All-genus WDVV recursion, quivers, and BPS invariants

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Intercontinental Moduli and Algebraic Geometry Zoominar

Recursion

Let X be a smooth projective surface and D be an ample divisor in X. Let β be a nonzero curve class of X. We use $\overline{M}_{g,m}(\mathcal{O}_X(-D), \beta)$ to denote the moduli space of m pointed genus g stable maps of class β to the total space of $\mathcal{O}_X(-D)$. Since D is ample, $\overline{M}_{g,m}(\mathcal{O}_X(-D), \beta)$ coincides with the moduli $\overline{M}_{g,m}(X, \beta)$. Let $[pt]$ be the point class of X. We consider the following primary Gromov-Witten invariants:

$$N_{g,\beta}^{\mathcal{O}_X(-D)} := \int_{[\overline{M}_{g,m}(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}}} \prod_{i=1}^{m} ev_i^*([pt])$$

By dimension constraint, we need $m = T_{\log} \cdot \beta$ where $T_{\log} = -K_X - D$. In particular, $T_{\log} \cdot \beta \geq 0$. We only consider those β such that $T_{\log} \cdot \beta > 0$.

By fixing β and summing over g, we get the following generating series

$$F_\beta^{\mathcal{O}_X(-D)} := \sum_{g \geq 0} N_{g,\beta}^{\mathcal{O}_X(-D)} h^{2g-2+T_{\log}\cdot\beta}$$

The Gopakumar-Vafa conjecture proven by Zinger, Ionel, Parker, Doan, Walpuski tells us that we can reorganized it as

$$F_\beta^{\mathcal{O}_X(-D)} = \sum_{g \geq 0} n_{g,\beta}^{\mathcal{O}_X(-D)} (2\sin(h/2))^{2g-2+T_{\log}\cdot\beta}$$

where $n_{g,\beta}^{\mathcal{O}_X(-D)} \in \mathbb{Z}$. In particular, $F_\beta^{\mathcal{O}_X(-D)} \in \mathbb{Q}((-q)^{-\frac{1}{2}})$ with $q = e^{ih}$.

For a divisor $D \subset X$, the *virtual genus* $g(D)$ of a divisor D is defined by

$$g(D) := 1 - \frac{1}{2} T_{\log} \cdot D$$

In this talk, we will further assume D to be of virtual genus 0. Then

Theorem (Bousseau-W.)

Let X be a smooth projective surface and D be an ample divisor in X with virtual genus 0. Then, we have the following recursive formula:

$$F_\beta^{\mathcal{O}_X(-D)} = \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} F_{\beta_1}^{\mathcal{O}_X(-D)} F_{\beta_2}^{\mathcal{O}_X(-D)} \left(q^{D\cdot\beta_1} + q^{-D\cdot\beta_1} - 2\right) \binom{T_{\log}\cdot\beta - 3}{T_{\log}\cdot\beta_1 - 1}$$

if $T_{\log} \cdot \beta \geq 3$.

Genus 0 recursion

If we specialize the above recursion to genus 0, we get

$$N_{0,\beta}^{\mathcal{O}_X(-D)} = -\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} N_{0,\beta_1}^{\mathcal{O}_X(-D)} N_{0,\beta_2}^{\mathcal{O}_X(-D)} (D\cdot\beta_1)^2 \binom{T_{\log}\cdot\beta-3}{T_{\log}\cdot\beta_1-1}$$

This genus zero recursion can also be deduced from the WDVV equation of relative GW theory together with local/relative correspondence:

$$(T_{\log}\cdot D) N_{0,\beta}^{\mathcal{O}_X(-D)} = -2\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} N_{0,\beta_1}^{\mathcal{O}_X(-D)} N_{0,\beta_2}^{\mathcal{O}_X(-D)} (D\cdot\beta_1)^2 \binom{T_{\log}\cdot\beta-3}{T_{\log}\cdot\beta_1-1}$$

So the requirement of D to be virtual genus zero, i.e., $T_{\log}\cdot D = 2$ is necessary. And this is also why we treat the previous recursion as all-genus WDVV recursion.

The genus 0 consideration can also be used to show that the requirement of D to be ample can not be relaxed to be nef.

