Donaldson-Thomas Theory of the Quantum Fermat Quintic

Intercontinental Moduli Zoominar
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Joint work with Yu-Hsiang Liu

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and Atsushi Kanazawa

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with contributions by Atharva Korde
1. The quantum Fano Quintic

Quantum projective 4-space: non-commutative graded algebra

\( \mathbb{P}^4_q : C \langle t_0, t_1, t_2, t_3, t_4 \rangle / t_i t_j = q^{n_{ij}} t_j t_i \), \( q \in C \) fixed \( 1/\sqrt{1} \)

\( N = (n_{ij}) \in M_{5 \times 5}(\mathbb{F}_q) \) skew-symmetric matrix

\[
N = \begin{pmatrix}
0 & 1 & -1 & -1 & -1 \\
-1 & 0 & 1 & -1 & -1 \\
1 & -1 & 0 & 1 & -1 \\
-1 & -1 & -1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{pmatrix}
\]

(to six formulas.)

(this is generic!)

\( t^5 \) are central elements: obtain the quantum Fano Quintic

\( Q_q^5 : C \langle t_0, t_1, t_2, t_3, t_4 \rangle / t_0^5 + \cdots + t_4^5 \)

(graded algebra) \( Q_q \subseteq \mathbb{P}^4_q \)

2. Non-commutative projective schemes

\( Q_q \) is a non-commutative projective scheme (Artin-Zhang)

(graded \( C \)-algebras \( S \)) \( \leftrightarrow \) (triples \( (C, U, (v)) \))

\( E \): abelian category

\( U \in E \): object

\( (v) : E \to E \) auto-equivalence

\( f \mapsto f(v) \)

\( S \mapsto \text{Proj } S = (\text{grg}(S), S, \text{shift}) \)

\( \text{grg}(S) \): category of tails of F.g. graded \( S \)-modules
\[ \bigoplus_n \text{Hom}(0, 0(n)) \leftarrow (\mathcal{E}, \mathcal{O}, \{1\}) \]

\[ a \cdot b = a(\deg b) \circ b \]

With enough conditions on \( \mathcal{E} \) and triples this gives an equivalence of categories (On \( \mathcal{A} \)-algebra side up to finite modules)

**Theorem (Kanezawa)**

For the quantum Fermat quintic \( (\text{any } N = (n_i)_i) \)

\[ qgr(\mathcal{Q}) \]

(i) has global dimension 3

(ii) is a Calabi-Yau 3 category iff \( \begin{bmatrix} 1 \end{bmatrix} \in \mathbb{P}^5 \) is an eigenvector of \( N \).

(i) \( \text{Ext}^i(\mathcal{E}, \mathcal{F}) = 0 \quad \forall i > 3 \)

(ii) \( \text{Ext}^i(\mathcal{E}, \mathcal{F})^\vee = \text{Ext}^{3-i}(\mathcal{F}, \mathcal{E}) \)

(i) \( \mathcal{Q} \) is smooth of dimension 3

(ii) \( \mathcal{Q} \) is a Calabi-Yau 3-fold.

So moduli spaces of objects in \( qgr(\mathcal{Q}) \) should admit a Donaldson-Thomas theory. We were not able to construct it using techniques from non-commutative projective geometry.

3. The quintic mirror.

Recall: The mirror family of the 101-dimensional family of smooth quintics is the 1-dimensional family of singular CY3s obtained as follows:
Start with $\mathcal{Q}_4 : \mathbb{C}[t_0, \ldots, t_4] / \ t_0^5 + \cdots + t_4^5 - 5\psi t_0 \cdots t_4$ (coincidence!).

A 1-parameter family of quartics in $\mathbb{P}^4$.

Act with the group $G = \frac{\text{ker} S^*}{\text{im} A'}$, $\mathbb{P}_5 \xrightarrow{\pi} \mathbb{P}_5 \xrightarrow{\pi^*} \mathbb{P}_5$, # $\mathbb{A}^2 (25)$

by $t_i \cdot t_j = q^{5i} t_j$.

and pass to the quotient: a 1-parameter family of

singular $C^{3,2}$, $\mathcal{W}_\psi$. $\mathcal{Q}_4 \longrightarrow \mathcal{W}_\psi$ degree 125

In fact $\mathcal{W}_\psi : \mathbb{C}[x_0, x_1, x_2, x_3, y] / \Sigma x_i - 54 y$, $x_5 = \pi x_i$

where $x_i = t_i^5$, $y = t_0 \cdots t_4$.

