# Donaldson-Thomas Theory of the Quantum Fermat Quintic

Intercontinental Moduli Zoominar

Feb 14, 2022

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arXiv: 2004.10346

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arXiv: 1409.4101

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## 1. The quantum Fermat Quintic

Quantum projective 4-space: non-commutative graded algebra

$$\mathbb{P}^4_q : \mathbb{C}\langle t_0, .., t_4\rangle / t_i t_j = q^{n_{ij}} t_j t_i \qquad q \in \mathbb{C} \text{ fixed } \sqrt[5]{1}$$

$N = (n_{ij}) \in M_{5\times 5}(\mathbb{F}_5)$ skew-symmetric matrix

$$N = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix} \qquad \text{(to fix formulas)}$$

(this is generic!)

$t_i^5$ are central elements: obtain the Quantum Fermat Quintic

$$Q_q : \quad \mathbb{C}\langle t_0, .., t_4\rangle_q \,/\, t_0^5 + .. + t_4^5$$

(graded algebra). $\qquad Q_q \hookrightarrow \mathbb{P}^4_q$

## 2. Non-commutative projective schemes

$Q_q$ is a non-commutative projective scheme (Artin-Zhang)

(graded $\mathbb{C}$-algebras $S$) $\longleftrightarrow$ (triples $(\mathcal{C}, \mathcal{O}, (1))$)

$\mathcal{C}$: abelian category

$\mathcal{O} \in \mathcal{C}$: object

$(1): \mathcal{C} \to \mathcal{C}$ auto-equivalence
$F \longmapsto F(1)$

$$S \longmapsto \operatorname{Proj} S = (\operatorname{qgr}(S), S, \text{shift})$$

$\operatorname{qgr}(S)$: category of tails of f.g. graded $S$-modules

$$\bigoplus_n \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(n)) \longleftarrow\!\!\shortmid (\mathcal{C}, \mathcal{O}, (1))$$

$$a \cdot b = a(\deg b) \circ b$$

With enough conditions on $S$ and triples this gives an equivalence of categories (On G-algebra side up to finite modules)

**Theorem (Kanazawa)**

For the quantum Fermat quintic (any $N = (n_{ij})$)

$\mathrm{qgr}(Q_q)$ (i) has global dimension 3

(ii) is a Calabi-Yau 3 category iff $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{F}^5$ is an eigenvector of $N$.

(i): $\mathrm{Ext}^i(E, F) = 0 \quad \forall i > 3$

(ii): $\mathrm{Ext}^i(E, F)^\vee = \mathrm{Ext}^{3-i}(F, E)$

(i) $Q_q$ is smooth of dimension 3

(ii) $Q_q$ is a Calabi-Yau 3-fold

So moduli spaces of objects in $\mathrm{qgr}(Q)$ should admit a Donaldson-Thomas theory. We were not able to construct it using techniques from non-commutative projective geometry.

## 3. The quintic mirror.

Recall: The mirror family of the 101-dimensional family of smooth quintics is the 1-dimensional family of singular CY3s obtaind as follows:

Start with $Q_\psi : \mathbb{C}[t_0, \dots, t_4] / t_0^5 + \dots + t_4^5 - 5\psi t_0 \dots t_4$ ($t_i, t_j$ commute!).

A 1-parameter family of quintics in $\mathbb{P}^4$,

Act with the group $G = \frac{\ker \Delta^*}{\operatorname{im} \Delta}$, $\mathbb{F}_5 \xrightarrow{\Delta} \mathbb{F}_5^5 \xrightarrow{\Delta^*} \mathbb{F}_5$, $\# G = 125$.

by $\vec{h} \cdot t_i = q^{h_i} t_i$.

and pass to the quotient: a 1-parameter family of singular CY3s, $W_\psi$. $\quad Q_\psi \longrightarrow W_\psi$ degree 125

In fact $W_\psi : \mathbb{C}[x_0, \dots, x_4, y] / \sum x_i - 5\psi y, \; y^5 - \prod x_i$

where $x_i = t_i^5$, $y = t_0 \dots t_4$.