For example, let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ and D be a fiber of X. We have the following recursion:

$$N_{0,\beta}^{\mathcal{O}_X(-D)} = - \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} N_{0,\beta_1}^{\mathcal{O}_X(-D)} N_{0,\beta_2}^{\mathcal{O}_X(-D)} (D \cdot \beta_1)^2 \binom{T_{\log} \cdot \beta - 3}{T_{\log} \cdot \beta_1 - 1} + (D \cdot \beta)^2 N_{0,\beta-f}^{\mathcal{O}_X(-D)}$$

if $T_{\log} \cdot \beta \geq 3$, where f stands for the fiber class. Still, it can be deduced from the WDVV equation of relative GW theory together with local/relative correspondence.

The appearance of the additional term $(D \cdot \beta)^2 N_{0,\beta-f}^{\mathcal{O}_X(-D)}$ follows from the calculation of WDVV equation, there will be a contribution from a splitting of curve class $\beta = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \neq 0$ but $D \cdot \beta_2 = 0$. Such kind of contribution can not appear if D is ample.

Deformation equivalence

According to Lanteri and Palleschi, the condition that X has an ample divisor D with virtual genus 0 actually forces (X, D) to be the following two types:

(1) $(X, D) = (\mathbb{P}^2, \text{line})$ or $(\mathbb{P}^2, \text{conic})$;

(2) X is a Hirzebruch surface and $\mathcal{O}_X(D) \otimes \mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(1)$ for any fiber f of the Hirzebruch surface X.

Let us give a more detailed description of case (2). First, we specialize X to be $F_n = \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$ where $n \in \mathbb{Z}_{\geq 0}$. Let C_n and C_{-n} be the sections of X with intersection numbers n and $-n$ respectively. The requirements that $\mathcal{O}_X(D) \otimes \mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(1)$ and D is ample will imply that $D = C_n + sf$ with $s > 0$.

A deformation of F_n to F_{n+2} is given by

$$\left\{([x_0 : x_1], [y_0 : y_1 : y_2], t) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C} \middle| x_0^{n+2} y_1 - x_1^{n+2} y_0 + t x_0^{n+1} x_1 y_2 = 0\right\}$$

Under such a deformation the divisor $C_n + (s+1)f$ of F_n will deform to be $C_{n+2} + sf$ of F_{n+2} and the curve class $d_1 C_{-n} + d_2 f$ deforms to be $d_1 C_{-n-2} + (d_1 + d_2)f$. So after a sequence of deformation, we have

$$\begin{aligned} \mathrm{GW}(\mathcal{O}_{F_n}(-C_n - sf)) &\simeq \mathrm{GW}(\mathcal{O}_{F_{n+2}}(-C_{n+2} - (s-1)f)) \simeq \\ &\cdots \simeq \mathrm{GW}(\mathcal{O}_{F_{n+2s-2}}(-C_{n+2s-2} - f)) \end{aligned}$$

The all-genus WDVV equation is compatible with the above deformation. So it is enough to consider only the following three types of pairs (X, D):

$$(\mathbb{P}^2, \text{line}), \quad (\mathbb{P}^2, \text{conic}), \quad (F_n, C_n + f),\ n \geq 0.$$

Sketch of the proof

The proof for the all-genus WDVV recursion goes as follows:

Local GW-theroy $\longrightarrow$ Relative (Log) GW-theory $\longrightarrow$ Quiver DT-theory

We first translate the recursion into a recursion for relative GW-invariants using the local/relative correspondence. We then further translate it into a recursion for quiver DT-invariants via a GW/quiver correspondence derived by Bousseau from the GW/Kronecker correspondence for log Calabi-Yau surfaces. The recursion on the quiver DT-side can then be deduced using the geometric properties of the quiver moduli.

Local/Relative correspondence

Let $\overline{M}_{g,m}(X/D,\beta)$ be the moduli space of m-pointed genus g relative stable maps of class β to (X,D) with only one contact condition of maximal tangency along D. We consider

$$N_{g,\beta}^{X/D} := \int_{[\overline{M}_{g,m}(X/D,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{m} ev_i^*([pt])(-1)^g \lambda_g$$

The genus 0 local/relative correspondence proven by van Garrel-Graber-Ruddat tells us that

$$N_{0,\beta}^{\mathcal{O}_X(-D)} = \frac{(-1)^{D\cdot\beta-1}}{D\cdot\beta} N_{0,\beta}^{X/D}$$