The singular locus of $\mathcal{W}_\psi$ is always the same:

1. $C_{012} = \{ x_0 + x_1 + x_2 = 0, x_3 = 0, x_4 = 0 \}$

$C_{ijh}$: similar

$(\frac{5}{3}) = 10$ lines in $\mathcal{W}_\psi$ where $\mathcal{W}_\psi$ has a transverse $A_4$ singularity.

2. $P_{01} = \{ x_0 + x_1 = 0, x_2 + x_3 + x_4 = 0 \}$

$P_{ij}$ similar

$(\frac{5}{2}) = 10$ points where 3 lines meet, more complicated singularity.

But $\mathcal{W}_\psi$ can be resolved to a Calabi-Yau threefold.

Kovary: back to non-commutative case: $\Pi t_i$ is also central, and $Q_{q, \psi} : \mathbb{C}[t_0, \ldots, t_4] / \Sigma t_i^5 - 5\psi \Pi t_i$ is still a smooth non-commutative Calabi-Yau threefold.
Note: \( W_q \) embeds also into \( Q_{q,q} \), in fact into the centre \( Z(Q_{q,q}) \)
because \( x_1 = x_5 \), \( y_1 = 0 \) are central in \( Q_{q,q} \).

Conjecture: In fact, \( W_q = Z(Q_{q,q}) \).

This would imply that \( Q_{q,q} \) is a non-commutative crepant resolution of the quintic minor \( W_q \):

- it is smooth Calabi-Yau threefold
- its centre is \( W_q \).

Note: In the literature "non-commutative crepant resolution" \( Z \subseteq A \)
is usually:

- local (affine, ungraded case)
- requires \( A = \text{End}_Z(M) \) for a reflexive \( Z \)-module \( M \).
(VandenBergh)

\( W_q \subset Q_{q,q} \) is:

- global
- not of this form, as \( 125 \) not a square.

We have not seriously studied \( Q_{q,q} \) for general \( q \).
But any enumerative theory should be deformation invariant,
so we should be justified in considering only the case \( q \to 0 \).

Question: Are we, in fact, dealing with the quintic minor,
rather than a quantum version of the quintic?

4. Sheaves of Frobenius algebras

\( Q_q \) has a central (commutative) subalgebra over which it is finite.
The 5-Veblenese subalgebra of $\mathbb{C}[t_{0,\ldots,t_{4}}]$ is a graded free module over $\mathbb{C}[t_{0,\ldots,t_{4}}]$ on the basis $t^k$, where $\sum k_i = 5$, $0 \leq k_i \leq 4$.

$\mathcal{A} \cong \mathcal{O}_X + \mathcal{O}_X(-1)^{121} + \mathcal{O}_X(-2)^{381} + \mathcal{O}_X(-3)^{121} + \mathcal{O}_X(-4)$

as $\mathcal{O}_X$-module (not as algebra).

Multiplication in $\mathcal{A}$, composed with projection to $\mathcal{A} \to \mathcal{O}_X(-4)$ defines a perfect pairing

$\mathcal{A} \otimes \mathcal{O}_X \mathcal{A} \to \mathcal{O}_X(-4) = \omega_X, \quad a \otimes b \mapsto \text{rk}(ab)$.

The pairing is symmetric if $t^k t^{4-k} = t^k t^{4-k}$, i.e., $t^1$ eigenvector of $N$.

**Definition.** $X$: smooth scheme, $\mathcal{A}$: locally free sheaf of $\mathcal{O}_X$-algebras with symmetric perfect pairing $\mathcal{A} \otimes \mathcal{O}_X \mathcal{A} \to \omega_X$ is a sheaf of Frobenius algebras over $X$.