The singular locus of $W_\psi$ is always the same:

① $C_{012} = \{ x_0 + x_1 + x_2 = 0, \; x_3 = 0, \; x_4 = 0 \}$

$C_{ijk}$ : similar

$\binom{5}{3} = 10$ lines in $W_\psi$ where $W_\psi$ has a transverse $A_4$ singularity

② $P_{01} = \{ x_0 + x_1 = 0, \; x_2 = x_3 = x_4 = 0 \}$

$P_{ij}$ similar

$\binom{5}{2} = 10$ points where 3 lines meet.

more complicated singularity

But $W_\psi$ can be resolved to a Calabi-Yau threefold.

Kanazawa: back to non-commutative case: $\prod t_i$ is also central, and

$Q_{q,\psi} : \quad \mathbb{C}\langle t_0, \dots, t_4 \rangle_q / \sum t_i^5 - 5\psi \prod t_i$ is still

a smooth non-commutative Calabi-Yau threefold.

<u>Note:</u> $\mathcal{W}_\psi$ embeds also into $Q_{q,\psi}$, in fact into the centre $Z(Q_{q,\psi})$ because $x_i = t_i^5$, $y_1 = t_0 \dots t_4$ are central in $Q_{q,\psi}$

<u>Conjecture:</u> In fact, $\mathcal{W}_\psi = Z(Q_{q,\psi})$.

This would imply that $Q_{q,\psi}$ is a non-commutative crepant resolution of the quintic mirror $\mathcal{W}_\psi$.

- it is smooth Calabi-Yau threefold
- it's centre is $\mathcal{W}_\psi$.

<u>Note:</u> In the literature "non-commutative crepant resolution" $Z \subset A$ is usually
- local (affine, ungraded case)
- requires $A = \mathrm{End}_Z(M)$ for a reflexive $Z$-module $M$.

(van den Bergh)

$\mathcal{W}_\psi \subset Q_{q,\psi}$ is
- global
- not of this form, as 125 not a square.

We have not seriously studied $Q_{q,\psi}$ for general $\psi$.
But any enumerative theory should be deformation invariant, so we should be justified in considering only the case $\psi \to 0$.

<u>Question:</u> Are we, in fact, dealing with the quintic mirror, rather than a quantum version of the quintic?

## 4. Sheaves of Frobenius algebras

$Q_q$ has a central (commutative) subalgebra over which it is finite:

$\mathbb{C}[t_0^5,\dots,t_4^5]/t_0^5+\dots+t_4^5 \hookrightarrow \mathbb{C}\langle t_0,\dots,t_4\rangle_q / t_0^5+\dots+t_4^5$

$= \mathbb{C}[x_0,\dots,x_4]/x_0+\dots+x_4$ $\qquad Q_q^{(5)}$: loc. free sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, rank $=625$

hyperplane $\mathbb{P}^3 \cong X \hookrightarrow \mathbb{P}^4$

The 5-Veronese subalgebra of $\mathbb{C}\langle t_0,\dots,t_4\rangle_q$ is a graded free module over $\mathbb{C}[t_0^5,\dots,t_4^5]$ on the basis $t^{\vec{k}}$, where $\sum k_i = 5$, $0 \le k_i \le 4$.

$$\rightsquigarrow \quad \mathcal{A} \cong \mathcal{O}_X + \mathcal{O}_X(-1)^{121} + \mathcal{O}_X(-2)^{381} + \mathcal{O}_X(-3)^{121} + \mathcal{O}_X(-4)$$

as $\mathcal{O}_X$-module (not as algebra).

Multiplication in $\mathcal{A}$, composed with projection $\mathrm{tr}: \mathcal{A} \to \mathcal{O}_X(-4)$ defines a perfect pairing

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \longrightarrow \mathcal{O}_X(-4) = \omega_X \qquad a \otimes b \mapsto \mathrm{tr}(ab).$$

pairing is symmetric $\iff t^{\vec{k}}\, t^{4-\vec{k}} = t^{4-\vec{k}}\, t^{\vec{k}}$

$\iff \vec{1}$ eigenvector of $N$.

**Definition.** $X$: smooth scheme, $\mathcal{A}$: locally free sheaf of $\mathcal{O}_X$-algebras with symmetric perfect pairing $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \omega_X$ is a <u>sheaf of Frobenius algebras</u> over $X$.

