Fixed point schemes as spectrum of equivariant cohomology and Kirillov algebras

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§ 1 Multiplicity algebras

Let $f : \overset{\mathbb{C}^\times \text{ with + weights}}{U} \cong \mathbb{C}^N \to \overset{\mathbb{C}^\times}{V} \cong \mathbb{C}^N$ homogeneous

(more generally $\mathbb{C}^\times \curvearrowright U, V$ affine, semiprojective, V smooth, U Cohen-Macaulay)
- $U^{\mathbb{C}^\times} = \{0_U\}$
- attracting

Def^n multiplicity algebra of f

$$Q_f := \mathbb{C}[f^{-1}(0_V)] = \mathbb{C}[U] / (f^{-1}(m_{0_V})) = \mathbb{C}[x_1, \dots, x_N] / (f_1, \dots, f_N)$$

- notion due to (Arnold et al 1982)

Thm/Def^n $\dim(Q_f) < \infty \Leftrightarrow f$ is proper $\Leftrightarrow f$ is finite flat ($\Rightarrow$ loc. free)

$\Leftrightarrow f^{-1}(0_V) = \{0\} :\Leftrightarrow$ f very stable

in this very stable case $m_f := \dim(Q_f)$ is multiplicity of $f^{-1}(0_V)$ = degree of f

- f $\mathbb{C}^\times$-equivariant $\Rightarrow$ $\mathbb{C}^\times \circlearrowright Q_f$ $\Rightarrow$ $Q_f = \bigoplus_{k=0}^{m} Q_f^k$ graded

<u>Thm</u> Q_f complete intersection $\Rightarrow$
- $Q_f^m = \mathbb{C}\mathfrak{J}_f$ 1-dim^l
- $Q_f^i \times Q_f^{m-i} \to Q_f^m$ non-degenerate

<u>Poincaré duality ring</u>

<u>Examples</u> of very stable maps:
- §2. spectrum of equivariant cohomology
- §3. fixed point scheme of regular actions
- §4 spectrum of Kirillov algebras
- §5 Hitchin map on very stable upward flows
- §6 spectrum of universal bundle of algebras

§ 2. Equivariant cohomology

- G cpx reductive, universal G-bundle $\begin{array}{c} G \circlearrowright EG \sim * \\ \downarrow \\ BG \end{array}$ defined up to $\sim$

<u>Thm</u> $H_G^* := H^*(BG;\mathbb{C}) \cong H^*(BT;\mathbb{C})^W \cong \mathbb{C}[\mathfrak{t}]^W \rightsquigarrow \operatorname{Spec} H_G^* = \mathfrak{t}/\!/W$

grading from $\mathbb{C}^\times \circlearrowright \mathfrak{t}$ weight 2 $\qquad \cong$ affine space by Chevalley

$T \subset G$ maximal torus $\quad W$ = Weyl group

- $G \circlearrowright X$ algebraic variety

<u>Borel construction:</u> $X_G := X \times EG/G \xrightarrow{\pi} EG/G = BG$

an X-bundle

<u>Def^n</u> $H_G^*(X;\mathbb{C}) := H^*(X_G;\mathbb{C})$

$\pi^* \uparrow$

H_G^*

graded commutative H_G^*-algebra

- when $\dim H^{odd}(X;\mathbb{C}) = 0 \Rightarrow G \circlearrowright X$ <u>equivariantly formal</u>

$\Leftrightarrow \quad \pi^*: H^*_G \to H^*_G(X)$ is a finite free module $\Leftrightarrow \quad H^*_G(X;\mathbb{C}) \cong H^*(X;\mathbb{C}) \times H^*_G$

$\Leftrightarrow \quad H^*(X;\mathbb{C}) \cong H^*_G(X;\mathbb{C}) \otimes_{H^*_G} \mathbb{C}$

$\Leftrightarrow \quad f: \operatorname{Spec}(H^*_G(X;\mathbb{C})) \to \operatorname{Spec}(H^*_G) \cong \mathfrak{t}/\!/W$ very stable

$\mathbb{C}^\times$ from grading $Q_f \cong H^*(X;\mathbb{C})$ Poincaré duality ring

Example $T \subset B \subset P \subset G$; e.g.

$$P_k = \begin{array}{c|cc|c} & k & n-k & \\ \hline k & * & * & k \\ n-k & 0 & * & n-k \\ \hline & k & n-k & \end{array} \subset GL_n$$

max torus, Borel, parabolic

$G \circlearrowright G/P$ partial flag variety e.g. $GL_n/P_k \cong Gr(k,n)$

$$H^*_G(G/P;\mathbb{C}) \cong H^*_{G\times P}(G;\mathbb{C}) \cong H^*_P \cong H^*_L = \mathbb{C}[\mathfrak{t}]^{W_L}$$