But for $g > 0$, we have correction terms:

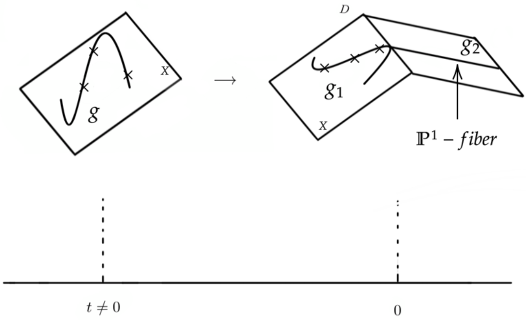
$$N_{g,\beta}^{\mathcal{O}_X(-D)} = \frac{(-1)^{D\cdot\beta-1}}{D\cdot\beta} N_{g,\beta}^{X/D} + \cdots$$

The higher genus generalization of the local/relative correspondence was proven by Bousseau-Fan-Guo-W. Fortunately, when D is ample and virtual genus 0 and $T_{\log}\cdot\beta > 0$, these correction terms can be explicitly calculated:

By fixing β with $T_{\log}\cdot\beta > 0$ and summing over g, we have

$$F_\beta^{X/D} := \sum_{g\geq 0} N_{g,\beta}^{X/D} h^{2g-1+T_{\log}\cdot\beta}$$

It is related to $F_\beta^{\mathcal{O}_X(-D)}$ as follows:



Theorem (Bousseau-W.)

$$F_\beta^{\mathcal{O}_X(-D)} = F_\beta^{X/D} \frac{(-1)^{D\cdot\beta-1}}{2\sin(\frac{(D\cdot\beta)h}{2})}$$

GW/quiver correspondence

After converting local invariants to relative invariants, we need further translate them into quiver DT-invariants.

A quiver Q consists of a finite set of vertices Q_0 together with a finite set of arrows $Q_1 = \{\alpha : i \to j | i, j \in Q_0\}$.

A representation of a quiver Q consists of a tuple of vector space $(V_i)_{i \in Q_0}$ indexed by the vertices, plus a tuple of linear morphisms $(V_\alpha : V_i \to V_j)_{\alpha : i \to j}$ indexed by the arrows. By fixing a dimension vector $\underline{d} = (\dim V_i)_{i \in Q_0}$ and a stability θ, we could construct a moduli of θ-stable (semistable) quiver representations with fixed dimension vector $\underline{d}$.

For all the quivers considered in this talk, the stability condition will always choose to be $\theta(\cdot) = \{\underline{d}, \cdot\}$ where $\{\cdot, \cdot\}$ is the antisymmetrized Euler form.

Given a projective moduli space Y of semistable quiver representations, the corresponding refined Donaldson-Thomas invariant $\Omega_Y(q)$ is defined as follows. If the stable locus of Y is not empty, then

$$\Omega_Y(q) = (-q^{1/2})^{-\dim_{\mathbb{C}} Y} \sum_{i=0}^{2\dim_{\mathbb{C}} Y} \dim \mathrm{IH}^i(Y, \mathbb{Q})(-q^{1/2})^i$$

i.e., $\Omega_Y(q)$ is the shifted Poincaré polynomial of the intersection cohomology of Y. Otherwise, if the stable locus of Y is empty, then $\Omega_Y(q) = 0$.

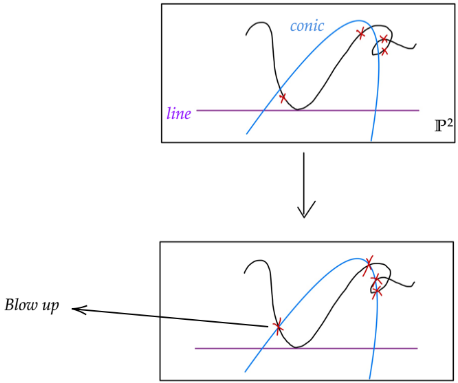
The correspondence between log GW-invariants of log Calabi-Yau surfaces and quiver DT invariants starts from the work of Gross-Pandharipande-Siebert on the GW-side and Reineke on the quiver side. This genus 0/DT correspondence was later generalized by Bousseau to higher genus/refined DT correspondence.