If the sheaf of algebras $\mathcal{A}/\mathcal{O}_X$ has finite global dimension $n = \dim X$ it has a dualizing bimodule $\omega_{\mathcal{A}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$

such that

$$\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{F},\mathcal{G}) = \mathrm{Ext}^{n-i}_{\mathcal{A}}(\mathcal{G}, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{F})^\vee \qquad \forall\ \mathcal{F},\mathcal{G} \in \mathrm{Coh}(\mathcal{A})$$

$\mathrm{Coh}(\mathcal{A})$ : left $\mathcal{A}$-modules which are coherent $\mathcal{O}_X$-modules.

A symmetric pairing $\mathcal{A}\otimes\mathcal{A} \to \omega_X$ identifies

$$\omega_{\mathcal{A}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X) = \mathcal{A} \qquad \text{as } \mathcal{A}\text{-bimodule}$$

so $\mathrm{Coh}(\mathcal{A})$ becomes a Calabi-Yau $n$-category.

Rmk: In our situation $\mathrm{qgr}(Q_q)$ $\quad Q_q = \mathbb{C}\langle t_0,\dots,t_4\rangle_q / t_0^5+\dots+t_4^5$

and $\mathrm{Coh}(\mathcal{A})$ $\quad \mathcal{A}$ : Frobenius algebra $/ X \cong \mathbb{P}^3$

are equivalent. Study $\mathrm{Coh}(\mathcal{A})$ instead.

$\mathrm{Coh}(\mathcal{A})$: coh. $\mathcal{O}_X$-modules with structure of left $\mathcal{A}$-module.

## 5. Moduli spaces for pairs $(X, \mathcal{A})$

$X$: smooth projective scheme $\mathcal{O}_X(1)$.

$\mathcal{A}$: locally free sheaf of Frobenius algebras over $X$

Assume $\mathcal{A}$ of finite global dimension $n = \dim X$.

$\mathcal{F} \in \mathrm{Coh}(\mathcal{A})$: Hilbert polynomial $p(\mathcal{F})(i) = \chi(X, \mathcal{F}(i))$.

**Definition / Theorem** (Simpson)

$\mathcal{F}$ is (semi)-stable if

(i) pure as $\mathcal{O}_X$-module

(ii) $\forall\ 0 < \mathcal{F}' < \mathcal{F}$ $\quad$ $\mathcal{A}$-submodule

$$\frac{p(\mathcal{F}')(i)}{\mathrm{rk}\,\mathcal{F}'} \underset{(\le)}{<} \frac{p(\mathcal{F})(i)}{\mathrm{rk}\,\mathcal{F}} \qquad \forall i \gg 0$$

- pure modules have Harder-Narasimhan filtrations
- semi stable modules have Jordan-Hölder filtrations

$\rightarrow$ S-equivalence for semi-stable modules

- $\mathcal{F}$ stable $\Rightarrow$ $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F}) = \mathbb{C}$.

Let $h$ be a polynomial.

$\mathfrak{M}^{ss,h}(X, \mathcal{A})$ : semi-stable $\mathcal{A}$-modules with Hilbert polynomial $h$
Artin stack of finite type with a good moduli space $M^{ss,h}(X, \mathcal{A})$

$M^{ss,h}(X, \mathcal{A})$ : projective scheme classifying S-equivalence classes (or polystable sheaves)

$\mathfrak{M}^{s,h}(X, \mathcal{A}) \longrightarrow M^{s,h}(X, \mathcal{A})$ is a $\mathbb{C}^*$-gerbe

$M^{s,h}(X, \mathcal{A}) \subset M^{ss,h}(X, \mathcal{A})$ open, classifies isomorphism classes.

<u>Hilbert schemes</u> $\mathrm{Hilb}^h(X, \mathcal{A}) \subset \mathrm{Quot}^h(X, \mathcal{A})$ closed subscheme classifying coherent $\mathcal{A}$-modules with an epimorphism $\mathcal{A} \twoheadrightarrow \mathcal{F}$.