If the sheaf of algebras $\mathcal{A}/\mathcal{O}_X$ has finite global dimension $n = \dim X$, it has a dualizing bimodule $\omega = \text{Hom}_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$ such that...
\[ \text{Ext}^i_A \left( \mathcal{F}, \mathcal{G} \right) = \text{Ext}^{n-i}_A \left( \mathcal{G}, \mathcal{O}_A \otimes_A \mathcal{F} \right)^* \quad \forall \mathcal{F}, \mathcal{G} \in \text{Coh}(A) \]

\[ \text{Coh}(A) : \text{left } A \text{-modules which are coherent } \mathcal{O}_A \text{-modules.} \]

A symmetric pairing \( \omega : A \otimes A \rightarrow k \) identifies

\[ \omega = \text{Hom}_{\mathcal{O}_X} (A, \mathcal{O}_X) = : A \quad \text{as } A \text{-bimodule} \]

so \( \text{Coh}(\mathcal{U}) \) becomes a Calabi-Yau \( n \)-category.

**Rmk.** In our situation \( \text{agr}(\mathcal{O}_X) \]

\[ \mathcal{Q}_\mathcal{O} = \mathcal{O} \oplus \mathcal{O} \]

and \( \text{Coh}(\mathcal{U}) \]

\( A : \text{Frobenius algebra} / X \subset \mathbb{P}^3 \)

are equivalent. Study \( \text{Coh}(\mathcal{U}) \) instead.

\( \text{Coh}(\mathcal{U}) : \text{coh } \mathcal{O}_X \text{-modules with structure of left } A \text{-module.} \)

5. Moduli spaces for pairs \( (X, \mathcal{U}) \)

\( X : \text{smooth projective scheme } \mathcal{O}_X. \)

\( A : \text{locally free sheaf of Frobenius algebras over } X. \)

Assume \( A \) of finite global dimension \( n = \dim X. \)

\( \mathcal{F} \in \text{Coh}(A) : \text{Hilbert polynomial } \text{p}(\mathcal{F})(i) = X(X, \mathcal{F}(i)) \).

**Definition / Theorem.** (Simpson)

\( \mathcal{F} \) is **(semi)**-**stable if**

(i) pure as \( \mathcal{O}_X \)-module

(ii) \( \forall 0 < \mathcal{F}' < \mathcal{F} \quad \text{A-submodule} \)

\[ \frac{\text{p}(\mathcal{F}')(i)}{\text{deg} \mathcal{F}'} < \frac{\text{p}(\mathcal{F})(i)}{\text{deg} \mathcal{F}} \quad \forall i > 0 \]

- pure modules have **Harder- Narasimhan filtrations**
- semi stable modules have **Jordan-Hölder filtrations**
\[\rightarrow \text{S-equivalence for semi-stable modules}\]

\[\text{If stable } \Rightarrow \text{Hom}_A(T, F) = 0\]

let \( h \) be a polynomial

\[M^{ss, h}(X, A)\]: semi-stable \( A \)-modules with Hilbert polynomial \( h \)

Artin stack of finite type with a good moduli space \( M^{ss, h}(X, A)\).

\[M^{ss, h}(X, A)\]: projective scheme classifying S-equivalence classes (or polystable sheaves)

\[M^{s, h}(X, A) \rightarrow M^{ss, h}(X, A)\] is a \( \mathbb{C}^* \)-grobe

\[M^{s, h}(X, A) \subset M^{ss, h}(X, A)\] open, classifies isomorphism classes.

**Hilbert Schemes**

\([\text{Hilb}^h(X, A) \subset \text{Quot}^h(X, A)\]: closed subscheme classifying coherent \( A \)-modules with an epimorphism \( A \rightarrow T\)

Would like a morphism, as in classical Donaldson-Thomas theory

\[\text{by } h \leq 1: \quad \text{Hilb}^h(X, A) \rightarrow M^{s, p^h}(X, A)\]

\[\begin{align*}
\text{p} &= p(A) \\
A \rightarrow T &\quad \rightarrow \ker (A - T)
\end{align*}\]

\[\text{Hilb}^h(X, A)\] easier to handle, \( M^{s, p^h}(X, A)\) better deformation theory

\(i)\) \(A \otimes \mathbb{C}(x)\) is a division ring, all non-zero submodules

\[0 \neq T' \subset \ker (A - T)\] have same rank as \( A\), so

\[p(T') (i) = p(\ker (A - T')) (i) \quad \forall i \Rightarrow 0 = \ker (A - T)\) stable.

So the morphism exists (commutative analogue: pure rank 1 sheaves automatically stable).
(iii) If $H^1(X,A) = 0$ the morphism is an open immersion.

(i), (iii) $\text{Hilb}^b(X,A)$ is a union of connected components of $\text{M}^b(X,A)$.