$L := P/U \cong u \sim *$ e.g. $L_k = P_k/U_k \cong GL_k \times GL_{n-k}$

Levi factor (reductive)   unipotent radical $W_{L_k} \cong S_k \times S_{n-k}$

$\Rightarrow \quad H^{odd}_G(G/P;\mathbb{C}) = 0 \quad \Rightarrow$ equivariantly formal

$$\begin{tikzcd}
\underline{\text{thus}}\ f: \mathrm{Spec}(H^*_G(G/P;\mathbb{C})) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Spec}\, H^*_G \arrow[d, phantom, "\cong" rotate=90] & \text{e.g. } \mathrm{Spec}(H^*_{GL_n}(Gr(k,n);\mathbb{C})) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Spec}\, H^*_{GL_n} \arrow[d, phantom, "\cong" rotate=90] \\
\mathfrak{t}/\!/W_L \arrow[r] & \mathfrak{t}/\!/W & \mathfrak{t}/\!/S_k\times S_{n-k} \arrow[r] & \mathfrak{t}/\!/S_n
\end{tikzcd}$$

very stable & $Q_J \cong H^*(G/P;\mathbb{C}) \overset{\text{e.g}}{\cong} H^*(Gr(k,n);\mathbb{C}) \cong$

$$\mathbb{C}[e_1,\dots,e_k,f_1,\dots,f_{n-k}] \leftarrow \mathbb{C}[a_1,\dots,a_n]$$

$$e(t)\cdot f(t) \leftarrow\!\shortmid\ a(t)=t^n+a_1t^{n-1}+\dots+a_n$$

§3 Fixed point schemes

- Observation:

((Yun) for $P=B$)

partial Grothendieck-Springer resolution $\to$ $\tilde{\mathfrak{g}}_P = \{(x,P)\in \mathfrak{g}\times G/P \mid x\in \mathfrak{p}\}$

$$\begin{tikzcd}
\tilde{S} \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "\cong"] & \tilde{\mathfrak{g}}_P \arrow[r] \arrow[d] & \mathfrak{l}/\!/L \cong \mathfrak{t}/\!/W_L \arrow[d] \arrow[r, phantom, "\cong"] & \mathrm{Spec}(H^*_G(G/P;\mathbb{C})) \arrow[d, "f"] \\
S \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & \mathfrak{g} \arrow[r, "\chi"] & \mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W \arrow[r, phantom, "\cong"] & \mathrm{Spec}(H^*_G)
\end{tikzcd}$$

$\|$

Kostant section: $e+\mathfrak{g}^f \subset \mathfrak{g}^{reg}$

- $\mathfrak{g}, \mathfrak{p}, \mathfrak{t}, \mathfrak{l} := \mathrm{Lie}(G, P, T, L)$
- $\mathfrak{g}^{reg} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \dim \mathfrak{t}\} \subset \mathfrak{g}$ $\overline{\text{codim } 3.}$
- $(e, f, h) \subset \mathfrak{g}^{reg} \subset \mathfrak{g}$ principal $\mathfrak{sl}_2$-triple
- $\mathcal{S} := e + \mathfrak{g}^f \subseteq \mathfrak{g}^{reg}$ <u>slice</u> $\mathcal{S} \cong \mathfrak{g}^{reg}/G$ Kostant <u>section</u> $\chi : \mathcal{S} \xrightarrow{\sim} \mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W$
- $G \circlearrowright X$ smooth proj. <u>regular</u> s.t. $g \in G^{reg}$ $|X^g| < \infty$
    - enough to check for principal unipotent $u = \exp(e) \in G$ $|X^u| < \infty \Leftrightarrow |X^u| = 1$
    - <u>examples</u>: partial flag varieties, some wonderful compactifications, spherical varieties
    - $$\begin{array}{c} V \\ \downarrow \\ \mathcal{S} \times X \subset \mathfrak{g} \times X \end{array}$$ vector field by infinitesimal action, $\mathcal{Z} := Z(V) \subset \mathcal{S} \times X$
      $\dfrac{\text{fixed point scheme}}{\text{zero scheme of } V}$
- example $\mathcal{Z} = \tilde{\mathcal{S}} \subset \mathcal{S} \times G/P$ for $G \circlearrowright X = G/P$

Thm (Hausel-Rychlewicz 2022)