Would like a morphism, as in classical Donaldson-Thomas theory

<u>$\deg h \leq 1$</u>:

$$\begin{aligned} \mathrm{Hilb}^h(X, \mathcal{A}) &\longrightarrow M^{s, p-h}(X, \mathcal{A}) \qquad p = p(\mathcal{A}) \\ \mathcal{A} \twoheadrightarrow \mathcal{F} &\longmapsto \ker(\mathcal{A} \to \mathcal{F}) \end{aligned}$$

$\mathrm{Hilb}^h(X, \mathcal{A})$ easier to handle, $M^{s,p-h}(X, \mathcal{A})$ better deformation theory

(i) if <u>$\mathcal{A} \otimes \mathbb{C}(X)$ is a division ring</u>, all non-zero submodules $0 \subsetneq \mathcal{F}' \subsetneq \ker(\mathcal{A} \to \mathcal{F})$ have same rank as $\mathcal{A}$, so

$p(\mathcal{F}')(i) < p(\ker(\mathcal{A} \to \mathcal{F}))(i) \quad \forall i \gg 0 \quad \Rightarrow \ker(\mathcal{A} \to \mathcal{F})$ stable.

so the morphism exists (commutative analogue: pure rank 1 sheaves automatically stable)

(ii) if $H^1(X, \mathcal{A}) = 0$ the morphism is an open immersion

(i), (ii) $\mathrm{Hilb}^h(X, \mathcal{A})$ is a union of connected components of $M^{s,p-h}(X, \mathcal{A})$.

(commutative analogue: $\mathrm{Hilb}^h(X, \mathcal{O}_X)$ is a moduli space of torsion-free rank 1 sheaves with trivial determinant)

## 6. Donaldson–Thomas theory for pairs $(X, \mathcal{A})$

$X$: smooth projective scheme $\mathcal{O}_X(1)$.

$\mathcal{A}$: locally free sheaf of Frobenius algebras over $X$

Assume $\mathcal{A}$ of finite global dimension $n = \dim X$.

**Theorem** (Liu)

$M^{s,h}(X, \mathcal{A})$ carries a symmetric ($\sim$ perfect of virtual dimension 0) obstruction theory

deformation space $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$

obstruction space $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$, dual to deformation space.

In particular, $M^{s,h}(X, \mathcal{A})$ carries a virtual fundamental class

$$[M^{s,h}(X, \mathcal{A})]^{vir} \in A_0(M^{s,h}(X, \mathcal{A}))$$

Rmk. universal family

$$\begin{tikzcd} \mathcal{F} / X \times \mathcal{M} \arrow[r] \arrow[d, "\pi"'] & X \times M \arrow[d] \\ \mathcal{M} \arrow[r, "\text{gerbe}"] & M \end{tikzcd}$$

obstruction theory is $R\pi_* R\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$

even though $\mathcal{F}$ may not descend the gerbe,

$R\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$ will descend: think of $\mathcal{F}$ as a twisted sheaf,

the two twists cancel out.

**Definition.** Suppose $h$ chosen such that $ss \Rightarrow s$ so that $M^{s,h}(X,\mathcal{A})$ is proper
(for example if $\mathcal{A} \otimes \mathbb{C}(X)$ is a division algebra and we consider sheaves of dimension $\dim X$ and rank $\mathrm{rk}\,\mathcal{A}$.)

$$DT\left(M^{s,h}(X,\mathcal{A})\right) = \int_{[M^{s,h}(X,\mathcal{A})]^{vir}} 1 \quad \in \mathbb{Z}$$

If (i), (ii) are satisfied, also

$$DT\left(\mathrm{Hilb}^h(X,\mathcal{A})\right) = \int_{[\mathrm{Hilb}^h(X,\mathcal{A})]^{vir}} 1 \quad \in \mathbb{Z} .$$

These are deformation invariants.

We are interested in the "partition function"

$$\sum_n DT\left(\mathrm{Hilb}^n(X,\mathcal{A})\right) t^n \qquad h = \text{constant} = n .$$

**Remark.** Since $[\ ]^{vir}$ is defined in terms of a symmetric obstruction theory:

$$DT\left(\mathrm{Hilb}^n(X,\mathcal{A})\right) = \chi^{top}\left(\mathrm{Hilb}^n(X,\mathcal{A}), \nu\right)$$

weighted Euler characteristic.