[Commutative analogue: $\text{Hilb}^b(X,O_X)$ is a moduli space of torsion-free rank 1 sheaves with trivial determinant.]

6. Donaldson–Thomas theory for pairs $(X,A)$

$X$: smooth projective scheme $O_X$.

$A$: locally free sheaf of Frobenius algebra over $X$.

Assume $A$ of finite global dimension $n = \dim X$.

**Theorem (Liu)**

$\text{M}^{(1)}(X,A)$ carries a symmetric (and perfect of virtual dimension 0) obstruction theory.

- Deformation space $\text{Ext}^1_A(F,F)$
- Obstruction space $\text{Ext}^2_A(F,F)$, dual to deformation space.

In particular, $\text{M}^{(1)}(X,A)$ carries a virtual fundamental class $[\text{M}^{(1)}(X,A)]^{\text{vir}} \in A_0(\text{M}^{(1)}(X,A))$.

**Remark**: universal family $F/\{x \times M \to X \times M$.

- Obstruction theory is $\text{Rm} \circ \text{Rfom}_A(F,F)$.

- Even though $F$ may not descend the gerbe, $\text{Rfom}_A(F,F)$ will descend. Think of $F$ as a twisted sheaf, the two twists cancel out.
Definition. Suppose \( h \) chosen such that \( ss \Rightarrow s \) so that \( M^{s,h}(X, A) \) is proper

(for example if \( A \otimes \mathbb{C}(x) \) is a division algebra and we consider sheaves of dimension \( \dim X \) and rank \( \text{rk} A \)).

\[
\text{DT}(M^{s,h}(X, A)) = \sum_{x \in \mathbb{Z}} \frac{1}{[M^{s,h}(X, A)]^x}
\]

If (i),(ii) are satisfied, also

\[
\text{DT}(\text{Hilb}^h(X, A)) = \sum_{x \in \mathbb{Z}} \frac{1}{[\text{Hilb}^h(X, A)]^x}
\]

These are deformation invariants.

We are interested in the "partition function"

\[
\sum_n \text{DT}(\text{Hilb}^n(X, A)) t^n \quad h = \text{constant} = n
\]

Remark. Since \([ \ ]^{\text{vir}} \) is defined in terms of a symmetric obstruction theory:

\[
\text{DT}(\text{Hilb}^n(X, A)) = \chi^{\text{top}}(\text{Hilb}^n(X, A), v)
\]

weighted Euler characteristic.

\( v \): generalized Milnor number: an integer invariant of a singularity/germ of an analytic space.
Main properties:

(i) Constructible $X \rightarrow \mathbb{P}^n$

(ii) $X = \text{Crit}(M, f)$, $f : M \rightarrow \mathbb{C}$ holomorphic on $M$, complex manifold then

$$\nu_X(p) = (-1)^{\dim M} \left( 1 - X^{\text{top}} \text{ (Milnor fibre of } f \text{ at } p \text{)} \right)$$

(iii) If $M$ admits a $C^\infty$-action with $P$ as isolated fixed point, $f$ homogeneous

$$\nu_X(p) = (-1)^{\dim T_x P}$$

We will compute $DT(\text{Hilb}^-(X, A))$ as a weighted Euler characteristic.
7. Computation of $Z_Y(t) = \sum_n D_T(\text{Hilb}^n Y) t^n$

$Y$: commutative quintic 3-fold.

$\text{Hilb}^n Y IP \subset \text{Hilb}^n Y$: punctual Hilbert scheme of subschemes of $Y$ of length $n$, supported at $P \in Y$.

$Z_Y(t) = \sum_n X(\text{Hilb}^n Y, \nu_{\text{Hilb}^n Y}) t^n$

$Z_{Y,IP}(t) = \sum_n X(\text{Hilb}^n Y IP, \nu_{\text{Hilb}^n Y}) t^n$

$Z_Y(t) = Z_{Y,IP}(t)^{X(Y)}$ \quad $X(Y) = -200$ \quad (cutting & pasting)

$\text{Germ} \left( \text{Hilb}^n Y IP, \text{Hilb}^n Y \right)$

$= \text{Germ} \left( \text{Hilb}^n \mathbb{C}^3 | 0, \text{Hilb}^n \mathbb{C}^3 \right)$