For this talk, we are interested in relative GW-invariants of the following three types of pairs (X, D):

$$(\mathbb{P}^2, \text{line}), \quad (\mathbb{P}^2, \text{conic}), \quad (F_n, C_n + f),\ n \geq 0$$

We are in a situation of Fano-like cases instead of Calabi-Yau. So the first step is to convert the Fano problems into Calabi-Yau ones.

Here is an illustration of how it works:

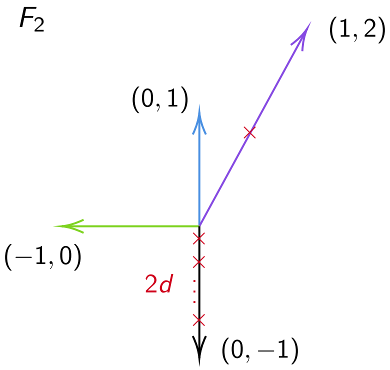


$$(\mathbb{P}^2, L) \longrightarrow (\mathbb{P}^2, L \sqcup C) \longrightarrow (Bl(\mathbb{P}^2), L \sqcup \widetilde{C})$$

Next step, find a toric model for $(Bl(\mathbb{P}^2), L \sqcup \widetilde{C})$:

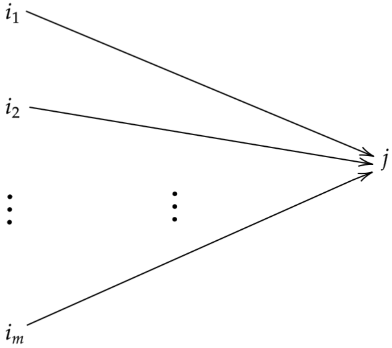
$$
\begin{array}{ccc}
 & \left(\widetilde{Z}, \widetilde{D}\right) & \\
 \swarrow^{\pi_{\widetilde{Z}}} & & \searrow^{\pi_{\bar{Z}}} \\
\left(Bl\left(\mathbb{P}^2\right), L \sqcup \widetilde{C}\right) & & (\bar{Z}, \bar{D})
\end{array}
$$

Here $(\bar{Z}, \bar{D})$ is a toric log Calabi-Yau surface with maximal boundary and $\pi_{\bar{Z}}$ is a sequence of interior blow-ups and $\pi_{\widetilde{Z}}$ is a sequence of corner blow-ups. The reason to find such a toric model is that on the GW-side, the GW/quiver correspondence is dealing with log GW-invariants of $(\widetilde{Z}, \widetilde{D})$. And a quiver can be constructed via the toric data of $(\bar{Z}, \bar{D})$ and $\pi_{\bar{Z}}$. For different toric models, the quivers are related via mutations.



The above gives a toric model for $(Bl(\mathbb{P}^2), L \sqcup \widetilde{C})$ by setting $\bar{Z} = F_2$ and $\bar{D}$ = union of toric divisors. $\pi_{\bar{Z}}$ is a sequence of blow-ups at the red points, and $\pi_{\widetilde{Z}}$ is a sequence blow-downs of divisors associated to the purple and blue rays.

We can then construct a quiver from the above toric data:



with $m = T_{\log} \cdot \beta = 2d$ and dimension vector $\underline{d} = \sum e_{i_k} + de_j \in \mathbb{N}Q_0$. Recall that, the stability θ is always given by $\theta(\cdot) = \{\underline{d}, \cdot\}$.

We use $M_{d[l]}^{\mathbb{P}^2/L}$ to denote the corresponding moduli of θ-semistable quiver representations with $[l]$ be the line class. Then the GW/quiver correspondence in this case becomes

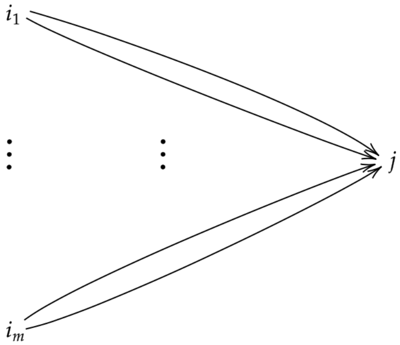
$$\Omega_{M_{d[l]}^{\mathbb{P}^2/L}}(q) = F_{d[l]}^{\mathbb{P}^2/L} \frac{(-1)^{d-1}}{(2\sin(h/2))^{2d-1}},\ q = e^{ih}$$