$\nu$: generalized Milnor number: an integer invariant of a singularity / germ of an analytic space

Main properties: (i) constructible $X \longrightarrow \mathbb{Z}$

(ii) $X = \mathrm{Crit}(M, f)$ $f: M \to \mathbb{C}$ holomorphic

on $M$: complex manifold then

$$\nu_X(P) = (-1)^{\dim M}\left(1 - \chi^{top}\left(\text{Milnor fibre of } f \text{ at } P\right)\right)$$

(iii) If $M$ admits a $\mathbb{C}^\times$-action with

$P$ as isolated fixed point, $f$ homogeneous

$$\nu_X(P) = (-1)^{\dim T_X|P}$$

We will compute $DT(\mathrm{Hilb}^n(X, A))$ as a weighted Euler characteristic.

7. Computation of $Z_Y(t) = \sum_n DT(\mathrm{Hilb}^n Y)\, t^n$

$Y$: commutative quintic 3-fold.

$\mathrm{Hilb}^n Y \mid P \subset \mathrm{Hilb}^n Y$ : punctual Hilbert scheme,

subschemes of $Y$ of length $n$, supported at $P \in Y$.

$$Z_Y(t) = \sum_n \chi(\mathrm{Hilb}^n Y, \nu_{\mathrm{Hilb}^n Y})\, t^n$$

$$Z_{Y|P}(t) = \sum_n \chi(\mathrm{Hilb}^n Y|P, \nu_{\mathrm{Hilb}^n Y})\, t^n$$

$$Z_Y(t) = Z_{Y|P}(t)^{\chi(Y)} \qquad \chi(Y) = -200 \qquad \text{(cutting \& pasting)}$$

$$\mathrm{Germ}(\mathrm{Hilb}^n Y|P,\ \mathrm{Hilb}^n Y)$$
$$= \mathrm{Germ}(\mathrm{Hilb}^n \mathbb{C}^3|0,\ \mathrm{Hilb}^n \mathbb{C}^3)$$

$$\rightsquigarrow\ \chi(\mathrm{Hilb}^n Y|P, \nu_{\mathrm{Hilb}^n Y}) = \chi(\mathrm{Hilb}^n \mathbb{C}^3|0, \nu_{\mathrm{Hilb}^n \mathbb{C}^3})$$

using $\mathbb{C}^\times$-action, Property (iii) of $\nu$

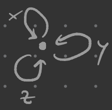
$$\chi(\mathrm{Hilb}^n \mathbb{C}^3|0, \nu_{\mathrm{Hilb}^n \mathbb{C}^3}) = (-1)^n \#\, \text{3D partitions of } n$$

$$\rightsquigarrow\ Z_{Y,P}(t) = Z_{\mathbb{C}^3|0}(t) = M(-t)$$

$$M(t) = \prod_{m=1}^{\infty} \frac{1}{(1-t^m)^m} \qquad \text{MacMahon function}$$

$$\boxed{Z_Y(t) = M(-t)^{-200}}$$

$\mathbb{C}[x,y,z]$ = Jacobi algebra of quiver




with potential $xyz - xzy$

(giving rise to relations $xy = yx$, $xz = zx$, $yz = zy$)

$\mathrm{Hilb}^n\, \mathbb{C}^3$ = length-$n$ quotient modules of $\mathbb{C}[x,y,z]$

= stable representations of quiver with relations of dimension vector $n$ with a framing vector

$\mathrm{Hilb}^n\, \mathbb{C}^3 | 0$ = nilpotent representations

The quiver is the Ext-quiver of the simple object $S = \mathcal{O}_p$ in $\mathrm{Coh}\, Y$.

1 vertex $\leftrightarrow S$

arrows $\leftrightarrow$ basis of $\mathrm{Ext}^1(S,S)^\vee \leftrightarrow$ coordinates of $Y$ near $P$.

path algebra: free algebra on $\mathrm{Ext}^1(S,S)^\vee$ $A = \mathbb{C}\langle x,y,z \rangle$

Yoneda product: $\mathrm{Ext}^1(S,S) \otimes \mathrm{Ext}^1(S,S) \longrightarrow \mathrm{Ext}^2(S,S)$

gives $\mathrm{Ext}^2(S,S)^\vee \longrightarrow \mathrm{Ext}^1(S,S)^\vee \otimes \mathrm{Ext}^1(S,S)^\vee$

3 quadratic relations in $A$ $\longrightarrow$ quotient $= \mathbb{C}[x,y,z]$.