$\nabla X(\text{Hilb}^n Y IP, \nu_{\text{Hilb}^n Y}) = X(\text{Hilb}^n \mathbb{C}^3 | 0, \nu_{\text{Hilb}^n \mathbb{C}^3})$

using $C^\infty$-action, Property (iii) of $\nu$

$X(\text{Hilb}^n \mathbb{C}^3 | 0, \nu_{\text{Hilb}^n \mathbb{C}^3}) = c^{-1} \pi^* \text{3D partitions of } n$

$\nabla Z_{Y,IP}(t) = Z_{\mathbb{C}^3 | 0}(t) = M(-t)$

$M(t) = \frac{1}{1} \prod_{m=1}^{\infty} \frac{1}{(1-tm)^m}$ \quad MacMahon function

$Z_Y(t) = M(-t)^{-200}$
$C[x,y,z] = \text{Jacobi algebra of quiver } \mathcal{G}$

with potential $xy = xz$

(giving rise to relations $xy = yx$, $xz = zx$, $yz = zy$)

$\text{Hilb}^n C^3 = \text{length } n \text{ quotient modules of } [C[x,y,z]]$

$= \text{stable representations of quiver}$

with relations

of dimension vector $n$

with a framing vector

$\text{Hilb}^n C^3 | \text{O} = \text{nilpotent representations}$

The quiver is the $\text{Ext}$-quiver of the simple object $S = \text{O}_P$ in $\text{Coh} Q$.

1 vertex $\leftrightarrow S$

arrows $\leftrightarrow$ basis of $\text{Ext}^1(S,S) \cong \text{coordinates of } Y \text{ near } P$

path algebra = free algebra on $\text{Ext}^1(S,S) \cong A = C[x,y,z]$

Yoneda product: $\text{Ext}^1(S,S) \otimes \text{Ext}^1(S,S) \rightarrow \text{Ext}^2(S,S)$

gives $\text{Ext}^2(S,S) \hookrightarrow \text{Ext}^1(S,S) \otimes \text{Ext}^1(S,S)$

3 quadratic relations in $A \rightarrow \text{quotient } = C[x,y,z]$
8. Computation of \( Z_n(\epsilon) = \sum \text{DT}(\text{Hilb}^n(X, A)) \epsilon^n \)

\( \leq \sum \text{DT}(\text{Hilb}^n(X, A), \nu) \epsilon^n \)

Think length \( A \)-modules have 0-dimensional support in \( X \)

\( \nu \) can study locally in \( X = \{ x_0 + \ldots + x_4 = 0 \} \subset \mathbb{P}^4 \)

Localize by setting \( x_0 = 1 \)

\( u_i = \frac{x_i}{x_0} = \frac{x_i}{x_0} \)

Then \( X_0 = \mathbb{C}[x_1, \ldots, x_4] / x_1 + \ldots + x_4 = 1 \)

\( \downarrow \) \( x_i = u_i^5 \)

\( A = \mathbb{C}[u_1, \ldots, u_4] / u_1^5 + \ldots + u_4^5 = 1 \), \( u_i u_j = q_{ij} u_j u_i \)

\( \overline{\eta}_{ij} = \eta_{ij} - \eta_{i0} - \eta_{0j} \), \( \overline{\eta} \in \mathbb{M}_{4 \times 4}(\mathbb{F}_5) \), skew symmetric, \( \overline{\eta}^T = \overline{\eta} \).

\( \overline{\eta} = \begin{pmatrix}
0 & 2 & 1 & -2 \\
2 & 0 & -1 & -1 \\
1 & 1 & 0 & -2 \\
2 & 1 & 2 & 0
\end{pmatrix} \)

Point modules: representations of \( A \) on \( \mathbb{C} \).

\( u_1, \ldots, u_4 \) turn into numbers (which commute)

non-trivial commutation relations

at most one of \( u_1, u_4 \) is non-zero.

Say \( u_2 = u_3 = u_4 = 0 \) and \( u_1^5 = -1 \), so \( u_1 = q_1 \), \( \text{ie \mathbb{F}_5} \).