Specialize $q = 1$, we get

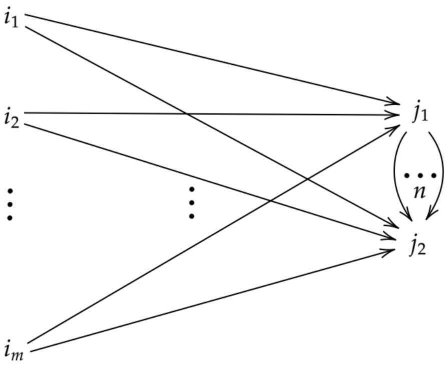
$$\chi_{IC}(M_{d[l]}^{\mathbb{P}^2/L}) = N_{0,d[l]}^{\mathbb{P}^2/L}$$

We remark that the above identity is first derived by Reineke and Weist using a direct computation on both GW and quiver sides. The higher genus generalization was given by Bousseau using the procedures I mentioned above.

Using Bousseau's method, we can find the quiver for $(\mathbb{P}^2, \text{conic})$:



Also the quiver for $(F_n, C_n + f)$:



For all of the three types of quivers, the number m of vertices on the LHS always equal to $T_{\log} \cdot \beta$ and the dimensions put on the vertices on the LHS are all 1. But the dimensions put on the vertices on the RHS will be determined by β.

For each of the above three types pairs (X, D), we use $M_\beta^{X/D}$ to denote the corresponding quiver moduli. Then we have

Theorem (Bousseau)

$$\Omega_{M_\beta^{X/D}}(q) = F_\beta^{X/D} \frac{(-1)^{D\cdot\beta+1}}{(2\sin(h/2))^{T_{\log}\cdot\beta-1}},\ q = e^{ih}$$

if $T_{\log} \cdot \beta > 0$.

BPS invariants

Combining local/relative correspondence with GW/quiver correspondence, we have

$$F_\beta^{\mathcal{O}_X(-D)} = \Omega_{M_\beta^{X/D}}(q)\frac{(2\sin(h/2))^{T_{\log}\cdot\beta-1}}{2\sin(\frac{(D\cdot\beta)h}{2})},\ q = e^{ih}$$

Together with

$$F_\beta^{\mathcal{O}_X(-D)} = \sum_{g\geq 0} n_{g,\beta}^{\mathcal{O}_X(-D)}(2\sin(h/2))^{2g-2+T_{\log}\cdot\beta}$$

It yields

$$\sum_{g\geq 0} n_{g,\beta}^{\mathcal{O}_X(-D)}(2\sin(h/2))^{2g} = \Omega_{M_\beta^{X/D}}(q)\frac{2\sin(h/2)}{2\sin(\frac{(D\cdot\beta)h}{2})}$$

Note that

$$\frac{2\sin(\frac{(D\cdot\beta)h}{2})}{2\sin(h/2)} = \frac{q^{(D\cdot\beta)/2} - q^{-(D\cdot\beta)/2}}{q^{1/2} - q^{-1/2}} = (-1)^{D\cdot\beta-1} P_{\mathbb{P}^{D\cdot\beta-1}}$$

where

$$P_{\mathbb{P}^{D\cdot\beta-1}} = (-q^{1/2})^{-(D\cdot\beta-1)} \sum_{i=0}^{D\cdot\beta-1} \dim \mathrm{H}^i(\mathbb{P}^{D\cdot\beta-1}, \mathbb{Q})(-q^{1/2})^i$$

So $\Omega_{M_\beta^{X/D}}(q)$ could divide $P_{\mathbb{P}^{D\cdot\beta-1}}$. This actually has a geometry meaning:

Let Q be a quiver corresponding to (X, D) as above. We construct a new quiver Q_- from Q by deducing the number of vertices on the LHS by one, and keep the dimensions putting on these vertices. We use $M_\beta^{\mathcal{O}_X(-D)}$ to denote the corresponding quiver moduli associated to Q_-. Then we have

$$\Omega_{M_\beta^{X/D}}(q) = P_{\mathbb{P}^{D\cdot\beta-1}} \Omega_{M_\beta^{\mathcal{O}_X(-D)}}$$

The reason is that the framed quiver moduli of Q_- actually gives a small resolution of $M_\beta^{X/D}$ and the framing quiver moduli is a $\mathbb{P}^{D\cdot\beta-1}$-bundle of $M_\beta^{\mathcal{O}_X(-D)}$ because $M_\beta^{\mathcal{O}_X(-D)}$ is smooth. This was first shown by Reineke and Weist for the quivers associated to $(\mathbb{P}^2, \text{line})$. Their arguments can be generalized to other two types of quivers.