$G \circlearrowright X$ smooth projective

G semisimple, $\circlearrowright$ regular

- $\pi : \mathcal{Z} \to \mathfrak{s} \cong \mathfrak{t}/\!/W$ finite flat
- $\mathbb{C}^\times \circlearrowright \mathcal{Z}$, $\mathbb{C}^\times \circlearrowright \mathfrak{s}$ equivariant
- $\mathcal{Z}$ reduced CM $\Rightarrow \pi$ very stable
- $\mathbb{C}[\mathcal{Z}] \cong H^*_G(X;\mathbb{C})$

Note proof first for Borel $B \subset G$, generalising (Brion-Carrell 2004) for $B = B(SL_2)$

§4 Kirillov algebras

- G semisimple cpx, $\lambda \in X^+$, $G \circlearrowright V^\lambda$ λ-highest weight irred. rep^n
- $\mathcal{Z}_\lambda(\mathfrak{g}) := \left(S(\mathfrak{g}) \otimes \operatorname{End} V^\lambda \right)^G$ classical Kirillov algebra

$$= \left(S(\mathfrak{g}^*) \otimes \operatorname{End} V^\lambda \right)^G = \operatorname{Maps}\left(\mathfrak{g} \to \operatorname{End} V^\lambda\right)^G$$

($\mathfrak{g} \cong \mathfrak{g}^*$ by Killing)

graded associative algebra / $S(\mathfrak{g}^*)^G \cong \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W = H^*_G$

from $S(\mathfrak{g}^*) = \bigoplus S^k(\mathfrak{g}^*)$ (we double degrees)

Thm (Kirillov 2000) $Z_\lambda(\mathfrak{g})$ commutative $\Leftrightarrow$ V^λ weight multiplicity free
$\dim|V^\lambda_\mu| = 1 \quad \mu \leq \lambda$

- e.g. $\lambda \in X^+$ minuscule, for SL_n $\omega_1, \dots, \omega_{n-1}$ fundamental rep^n

- e.g. $SL_n \circlearrowright \mathbb{C}^n = V^{\omega_1}$ standard representation

$M : \mathfrak{g} \to \operatorname{End} V^\lambda$ given by repr^n

$$Z_{\omega_1}(\mathfrak{sl}_n) \cong \mathbb{C}[M] \overset{?}{\longleftarrow} H^*_G$$

Thm (Panyushev 2004) - for $\lambda \in X^+$ weight multiplicity free

- $Z_\lambda(\mathfrak{g}) \leftarrow H^*_G$ finite flat
- reduced Cohen-Macaulay ($\Rightarrow \operatorname{Spec}(Z_\lambda(\mathfrak{g})) \to \mathfrak{t}/\!\!/W$)
very stable

- $\lambda \in X^+$ minuscule minuscule flag variety

$$Z_\lambda(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{t}]^{W_\lambda} \cong H^*_G(G/P_\lambda ; \mathbb{C}) \quad \text{as } H^*_G\text{-algebras}$$

- example $G = SL_n$, $\omega_k = \Lambda^k \mathbb{C}^n \in X^+(SL_n)$, $G/P_{\omega_k} \cong Gr(k,n)$

construct SL_n-equivariant maps $\mathfrak{sl}_n \to End(\Lambda^k \mathbb{C}^n)$

- $A \in \mathfrak{sl}_n \rightsquigarrow \Lambda^k(t \cdot I - A) : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$

$$\Lambda^k(t \cdot I - A) = t^k + E_1 t^{k-1} + \dots + E_k \qquad \text{e.g. } E_1 = Lie(\rho_{\omega_k})(A) = M(A)$$

and $(\Lambda^{n-k}(t \cdot I - A))^* : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$

$$(\Lambda^{n-k}(t \cdot I - A))^* = t^{n-k} + F_1 t^{n-k-1} + \dots + F_{n-k}$$

$$M_i : \mathfrak{sl}_n \to End(\Lambda^k(\mathbb{C}^n)) \qquad N_i : \mathfrak{sl}_n \to End(\Lambda^k(\mathbb{C}^n))$$
$$A \mapsto E_i \qquad\qquad A \mapsto F_i$$

they are commuting G-equivariant maps:

$$(t^k + E_1 t^{k-1} + \dots + E_k)(t^{n-k} + F_1 t^{n-k-1} + \dots + F_{n-k}) = \det(t \cdot I - A)\, Id_{\Lambda^k \mathbb{C}^n}$$

<u>Thm</u>

$$Z_{\omega_k}(\mathfrak{sl}_n) \cong \langle M_1, \dots, M_k, N_1, \dots, N_{n-k} \rangle \cong H^*_{SL_n}(Gr(k,n); \mathbb{C})$$

