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Gopakumar-Vafa invariants of local curves

joint with T. Kinjo & B. Davison.

§1. Intro

X : CY3. $\exists$ several curve counting theories:

(Gromov - Witten): counts $f : C \to X$ from curves C

(Donaldson-Thomas / Pandharipande-Thomas): counts $s : \mathcal{O}_X \to F$

F : 1-dim'l shv's

(Gopakumar - Vafa): counts F : 1-dim'l shv's

Conj (Maulik-Nekrasov-Okounkov-Pandharipande, Maulik-Toda):

$$GW = DT = PT = GV.$$

• [Bridgeland, Toda]: $DT = PT$ for any sm pj. CY3.

• [Pardon] : GW = DT for any sm prj CY3

Pardon's strategy :

Step 1 : Reduce everything to the case of local curves

i.e. $X = \mathrm{Tot}_C(N)$, where C : sm prj curve

N : rk 2 bdl w/ $\det(N) \cong \omega_C$

Step 2 : Compute GW & DT for local curves.

[Bryan – Pandharipande, Okounkov – Pandharipande]

$\rightsquigarrow$ local curves are fundamental objects.

• GW = GV : widely open

Advantages in GW theory : a lot of computational tools

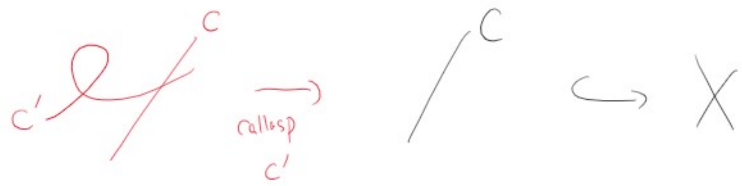
e.g. : degeneration & torus localization

Issues in GW theory :

$$M_{g,\beta}(X) = \left\{ f : C \to X \;\middle|\; \begin{array}{l} f_*[C] = \beta,\ g(C) = g \\ f : \text{stable} \end{array} \right\}$$

a): $\mathcal{M}_{g,\beta}(X)$ is a stack $\rightsquigarrow$ GW invariants are rational

b): Consider an embedded curve $C \subset X$:



$\rightsquigarrow$ C can contribute to ∞-many GW inv's

Conj (Gopakumar – Vafa)

$\exists$ another invariants $n_{g,\beta}$ s.t.

(integrality): $n_{g,\beta} \in \mathbb{Z}$

(finiteness): For a fixed β, $n_{g,\beta} = 0$ for $g \gg 0$

(GW = GV): $n_{g,\beta}$ have equivalent info w/ GW invariants.

§2. Gopakumar – Vafa theory:

X: CY3, $\beta \in H_2(X, \mathbb{Z})$, $\chi \in \mathbb{Z}$.

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$\rightsquigarrow$ $M = M_X(\beta, \chi)$: the moduli space of

μ-semistable 1-dim'l shv's F w/ $[F] = \beta$, $\chi(F) = \chi$

(w.r.t. an ample divisor)

<u>Key</u> : M can be locally written as a critical locus

$$\begin{tikzcd} A^{sm} \arrow[d, "f"'] & \supset \{df = 0\} \simeq U \underset{\text{open}}{\subset} M \\ \mathbb{C} & \end{tikzcd}$$

$\rightsquigarrow$ ϕ_f : a natural perverse sheaf of vanishing cycles on U

[Joyce et al] : ϕ_f glue to a global perverse shf ϕ_M on M.

- $\pi : M_X(\beta, X) \longrightarrow \mathrm{Chow}_\beta(X)$: the Hilbert–Chow morph.

<u>Def</u> (Maulik–Toda) :

Define $n_{g,\beta,\chi}(X)$ by the following equality :

$$\sum_{i\in\mathbb{Z}} \chi\left({}^{p}\mathcal{H}^i(\pi_*\phi_M)\right) q^i = \sum_{g\geq 0} n_{g,\beta,\chi}\cdot\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)^{2g}$$

Idea: For a smooth curve C of genus g,

- $\pi^{-1}([C]) \cong J(C)$
- $\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)^{2g}$ is (shifted) Poincare poly of $J(C)$

get rid of these contribution.

Fact:
- For $\chi=1$, $n_{g,\beta,\chi=1} \in \mathbb{Z}$
- $n_{g,\beta,\chi} = 0$ for $g>>0$

Conj (Maulik - Toda) (χ- independence)

$$n_{g,\beta,\chi} = n_{g,\beta,\chi'}$$

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Thm 1 (Kinjo – K.)

χ-independence holds for any local curves.

Thm 2 (Davison – K.)

GW = GV holds for $X = \mathrm{Tot}_C(N)$, $\begin{cases} C : \text{genus } 2, \\ N : \text{generic} \end{cases}$

$\beta = 2[C]$, $C \hookrightarrow X$: zero section.

§3. Local curves :

$X = \mathrm{Tot}_C(N) \rightsquigarrow \beta = r[C]$.

$$\boxed{\begin{aligned} &N = L^{-1} \otimes \omega_C \oplus L \\ &\Rightarrow X \cong \mathrm{Tot}_{\mathrm{Tot}(L)}\left(\omega_{\mathrm{Tot}(L)}\right) \end{aligned}}$$

$M_N(r, \chi) := M_X(r[C], \chi)$.

- N fits into: $0 \to L^{-1} \otimes \omega_C \to N \to L \to 0$, $\deg(L) \gg 0$

  $\boxed{X = \mathrm{Tot}_C(N) \to \mathrm{Tot}_C(L)}$

- $M_L(r, X)$: the similar moduli space on $\mathrm{Tot}_C(L)$

Thm (Kinjo–Masuda, Kinjo–K.)