Theorem (Bousseau-W.)

For each of the above three types pairs (X, D), we have

$$\sum_g n_{g,\beta}^{\mathcal{O}_X(-D)}(2\sin(h/2))^{2g} = (-1)^{D\cdot\beta-1}\Omega_{M_\beta^{\mathcal{O}_X(-D)}}(q)$$

if $T_{\log}\cdot\beta > 0$.

Note that using the deformation equivalence, the above theorem can be easily generalized to all the pairs (X, D) such that D is ample and virtual genus 0. The geometry properties of $M_\beta^{\mathcal{O}_X(-D)}$ will have some interesting consequence.

We define the BPS Castelnuovo number to be

$$g_\beta^{\mathcal{O}_X(-D)} := \sup\{g \mid n_{g,\beta}^{\mathcal{O}_X(-D)} \neq 0\}$$

Corollary

(1) $g_\beta^{\mathcal{O}_X(-D)} = \frac{(K_X+\beta)\cdot\beta}{2} + 1$;

(2) $n_{g,\beta}^{\mathcal{O}_X(-D)} = (-1)^{g+D\cdot\beta-1}$, *if* $g = \frac{(K_X+\beta)\cdot\beta}{2} + 1 \geq 0$.

if $T_{\log}\cdot\beta > 0$ *and* $M_\beta^{\mathcal{O}_X(-D)} \neq \emptyset$.

Note that case (1) matches with the genus-degree formula, and case (2) actually follows from the geometric fact that the moduli space $M_\beta^{\mathcal{O}_X(-D)}$ is connected.

Proof of the recursion

After replacing $\Omega_{M_\beta^{X/D}}(q)$ by $\Omega_{M_\beta^{\mathcal{O}_X(-D)}}(q)$ in

$$F_\beta^{\mathcal{O}_X(-D)} = \Omega_{M_\beta^{X/D}}(q)\frac{(2\sin(h/2))^{T_{\log}\cdot\beta-1}}{2\sin(\frac{(D\cdot\beta)h}{2})},\ q = e^{ih},$$

we get

$$F_\beta^{\mathcal{O}_X(-D)} = (-1)^{D\cdot\beta-1}\Omega_{M_\beta^{\mathcal{O}_X(-D)}}(q)(2\sin(h/2))^{T_{\log}\cdot\beta-2}$$

After plugging into the recursion

$$F_\beta^{\mathcal{O}_X(-D)} = \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} F_{\beta_1}^{\mathcal{O}_X(-D)}F_{\beta_2}^{\mathcal{O}_X(-D)}\left(q^{D\cdot\beta_1}+q^{-D\cdot\beta_1}-2\right)\binom{T_{\log}\cdot\beta-3}{T_{\log}\cdot\beta_1-1}$$

It becomes

$$\Omega_{M_\beta^{\mathcal{O}_X(-D)}} = \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2>0}} \Omega_{M_{\beta_1}^{\mathcal{O}_X(-D)}} \Omega_{M_{\beta_2}^{\mathcal{O}_X(-D)}} \left(P_{\mathbb{P}^{D\cdot\beta_1-1}}\right)^2 \binom{T_{\log}\cdot\beta-3}{T_{\log}\cdot\beta_1-1}$$

Here we recall that

$$P_{\mathbb{P}^{D\cdot\beta_1-1}} = (-1)^{D\cdot\beta_1-1} \frac{q^{\frac{D\cdot\beta_1}{2}} - q^{-\frac{D\cdot\beta_1}{2}}}{q^{1/2}-q^{-1/2}}$$

When $(X,D) = (\mathbb{P}^2, \text{line})$, the above recursion was first derived by Reineke and Weist. Their arguments can actually be generalized to give a proof of the recursion for other two types of quivers.

The key formula used in Reineke and Weist's proof is a formula relating DT-invariants of framed moduli spaces to unframed ones.

