{J,r} = B_{J,r}^{(\rho)} \in \mathbb{C}[[w^{\frac{1}{2}}]]$ for all $J \subset [\rho - 1]$ such that $\chi^{\mathrm{vir}}(M_S^H(\rho, c_1, c_2), \mu(L) \otimes E^{\otimes r})$ equals the coefficient of $w^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_S)+K_S^2}\, G_r^{\chi(L)}\, F_r^{\frac{1}{2}\chi(\mathcal{O}_S)} \sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_S)}\, \varepsilon_\rho^{\|J\| K_S c_1}\, A_{J,r}^{K_S L}\, B_{J,r}^{K_S^2}.$$

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Here $G_r(w) = 1 + v, \quad F_r(w) = \dfrac{(1+v)^{\frac{r^2}{\rho^2}}}{1 + \frac{r^2}{\rho^2} v}$ *with* $w = v(1+v)^{\frac{r^2}{\rho^2}-1}$

Furthermore, $A_{J,r}$, $B_{J,r}$ are all algebraic functions.

In general one gets the following (which we will need later)

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Let $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$
There exist $A_r, B_r, A_{i,r}, B_{ij,r} \in \mathbb{C}[[w^{\frac{1}{2}}]]$ for all $1 \leq i \leq j \leq \rho - 1$
sth. $\chi^{\mathrm{vir}}(M_S^H(\rho, c_1, c_2), \mu(L) \otimes E^{\otimes r})$ is the coefficient of $w^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(s)} G_r^{\chi(L)} F_r^{\frac{1}{2}\chi(s)} A_r^{LK_S} B_r^{K_S^2} \sum_{(a_1,\ldots,a_{\rho-1})} \prod_{j=1}^{\rho-1} \varepsilon_\rho^{ja_jc_1} SW(a_i)\, A_{j,r}^{a_jL} \prod_{1\leq j\leq k\leq \rho-1} B_{jk,r}^{a_ja_k}.$$

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Here $SW : H^2(S,\mathbb{Z}) \to \mathbb{Z}$ are the Seiberg-Witten invariants
If S has a smooth connected canonical divisor only nonzero SW invariants are $SW(0) = 1$, $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$
This gives previous version with

$$A_{J,r} := A_r \prod_{j\in J} A_{j,r}, \quad B_{J,r} := B_r \prod_{i\leq j\in J} B_{ij,r}$$

For any class $\alpha \in K^0(S)$, we define with $\pi_M : S \times M \to M$, $\pi_S : S \times M \to S$,

$$\alpha_M := -\pi_{M!}(\pi_S^*\alpha \cdot \mathcal{E} \cdot \det(\mathcal{E})^{-\frac{1}{\rho}})$$

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On $M := M_S^H(1, 0, n) \cong S^{[n]}$, we have $\alpha_M = \alpha^{[n]}$
For $\alpha \in K^0(S)$, the *virtual Segre number* of M is

$$\int_{[M]^{\mathrm{vir}}} c_{\mathrm{vd}}(\alpha_M) \in \mathbb{Z}$$

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and S has a smooth connected canonical divisor
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Conjecture (GK)

Let $\rho \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$. There exist V_s, W_s, $X_s \in \mathbb{C}[[z]]$, $Y_{J,s}$, $Z_{J,s} \in \mathbb{C}[[z^{\frac{1}{2}}]]$, for all $J \subset [\rho - 1]$ s.th. for all S as above, any $\alpha \in K^0(S)$ with $\mathrm{rk}(\alpha) = s$ we have that

$$\int_{[M_S^H(\rho, c_1, c_2)]^{\mathrm{vir}}} c_{\mathrm{vd}}(\alpha_M)$$

is the coefficient of $z^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_S)+K_S^2} \, V_s^{c_2(\alpha)} \, W_s^{c_1(\alpha)^2} \, X_s^{\chi(\mathcal{O}_S)}$$
$$\sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_S)} \, \varepsilon_\rho^{\|J\| K_S c_1} \, Y_{J,s}^{c_1(\alpha) K_S} \, Z_{J,s}^{K_S^2}.$$