§5 Hitchin map on very stable upward flows

- C smooth proj. curve ; $G := PGL_n$; $G^\vee = SL_n$
- $\mathcal{M}_G$ – moduli space of semistable rank n degree d Higgs bundles

  (E, Φ) $\Phi \in H^0(C; \operatorname{End} E_0 \otimes K)$ modulo line bundles $L \in \operatorname{Pic}(C)$

  rank n Higgs field trace free $(LE, \Phi) \sim (E, \Phi)$

- alg. symplectic $\omega \in \Omega^2(\mathcal{M}_G)$ (even hyperkähler)

- $h : \mathcal{M}_G \to \mathcal{A} := \bigoplus_{i=2}^{n} H^0(C; K^i) \circlearrowleft \mathbb{C}^\times$

  $(E, \Phi) \mapsto \det(x - \Phi)$ char. polynomial

  Hitchin map : proper, completely integrable Hamiltonian system

- $\mathbb{C}^\times \circlearrowright \mathcal{M}_G$ $(E, \Phi) \mapsto (E, \lambda\Phi)$

  semiprojective :

- $\mathbb{C}^\times \curvearrowright \mathcal{M}_G$ linear
- $\mathcal{M}_G^{\mathbb{C}^\times}$ proper $\Longrightarrow$ $\mathcal{M}_G = \coprod_{\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times}} W_\mathcal{E}^+$ BB partition
- $\lim_{\lambda \to 0} \lambda \mathcal{F}$ exists for all $\mathcal{F} \in \mathcal{M}_G$ — $W_\mathcal{E}^+ := \{ \mathcal{F} \in \mathcal{M}_G \mid \lim_{\lambda \mapsto 0} \lambda \cdot \mathcal{F} = \mathcal{E} \} \subset \mathcal{M}_G$

<u>upward flow</u>

<u>Thm</u> $\mathcal{E} \in \mathcal{M}^{s\mathbb{C}^\times}$ $\mathcal{M}_G \supset W_\mathcal{E}^+ \cong_{\mathbb{C}^\times} T_\mathcal{E}^+$ & Lagrangian

loc. closed

<u>Def^n</u> $\mathcal{E} \in \mathcal{M}^{s\mathbb{C}^\times}$, $W_\mathcal{E}^+$ very stable $\Leftrightarrow$ $h_\mathcal{E} := h|_{W_\mathcal{E}^+} : W_\mathcal{E}^+ \longrightarrow \mathcal{A}$ very stable

$\Updownarrow$

notion of Drinfeld, Laumon 1988 for $\mathcal{E} = (E, 0)$ $\Leftrightarrow$ $W_\mathcal{E}^+ \cap h^{-1}(0) = \{\mathcal{E}\}$ $\Leftrightarrow$ finite flat

<u>Examples of</u> very stable Higgs bundles

<u>Example 1</u>

$$\Phi_0 = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$\mathcal{E}_0 = (E_0, \Phi_0)$ $\quad E_0 := \mathcal{O} \oplus K^{-1} \oplus \dots \oplus K^{-n+1} \to K \oplus \mathcal{O} \oplus \dots \oplus K^{-n+2} = E_0 \otimes K$

<u>canonical uniformising Higgs bundle</u>

$$W_0^+ := W_{\mathcal{E}_0}^+ = \left\{ \underset{a \in \mathcal{A}}{\mathcal{E}_a = (E_0, \Phi_a)} \;\middle|\; \Phi_a = \begin{pmatrix} 0 & & & -a_n \\ 1 & 0 & & \vdots \\ & \ddots & & -a_2 \\ & & 1 & 0 \end{pmatrix} \right\}$$ <u>Hitchin section</u>

companion matrix of a

$h(\mathcal{E}_a) = a$

$h_0 : W_0^+ \xrightarrow{\cong} \mathcal{A}$ is a section of Hitchin map

$\Rightarrow$ very stable

<u>Example 2</u>

- $c \in C$
- $k = 1, 2, \dots$ or $n-1$,
- $E_k := \mathcal{O} \oplus K^{-1} \oplus \dots \oplus K^{-k+1} \oplus K^{-k}(c) \oplus \dots \longrightarrow E_k \otimes K$

$$\Phi_k = \begin{array}{c} \scriptstyle k \\ \begin{pmatrix} 0 & & & & & 0 \\ 1 & \ddots & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & s_c & & \\ & & & 1 & \ddots & \\ & & & & \ddots & 0 \\ 0 & & & & 1 & 0 \end{pmatrix} \end{array}$$