- $M_{\mathrm{Tot}_C(N)}(r, X) \cong \{df = 0\} \subset M_L(r, X) \xrightarrow{\exists f} \mathbb{C}$
- $\phi_{M_X} \cong \phi_f(IC_{M_L})$

$\Leftarrow$ $D^b(\mathrm{Coh}_{\leq 1}(X))$ is equivalent to a $\boxed{\text{deformed}}$ CY completion of $D^b(\mathrm{Coh}_{\leq 1}(\mathrm{Tot}_C(L)))$

$X = \mathrm{Tot}_C(N)$

$[N] \in \mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$

$\cong \mathrm{Hom}(L^{-1} \otimes \omega_C, L \otimes \omega_C)^*$

$= H^0(L^{\otimes 2})^* \ni l$

$B_L \to H^0(L^{\otimes 2}) \xrightarrow{l} \mathbb{C}$

- The Hilbert–Chow for M_L is called *Hitchin map*:

$$h : M_L \longrightarrow B_L = \underbrace{\bigoplus_{i=1}^{r} H^0(L^{\otimes i})}_{\text{affine space}}$$

$\rightsquigarrow$ $f : M_L \xrightarrow{h} B_L \xrightarrow{l} \mathbb{C}$, where l is linear

Need to understand: $\phi_l(h_* IC_{M_L})$

- For Thm 1 (X-independence), it is enough to show:

- For <u>Thm 1</u> (χ-independence), it is enough to show:

$$h_* IC_{M_L(r,\chi)} \simeq h_* IC_{M_L(r,\chi')}$$

This holds by Maulik–Shen.

<u>Key</u>: • (decomposition thm): $h_* IC_{M_L} \simeq \bigoplus_{i \in \mathbb{Z}} P_i[i]$, P_i: perverse sh's

• (Ngo's support thm.): Supp of P_i are <u>full</u>.

- For <u>Thm 2</u> (GW = GV), need to care about ϕ_ℓ.

Consider the distinguished triangle: P = one of P_i's

$$\mathbb{H}(\phi_\ell(P)) \to \mathbb{H}(P_0) \to \mathbb{H}(\psi_\ell(P))$$

where • P_0: the stalk at the origin

• ψ_ℓ: the nearby cycle functor.

$(B_L = H^0(L) \oplus H^0(L^{\otimes 2}))$

- $\mathbb{H}(B_L, P_0) \longleftrightarrow H^*(M_L, \mathbb{Q})$ studied by

<u>Expect</u>: $\underline{X = \mathrm{Tot}(L_1 \oplus L_2)}$

$n_{g,\beta} \iff$ PT using Behrend fct. [Diagonescue ...]

$\iff$ PT w/ <u>anti-diagonal $\mathbb{C}^*$</u>

$\big(B_L = H^0(L) \oplus H^0(L^{\otimes 2}) \big)$ $\qquad$ $\Leftrightarrow$ PT w/ anti-diagonal $\mathbb{C}^*$ ...]

$\Leftrightarrow$ GW w/ anti-diagonal $\mathbb{C}^*$

- $\mathbb{H}(B_L, P_0) \longleftrightarrow H^*(M_L, \mathbb{Q})$ studied by
  [de Cataldo - Hausel - Migliorini, P = W paper]

- $\mathbb{H}(M_L, \psi_\ell(P)) \simeq \mathbb{H}(P|_{U_\varepsilon})$, $\quad U_\varepsilon = \ell^{-1}(\varepsilon)$, $0 < |\varepsilon| \ll 1$.

$$U_\varepsilon \subset B_L \xrightarrow{\ell} \mathbb{C}$$

$\mathcal{C} \longrightarrow U_\varepsilon$: the universal spectral curve.

$p : \overline{J}(\mathcal{C}/U_\varepsilon) \longrightarrow U_\varepsilon$: the relative compactified Jacobian

$\rightsquigarrow$ $p_* IC_{\overline{J}} \simeq \bigoplus_i P_i|_{U_\varepsilon}[-i]$

For $a \in U_\varepsilon$, define $\nu_g(a)$ using $\chi(P_i|_{U_\varepsilon}[-i])$.

$\rightsquigarrow$ get a constructible fct. $\nu_g : U_\varepsilon \to \mathbb{Z}$.

We want to compute:

$$n_g^{U_\varepsilon} = \sum_{k \in \mathbb{Z}} k \cdot e(\nu_g^{-1}(k))$$

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For $\nu_g(a)$:

- [Maulik-Yun, Migliorini-Shende]: GV = PT for locally planar singularities
- [Oblomkov-Shende, Maulik]: PT = HOMFLY

For $e(\nu_g^{-1}(k))$:

$$\{\nu_g^{-1}(k)\}_{k \in \mathbb{Z}} \longleftrightarrow \{S_\lambda\}$$ the stratification of U_ε given by

singularity types of C_a

$$\begin{tikzcd} U_\varepsilon \arrow[dr, bend right] \arrow[r, phantom, "\subset"] & H^0(L^{\otimes 2}) \setminus \{0\} \arrow[d] \\ & \mathbb{P}H^0(L^{\otimes 2}) \arrow[r, hook] & \mathrm{Sym}^d(C) \supset T_\lambda, \ \lambda \vdash d \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P}H^0(L^{\otimes 2}) \arrow[r, hook] \arrow[d] & \mathrm{Sym}^d(C) \supset T_\lambda, \ \lambda \vdash d \arrow[d] \\ \{L^{\otimes 2}\} \arrow[r, hook] & \mathrm{Pic}^d(C) \end{tikzcd}$$

$(d = \deg(L^{\otimes 2}))$

$\rightsquigarrow \quad S_\lambda = U_\varepsilon \cap T_\lambda.$

<u>Idea</u> : realize S_λ as degeneracy locus.