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With $z = t(1+(1-\frac{s}{\rho})t)^{1-\frac{s}{\rho}}$, we have

$$V_s(z) = (1+(1-\tfrac{s}{\rho})t)^{\rho-s}(1+(2-\tfrac{s}{\rho})t)^{s},$$

$$W_s(z) = (1+(1-\tfrac{s}{\rho})t)^{\frac{1}{2}(s-1-\rho)}(1+(2-\tfrac{s}{\rho})t)^{\frac{1}{2}(1-s)},$$

$$X_s(z) = (1+(1-\tfrac{s}{\rho})t)^{\frac{1}{2}(s^2-(\rho+\frac{1}{\rho})s)}(1+(2-\tfrac{s}{\rho})t)^{-\frac{1}{2}s^2+\frac{1}{2}}(1+(1-\tfrac{s}{\rho})(2-\tfrac{s}{\rho})t)^{-\frac{1}{2}}.$$

Furthermore, $Y_{J,s}$, $Z_{J,s}$ are all algebraic functions

Theorem (Oberdieck 2022)

This conjecture is true for K3 surfaces

We get the following analogue of the Segre-Verlinde correspondence for Hilbert schemes

Conjecture (GK)

For any $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$, for all $J \subset [\rho - 1]$, we have

$$A_{J,r}(w) = W_{\rho+r}(z)\, Y_{J,\rho+r}(z), \quad B_{J,r}(w) = Z_{J,\rho+r}(z),$$

with

$$w = v(1+v)^{\frac{r^2}{\rho^2}-1}, \quad z = t(1+(1-\tfrac{s}{\rho})t)^{1-\frac{s}{\rho}}, \quad v = t(1-\tfrac{r}{\rho}t)^{-1}.$$

Blowup formulas

Let $\pi : \widehat{S} \to S$ blowup of S in a point with exceptional divisor D

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Conjecture

Let $L \in Pic(S)$, let $c \in K^0(S)$ be a class of rank ρ. Let $|r| \leq \rho$. Then

$$\chi^{\mathrm{vir}}(M_{\widehat{S}}(\pi^*c), \mu(\pi^*L+kD)\otimes E^{\otimes r}) = \chi^{\mathrm{vir}}(M_S(c), \mu(L)\otimes E^{\otimes r}), \ \ k = 0, \dots, \rho$$

$$\chi^{\mathrm{vir}}(M_{\widehat{S}}(\pi^*c - \ell\mathcal{O}_D), \mu(\pi^*L+(k+r-\tfrac{\ell}{\rho})D)\otimes E^{\otimes r}) = 0, \ \ k, \ell = 1, \dots, \rho-1$$

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There are similar s formulas for the Segre invariants
Based on computations with Mochizuki's formula
Note: similar formulas shown earlier by Nakajima-Yoshioka for eq. sheaves on $\mathbb{A}^2$ vs $\widehat{\mathbb{A}}^2$. Tannaka-Kuhn showed this generalizes to virtual invariants of moduli spaces of sheaves
use this: working on a proof

Using the structure conjecture this gives

Conjecture

Let $|r| \leq \rho$. *Recall* $[\rho-1] = \{1, \ldots, \rho-1\}$ *and* $\|J\| = \sum_{j \in J} j$

1. *For* $a = -\rho, \ldots, 0$ *we have*

$$\sum_{J \subset [\rho-1]} A_{J,r}(w)^a B_{J,r}(w)^{-1} = (1+v)^{\binom{a+1}{2}}, \qquad w = v(1+v)^{\frac{r^2}{\rho^2}-1}$$

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$$\sum_{J\subset[\rho-1]} \epsilon_\rho^{\ell\|J\|} A_{J,r}^a B_{J,r}^{-1}=0$$

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There are similar formulas for the Segre invariants related to the formulas for the Verlinde invariants by the Verlinde-Segre correspondence

Making suitable assumptions, the blowup formulas allow computation of coefficients of $A_{J,r}, B_{J,r}, \ Y_{J,s}, Z_{J,s}$

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In case $s = 0$ we can consider the case $\alpha = 0$, thus $\int_{[M^{\mathrm{vir}}]} c(\alpha_M) = \int_{[M^{\mathrm{vir}}]} 1$, and $\int_{[M^{\mathrm{vir}}]} 1 = 0$ unless $\mathrm{vd}(M) = 0$
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Thus $Z_{J,0}(z) = B_{J,-\rho}(w)$ are constant (independent of w and z)
Using blowup formulas and extensive computations for $\rho \leq 10$ we find