# 8. Computation of $Z_Q(t) = \sum_n DT(\mathrm{Hilb}^n(X,\mathcal{A}))\, t^n$

$$= \sum_n \chi(\mathrm{Hilb}^n(X,\mathcal{A}), \nu)\, t^n$$

Finite length $\mathcal{A}$-modules have 0-dimensional support in $X$

$\rightsquigarrow$ can study locally in $X = \{x_0 + \dots + x_4 = 0\} \subset \mathbb{P}^4$

$\rightsquigarrow$ Localize by setting $x_0 = 1$

$$u_i = \frac{t_0^4 t_i}{t_0^5} = \frac{t_0^4 t_i}{x_0}$$

Then $X_0 = \mathbb{C}[x_1, \dots, x_4] / x_1 + \dots + x_4 = -1$

$$\Big\downarrow x_i = u_i^5$$

$$A = \mathbb{C}[u_1, \dots, u_4] / u_1^5 + \dots + u_4^5 = -1, \quad u_i u_j = q^{\bar{n}_{ij}} u_j u_i$$

$\bar{n}_{ij} = n_{ij} - n_{i0} - n_{0j}$ $\quad \bar{N} \in M_{4\times 4}(\mathbb{F}_5)$, skew symmetric, $\bar{N}\vec{1} = \vec{0}$.

$$\bar{N} = \begin{pmatrix} 0 & -2 & -1 & -2 \\ 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

Point modules: representations of $A$ on $\mathbb{C}$.

$u_1, \dots, u_4$ turn into numbers (which commute)

non-trivial commutation relations

$\rightsquigarrow$ at most one of $u_1, \dots, u_4$ is non-zero.

Say $u_2 = u_3 = u_4 = 0$ and $u_1^5 = -1$, so $u_1 = -q^i$, $i \in \mathbb{F}_5$.

$\rightsquigarrow$ point modules $S_0, \dots, S_4$ supported at $\langle 1, -1, 0, 0, 0 \rangle \in X$

there are $\binom{5}{2} = 10$ such points in $X \subset \mathbb{P}^4$

$\rightsquigarrow$ 50 point modules for $Q_q = (X, \mathcal{A})$

$\rightsquigarrow$ $DT(\mathrm{Hilb}^1(X,\mathcal{A})) = 50$ $\quad$ (contrast with 200 in commutative case)

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Consider $\mathcal{A}$ near $P = \langle 1, -1, 0, 0, 0 \rangle$

Expectation: (assuming all simple $\mathcal{A}$-modules at $P$ are point modules)

$\mathrm{Germ}(\mathrm{Hilb}^n \mathcal{A} | P, \mathrm{Hilb}^n \mathcal{A})$

$= \mathrm{Germ}\left( \prod_{|\vec{d}|=n} M^s(Q, \vec{d}, v) | 0, \prod_{|\vec{d}|=n} M^s(Q, \vec{d}, v) \right)$

$(Q, f)$ Ext quiver of $S = S_0 \oplus \cdots \oplus S_4$, with potential $f$

$\vec{d}$ : dimension vector

$v$ : framing

Rmk. (Toda)

On a commutative Calabi-Yau 3-fold $Y$

$\mathrm{Germ}(\mathcal{M}^{ss}_{\omega} | P, \mathcal{M}^{ss}_{\omega})$

$= \mathrm{Germ}(\mathcal{M}_Q | 0, \mathcal{M}_Q)$

$\mathcal{M}_{\omega}$ : stack of Giesecker semistable sheaves / $Y$

$\mathcal{M}_{\omega} | P$ : fix the associated polystable sheaf $\bigoplus_i \mathcal{F}_i^{\oplus k_i}$

$\mathcal{M}_Q$ : representations of the Ext-quiver of $\oplus \mathcal{F}_i$ with potential with dimension vector $\vec{k}$