$s_c \in H^0(C, \mathcal{O}(c)) \cong$
$\cong H^0(C, K^{k-1} K^{-k}(c) K)$

(Hausel – Hitchin 2021) $\Rightarrow$ $\mathcal{E}_k := (E_k, \Phi_k) \in \mathcal{M}^{s\mathbb{C}^\times}$ very stable

i.e. $h_k : W_k^+ := W_{\mathcal{E}_k}^+ \longrightarrow \mathcal{A}$ very stable

$\mathcal{H}_k(W_0^+) \cong W_k^+$ by k-th fundamental Hecke transformation

$\Downarrow$

$$\begin{tikzcd}
\mathcal{A}_0 \subset W_k^+ \arrow[r, two heads] \arrow[d, "h_k"'] & \mathrm{Spec}(H^*_G(Gr(k,n),\mathbb{C})) \arrow[d, "f"] \\
\mathcal{A}_0 \subset \mathcal{A} \arrow[r, two heads] & \mathrm{Spec}(H^*_G)
\end{tikzcd}$$

" Hitchin map $h_k : W_k^+ \to \mathcal{A}$ is modelled by f, spectrum of $H^*_G(Gr(k,n);\mathbb{C})$ "

e.g. $Q_{h_k} \cong H^*(Gr(k,n),\mathbb{C})$

§6 Universal bundle of algebras

- $G = PGL_n$, $G^\vee = SL_n$
- $$\begin{tikzcd}[column sep=small]
\mathcal{M}_G \arrow[dr, "h_G"'] & & \mathcal{M}_{G^\vee} \arrow[dl, "h_{G^\vee}"] \\
& \mathcal{A} &
\end{tikzcd}$$ SYZ mirror $\overset{\text{classical limit}}{\rightsquigarrow}$ " $\mathfrak{S} : D(\mathcal{M}_G) \sim D(\mathcal{M}_{G^\vee})$ "

generically by relative Fourier-Mukai

Conjecture (Hausel – Hitchin 2021) $\quad c \in C$

$$\mathfrak{S}(\mathcal{O}_{W_k^+}) = \Lambda^k \mathbb{E}^\vee_c$$

universal SL_n-bundle $\quad \begin{array}{c}\mathbb{E}^\vee \\ \downarrow \\ \mathcal{M}_{G^\vee} \times C\end{array}$

in k-th fundamental representation

<u>check on conjecture</u>

relative FM should induce $h_{G*}(\mathcal{O}_{W_k^+}) \cong \Lambda^k \mathbb{E}_C^\vee|_{{}^\vee W_0^+}$

thus a fundamental universal bundle of algebras on

<u>indeed</u> $(\mathbb{E}^\vee, \Phi)|_{a \in \mathcal{A} \cong {}^\vee W_0^+} = (E_0^\vee, \Phi_a)$

$$\Phi_a = \begin{pmatrix} 0 & & & & -a_n \\ 1 & \ddots & 0 & & \vdots \\ & & 1 & 0 & 0 & -a_2 \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} : K^{-1} \to End(E_0^\vee)$$

$$E_0^\vee = K^{-\frac{n-1}{2}} \oplus \dots \oplus K^{\frac{n-1}{2}} \longrightarrow K(K^{-\frac{n-1}{2}} \oplus \dots \oplus K^{\frac{n-1}{2}})$$

apply $M_i(\Phi_a) : K^{-i} \to \mathrm{End}(\Lambda^k \mathbb{E})$ from Kirillov's $Z_{\omega_k}(\mathfrak{sl}_n)$

Combine to a bundle of algebra structure $\Lambda^k \mathbb{E}^\vee|_{\nu_{W_0^+}} \otimes \Lambda^k \mathbb{E}^\vee|_{\nu_{W_0^+}} \to \Lambda^k \mathbb{E}^\vee|_{\nu_{W_0^+}}$

<u>Thm</u>

$$\begin{tikzcd}
\mathrm{Spec}_{\mathcal{A}}(\Lambda^k \mathbb{E}_c^\vee) \arrow[r, phantom, "\cong"] \arrow[d, "\pi"'] & W_k^+ \arrow[r, two heads] \arrow[d, "h_k"'] & \mathrm{Spec}(H_G^*(Gr(k,n);\mathbb{C})) \arrow[d, "f"] \\
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{A} \arrow[r, two heads] & \mathrm{Spec}(H_G^*)
\end{tikzcd}$$

"spectrum of Kirillov algebras" = "fixed point scheme" = "spectrum of equivariant cohomology"