Conjecture

Let ξ primitive 4ρ-th root of unity. For $i, j \leq \rho - 1$ let $\beta_{ij} = \frac{\xi^{|i+j|} - \xi^{-|i+j|}}{\xi^{|i-j|} - \xi^{-|i-j|}}$.
For $J \subset [\rho - 1]$ put

$$\beta_J = \prod_{\substack{i \in J \\ j \in [\rho-1] \setminus J}} \beta_{ij}, \qquad B_I = \sum_{J \in [\rho-1]} \frac{\beta_J}{\beta_I}.$$

Then $B_{I,-\rho} = Z_{I,0} = B_I$.

Application: Donaldson invariants in arbitrary rank

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For any $\gamma \in H^k(S, \mathbb{Q})$ let
$\mu(\gamma) := \left(c_2(\mathcal{E}) - \frac{\rho-1}{2\rho} c_1(\mathcal{E})^2\right)/\mathrm{PD}(\gamma) \in H^k(M, \mathbb{Q})$. Let $L \in H^2(S, \mathbb{Q})$.
The rank ρ Donaldson invariants of S with respect to H, c_1 are

$$D^{S,H}_{\rho,c_1,c_2}(L + u\,\mathrm{pt}) = \int_{[M^H_S(\rho,c_1,c_2)]^{\mathrm{vir}}} \exp(\mu(L) + \mu(\mathrm{pt})u).$$

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$D^{S,H}_{\rho,c_1,c_2}(L + u\,\mathrm{pt})$ is the coefficient of z^{vd} of

$$\rho^{2-\chi(\mathcal{O}_S)} B_{\emptyset}^{K_S^2} e^{(\frac{1}{2}L^2+\rho u)z^2} \sum_{(a_1,\ldots,a_{\rho-1})} \prod_{j=1}^{\rho-1} \varepsilon_\rho^{j\cdot(a_j,c_1)} \widetilde{SW}(\widetilde{a}_j) e^{-\sin(\pi\frac{j}{\rho})(\widetilde{a}_j L)z} \prod_{1\leq i<j\leq\rho-1} \beta_{ij}^{\frac{1}{2}\widetilde{a}_i(\widetilde{a}_j-\widetilde{a}_i)}$$

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Here $\widetilde{a} := 2a - K_S$ for $a \in H^2(S, \mathbb{Z})$, and $\widetilde{SW}(\widetilde{a}) := SW(a)$ are the Seiberg Witten invariants in differential/algebraic geometry

In joint work with Kool, we had observed the following:

Conjecture (GK)

For all $\rho > 0$, $r \in \mathbb{Z}$, $A_{l,r}$ starts with 1, and the coefficient of $v^{\frac{n}{2}}$ of $A_{l,r}$, $B_{l,r}$ is a polynomial of degree $\leq n$ in r

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Satisfy alg. equations of degree $2^{\rho-1}$ (recall $[\rho-1] = \{1, \ldots, \rho-1\}$)

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$$\prod_{J\subset[\rho-1]} (y - A_{J,r}) \in \mathbb{Q}(v)[y], \qquad \prod_{J\subset[\rho-1]} (y - B_{J,r}) \in \mathbb{Q}(v)[y].$$

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We determined these polynomials for $|r| \leq \rho \leq 6$
this gives conjectural Segre and Verlinde formula for $|r| \leq \rho \leq 6$

These relations are complicated and difficult to generalize
e.g. rank 4, $r = 1$

$$\prod_{J\subset[3]}(y - A^{(4)}_{J,1}) = y^8 - \frac{8+8v+v^2}{(1+v)^2}y^7 + \frac{28+20v}{(1+v)^3}y^6 - \frac{56+120v+71v^2+8v^3}{(1+v)^6}y^5 + \frac{70+160v+104v^2+16v^3+v^4}{(1+v)^4}y^4 - \frac{56+120v+71v^2+8v^3}{(1+v)^{10}}y^3 + \frac{28+20v}{(1+v)^{11}}y^2 - \frac{8+8v+v^2}{(1+v)^{14}}y + \frac{1}{(1+v)^{16}}$$

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This polynomial is essentially palindromic

Recall that

$$A_{J,r} := A_r \prod_{j\in J} A_{j,r}, \quad B_{J,r} := B_r \prod_{i\leq j\in J} B_{ij,r}$$

Proceed as follows:

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(and similar for B; work in progress, only discuss the A case)

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Step 1: the equation for the $A_{J,r}$ is essentially palindromic.