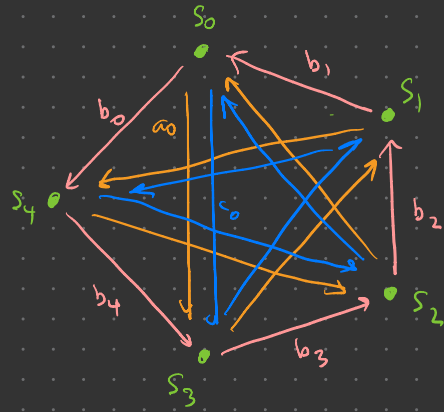
$\mathcal{M}_Q | 0$ : nilpotent representations

Theorem (Liu)

The expectation holds.

The quiver is:

vertices $\leftrightarrow$ point modules $S_0, \dots, S_4$




arrows $\leftrightarrow$ basic extensions between $S_i$

$a_i \in \mathrm{Ext}^1(S_i, S_{i-2})$

$$S_{i-2} \to a_i \to S_i$$

$$u_1 = \begin{pmatrix} -q^{i-2} & 0 \\ 0 & -q^{i} \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad u_3 = u_4 = 0.$$

$$u_1 u_2 = \begin{pmatrix} -q^{i-2} & 0 \\ 0 & -q^{i} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -q^{i-2} \\ 0 & 0 \end{pmatrix}$$

$$u_2 u_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -q^{i-2} & 0 \\ 0 & -q^{i} \end{pmatrix} = \begin{pmatrix} 0 & -q^{i} \\ 0 & 0 \end{pmatrix}$$

So $u_1 u_2 = q^{\bar{n}_{12}} u_2 u_1 = q^{-2} u_2 u_1$ is satisfied.

$b_i \in \mathrm{Ext}^1(S_i, S_{i-1})$ $\quad c_i \in \mathrm{Ext}^1(S_i, S_{i-2})$ similar.

Potential: $f = \left(\sum q^{i-1} b_i\right)\left(\sum a_i\right)\left(\sum c_i\right) - q^{-1}\left(\sum q^{i-1} b_i\right)\left(\sum c_i\right)\left(\sum a_i\right)$

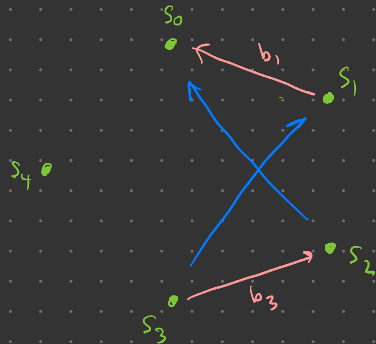
analytically locally near $P = \langle 1, -1, 0, 0, 0 \rangle$

$A \cong J(Q, f)$ $\qquad u_2, u_3, u_4 \longmapsto \sum a_i, \sum b_i, \sum c_i$

commutation relations among $u_2, u_3, u_4$ give relations among $\sum a_i, \sum b_i, \sum c_i$.

15 relations, e.g. $\partial_{a_i} : q^{i+2} c_{i+2} b_{i+3} = q^{-1} q^{i} b_{i+1} c_{i+3}$

e.g. $i=0$ : $q^3 c_2 b_3 = b_1 c_3$




Framing vector: $\vec{1} = (1, \dots, 1)$.

Corollary: $Z(A|P)(t) = Z(Q, f, \vec{1})(t, \dots, t) =: Z(Q,f)(t)$

So the 10 special points $\langle 1,-1,0,0,0 \rangle$ contribute $Z(Q,f)(t)^{10}$

There is a (complicated) box counting problem giving $Z(Q,f)(t)$ but we were not able to get a formula.

Generically: Away from the 10 special points

$$A \cong M_{5\times 5}\left(\mathcal{O}_X\left(\sqrt[5]{x_3}, \sqrt[5]{x_4}\right)\right) \quad \text{if } x_1 \neq 0, x_2 \neq 0.$$

So $A$ is Morita equivalent to a commutative algebra.

To study modules, ignore $M_{5\times5}$ up to rescaling the length by $5$:

Final Answer: $Z(X,A)(t) = Z(Q,f)(t)^{10} \; M(-t^5)^{-50}$.