Point modules \( S_0, S_4 \), supported at \( (1, 1, 0, 0, 0) \in X \)
there are \( \binom{5}{2} = 10 \) such points in \( X \subset \mathbb{P}^4 \)

50 point modules for \( Q_0 = (X, A) \)

\( \text{DT}(\text{Hilb}^n(X, A)) = 50 \) (contact with 200 in commutative case)
Consider $A$ near $P = \langle 1, 1, 0, 0, 0 \rangle$

**Expectation**: (assuming all simple $A$-modules at $P$ are point modules)

\[
\text{Germ} \left( \text{Hilb}^n A | P, \text{Hilb}^n A \right) = \text{Germ} \left( \prod_{d \in \mathbb{Z}} \text{M}^s(Q, d, v) | 0, \prod_{d \in \mathbb{Z}} \text{M}^s(Q, d, v) \right)
\]

$(Q, f)$ Ext quiver of $S = S_0 \oplus \cdots \oplus S_4$, with potential $f$

$d$: dimension vector

$v$: framing

**Rank**: (Today)

On a commutative Calabi-Yau 3-fold $Y$

\[
\text{Germ} \left( \text{M}_\omega | P, \text{M}_\omega \right) = \text{Germ} \left( \text{M}_\omega | 0, \text{M}_\omega \right)
\]

$\text{M}_\omega$: stack of Gieseker-semistable sheaves $/ Y$

$\text{M}_\omega | P$: fix the annotated polystable sheaf $\oplus \mathcal{F}_i$

$\text{M}_\omega | 0$: representations of the Ext-quiver of $\oplus \mathcal{F}_i$

with potential $f$

with dimension vector $k$

$\text{M}_\omega | 0$: nilpotent representations

**Theorem (Liu)**

The expectation holds.

The quiver is:
vertices $\leftrightarrow$ point modules $S_0, \ldots, S_4$

arrows $\leftrightarrow$ basic extensions between $S_i$

$a_i \in \text{Ext}^1(S_{i-1}, S_{i-2})$

$S_{i-1} \rightarrow a_i \rightarrow S_i$

\[
u_1 = \begin{pmatrix} -q^{-2} & 0 \\ 0 & q^{-i} \end{pmatrix}
\]

\[
u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \nu_3 = \nu_4 = 0
\]

\[
u_1 \nu_2 = \begin{pmatrix} -q^{-i-2} & 0 \\ 0 & q^{-i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -q^{-i-2} \\ 0 & 0 \end{pmatrix}
\]

\[
u_2 \nu_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -q^{-i-2} & 0 \\ 0 & q^{-i} \end{pmatrix} = \begin{pmatrix} 0 & -q^{-i} \\ 0 & 0 \end{pmatrix}
\]

So $\nu_1 \nu_2 = q^{-i+2}, \nu_2 \nu_1 = q^{-2} \nu_2 \nu_1$ is satisfied.

$b_i \in \text{Ext}^1(S_i, S_{i-2})$, $c_i \in \text{Ext}^1(S_i, S_{i-1})$, similar.

Potential: $f = (\sum q^{-i} b_i)(\sum c_i)$

analytically locally near $P = <i, 0, 0, 0>$

$A \cong \mathcal{O}(\mathcal{O}, f)$, $\nu_2, \nu_3, \nu_4 \rightarrow \Sigma a_i, \Sigma b_i, \Sigma c_i$

commutation relations among $\nu_2, \nu_3, \nu_4$ give relations among $\Sigma a_i, \Sigma b_i, \Sigma c_i$. 
15 relations, e.g. $d_\alpha q^3 c_{i2} b_{i+3} = q^3 q^i_1 b_{i1} c_{i3}$

e.g. $i=0$: $q^3 c_{i2} b_{3} = b_1 c_3$

Framing vector: $\mathbf{I} = (1, \ldots, 1)$.

**Corollary:**

$Z(A_{IP}) (t) = Z(G, t, \mathbf{I}) (t, \ldots, t) = Z(G, t)^{10}$

So the 15 special points $<1, -1, 0, 0, 0>$ contribute $Z(G, t)^{10}$

There is a (complicated) box country problem giving $Z(G, t)(t)$ but we were not able to get a formula.

**Generically:** Away from the 10 special points

$A \cong M_{5\times 5} \left( \mathcal{O}_x (\sqrt{x_1}, \sqrt{x_2}) \right)$ if $x_1 \neq 0$, $x_2 \neq 0$.

So $A$ is Morita equivalent to a commutative algebra.

To study modules, ignore $M_{5\times 5}$ up to rescaling the length by $k$.

**Find Answer:** $Z(\chi_A)(t) = Z(G, t)^{10} \cdot M (-t^5)^{-50}$.