Conjecture

$$\overline{p}_{\rho,r}(y,v) := \prod_{J\subset[\rho-1]} \left(y - (1+v)^{\frac{\rho+r-1}{2}} A_{J,r}\right)$$

satisfies $y^{2^{\rho-1}}\overline{p}_{\rho,r}(\frac{1}{y},v) = \overline{p}_{\rho,r}(y,v)$; *equiv.* $A_{J,r} = \dfrac{\prod_{i\in J} A_{i,r}^{\frac{1}{2}}}{(1+v)^{\frac{\rho+r-1}{2}} \prod_{j\notin J} A_{j,r}^{\frac{1}{2}}}$

Simpler relations for factors

Step 1: $A_{I,r} = \dfrac{\prod_{i \in I} A_{i,r}^{\frac{1}{2}}}{(1+v)^{\frac{\rho+r-1}{2}} \prod_{j \notin I} A_{j,r}^{\frac{1}{2}}}$

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Let $i \in [\rho - 1]$, $r < \rho$, let ξ primitive 4ρ-th root of unity
$(A_{i,r})^{\frac{1}{2\rho}}$ satisfies the equation $\frac{y^{\rho} - y^{-\rho}}{\xi^{2i-\rho} y^r + \xi^{\rho-2i} y^r} = v^{\frac{1}{2}}$

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$$(A_{i,r})^{\frac{1}{2\rho}} = \exp\left(\left(\frac{exp(\rho v^{\frac{1}{2}}) - exp(-\rho v^{\frac{1}{2}})}{\xi^{2i-\rho}\exp(rv^{\frac{1}{2}}) + \xi^{\rho-2i}\exp(-rv^{\frac{1}{2}})}\right)^{-1}\right)$$

(compositional inverse)

Result: Assume $0 \leq |r| \leq \rho$. Get Conjectural Verlinde formula for $\chi^{\mathrm{vir}}(M_S^H(\rho, c_1, c_2), \mu(L) \otimes E^{\otimes r})$ for $p_g(S) > 0$ $q(S) = 0$ and $K_S^2 = 0$ (resp. formula for $\int_{[M]^{\mathrm{vir}}} c(\alpha_M)$ for $0 \leq s = \mathrm{rk}(\alpha) \leq 2\rho$)
Working on the case $K_S^2 \neq 0$.

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For $\alpha \in K_c := \{v \in K(S) \,:\, \chi(S, c \otimes v) = 0\}$

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Note that c has rank ρ and α has rank r, then $\lambda(\alpha) = \mu(L) \otimes E^{\otimes r}$ and $\lambda(c) = \mu(L') \otimes E^{\otimes \rho}$, so strange duality exchanges r and ρ.

Use virtual strange duality and the algebraic equations for $A_{l,r}^{(\rho)}$, $B_{l,r}^{(\rho)}$ to order by order determine the coefficients and thus also algebraic equations for the $A_{l,\rho}^{(r)}$, $B_{l,\rho}^{(r)}$

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Conjecture

Assume $\rho > r > 0$.

1. *Write $p_{\rho,r}(v,y) = \prod_{I\subset[\rho-1]}(y - A^{(\rho)}_{I,r}) \in \mathbb{Q}(v)[y]$ Then for $J \subset [r-1]$, $(A^{(r)}_{J,\rho})^{\frac{\rho}{r}}$, $J \subset [r-1]$ is a zero of $p_{\rho,r}(\frac{1}{v}, y)$*
2. *Write $q_{\rho,r}(v,y) = \prod_{I\subset[\rho-1]}(y - B^{(\rho)}_{I,r}) \in \mathbb{Q}(v)[y]$ Then for $J \subset [r-1]$, $B^{(r)}_{J,\rho}$, is a zero of $q_{\rho,r}(\frac{1}{v}, y)$.*

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   Then for $J \subset [r-1]$, $B_{J,\rho}^{(r)}$, is a zero of $q_{\rho,r}(\frac{1}{v}, y)$.

Thus if we know equations for the $A_{I,r}^{(\rho)}$, $B_{I,r}^{(\rho)}$ for $\rho > r > 0$, we also know them for $r > \rho$.

In case $\rho > r$ have $p_{\rho,r}(v,y) = \prod_{I\subset[\rho-1]}(y - A_{I,r}^{(\rho)}) \in \mathbb{Q}(v)[y]$
Now assume $r > \rho$
The $(A_{I,r}^{(\rho)})^{\frac{r}{\rho}}$ with $I \subset [\rho-1]$ are $2^{\rho-1}$ of the 2^{r-1} zeros of $p_{\rho,r}(\frac{1}{v}, y)$

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Carry out steps 1 and 2 above: Step 1:

Conjecture

$$A_{J,r} = \frac{\prod_{i\in J} A_{i,r}^{\frac{1}{2}}}{v^{\frac{\rho}{2}}(1+v)^{\frac{\rho+r-1}{2}} \prod_{j\in[r-1]\setminus J} A_{j,r}^{\frac{1}{2}}}, \quad J \subset [r-1]$$

Step 2 (in progress): We find that $A_{i,r}$ satisfy equations similar to case $|r| < \rho$ (worked out until now for $gcd(r, \rho) = 1$, or $\rho | r$)
Step 3 (in progress): Extend this to the $B_{I,r}$: expect to find complete Verlinde and Segre conjecture for surfaces with $p_g > 0$, $q = 0$

Finally for Hilbert schemes of points we can get (and partially prove) the complete Verlinde (and Segre) formula

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$$g_r(w) = 1 + v, \quad f_r(w) = \frac{(1+v)^{r^2}}{1 + r^2 v}, \quad w = v(1+v)^{r^2-1}$$

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equivalently the $A_{i,r}(w)^{\frac{1}{2}}$ are the solutions of $\frac{y^{-1}+(-1)^r y}{y^{-r}-y^r} = v^{\frac{1}{2}}$, and $A_r(w) = \frac{1}{v^{\frac{1}{2}}(1+v)^{\frac{r}{2}} \prod_{i\in[r-1]} A_{i,r}^{\frac{1}{2}}}$

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Method of proof: by cobordism invariance can assume S is toric, use localization. This expresses answer in terms of combinatorics of partitions. Use results and methods of Anton Mellit on symmetric functions to study generating functions (would be talk by itself)

We can conjecturally also determine the *B*-series.

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Conjecture

$$B_r(w)^8 = \left(\frac{\prod_{i=1}^{r-1} A_{i,r}}{v}\right)^{4r+2} (1+v)^{r^2+2r}(1+r^2 v)^3 \cdot \prod_{i,j=1}^{r-1}(1 - A_{i,r}A_{j,r})^2 \prod_{\substack{i,j=1\\ i\neq j}}^{r-1}(1 - A_{i,r}^r A_{j,r}^r)^2$$

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Expect that for all ρ, r, the $B_{l,r}^{(\rho)}$ can be expressed in terms of the $A_{i,r}^{(\rho)}$

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c_{2n}(\alpha^{[n]}) = V_s^{c_2(\alpha)}\, W_s^{c_1(\alpha)^2}\, X_s^{\chi(\mathcal{O}_S)}\, Y_s^{c_1(\alpha)K_S}\, Z_s^{K_S^2}.$$

with V_s, W_s, X_s known.

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Conjecture (Johnson, Marian-Oprea-Pandharipande)

For any $r \in \mathbb{Z}$, we have

$$A_r(w) = W_s(z)\, Y_s(z), \quad B_r(w) = Z_s(z),$$

where $s = 1 + r$ and $w = v(1+v)^{r^2-1}$, $z = t(1+(1-s)t)^{1-s}$, and $v = t(1-rt)^{-1}$.

Theorem

This conjecture is true.

The methods for now are not strong enough to prove formula for B and Z series, but enough to show they become the same after change of variables