Let Q be a quiver. A framed quiver of $\widehat{Q}$ can be derived from Q by adding an additional vertex i_0 and n_i arrows from i_0 to $i \in Q_0$. By putting dimension 1 to i_0, we can extend the dimension vector $\underline{d}$ of Q to a dimension vector $\underline{\hat{d}}$ of $\widehat{Q}$. Assume that the stability θ of Q is normalized, i.e., $\theta(\underline{d}) = 0$. We then also extend the stability θ to a stability $\hat{\theta}$ of $\widehat{Q}$ by adding the entry 1 for the vertex i_0.

Then the moduli space $M^{\theta,\mathrm{fr}}_{\underline{d},\underline{n}}(Q)$ of θ-semistable $\underline{n}$-framed representations of Q with dimension $\underline{d}$ is simply the moduli space $M^{\hat{\theta}-sst}_{\underline{\hat{d}}}(\widehat{Q})$ of $\hat{\theta}$-semistable representations of $\widehat{Q}$ with dimension $\underline{\hat{d}}$.

The formula relating DT-invariants of framed moduli spaces to unframed ones can be stated as follows.

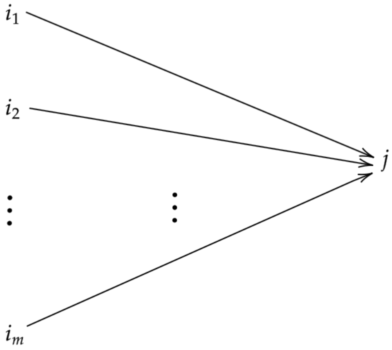
Let Q be a quiver with stability θ. We use Λ_0^+ to denote the set of nonzero dimension vectors $\underline{d}$ such that $\theta(\underline{d}) = 0$. Then

$$1 + \sum_{\underline{d} \in \Lambda_0^+} \Omega_{M^{\theta,\mathrm{fr}}_{\underline{d},\underline{n}}} (-1)^{\underline{n}\cdot\underline{d}} x^{\underline{d}} = \mathrm{Exp}\left(\sum_{\underline{d} \in \Lambda_0^+} P_{\mathbb{P}^{\underline{n}\cdot\underline{d}-1}} \Omega_{M^{\theta-sst}_{\underline{d}}} (-1)^{\underline{n}\cdot\underline{d}} x^{\underline{d}} \right)$$

where $\mathrm{Exp}(\cdot)$ is the plethystic exponential:

$$\mathrm{Exp}(f) = \exp\left(\sum_{k=1}^{\infty} \frac{f(x^k)}{k}\right)$$

Recall that the quivers Q associated to the pair $(\mathbb{P}^2, \text{line})$ are



with $\underline{d} = \sum e_{i_k} + de_j$, $\theta = \sum e^*_{i_k} - 2e^*_j$. By specifying the number of vertices on the LHS , we also use $M^L_{m,d}$ to denote the corresponding quiver moduli, and use $M^{L,\text{fr}}_{m,d}$ to denote the corresponding framed moduli.

By further setting $m = 2d$, $\underline{n} = e_j$ and taking the coefficient of $x_{i_1} \cdots x_{i_{2d}} x_j^d$ on both sides of the formula relating DT-invariants of framed moduli spaces to unframed ones:

$$1 + \sum_{\underline{d} \in \Lambda_0^+} \Omega_{M^{\theta,\mathrm{fr}}_{\underline{d},\underline{n}}} (-1)^{\underline{n}\cdot\underline{d}} x^{\underline{d}} = \mathrm{Exp}\left(\sum_{\underline{d} \in \Lambda_0^+} P_{\mathbb{P}^{\underline{n}\cdot\underline{d}-1}} \Omega_{M^{\theta-sst}_{\underline{d}}} (-1)^{\underline{n}\cdot\underline{d}} x^{\underline{d}} \right)$$

we have

$$\Omega_{M^{L,\mathrm{fr}}_{2d,d}} = \sum_{\substack{a_1 + a_2 \cdots + a_d = d \\ a_i \geq 0}} \frac{(2d)!}{\prod_{k=1}^{d} \left((2k)!\right)^{a_k} (a_k)!} \prod_{k=1}^{d} \left(P_{\mathbb{P}^{k-1}} \Omega_{M^L_{2k,k}} \right)^{a_k}$$

Note that

$$M_{d[l]}^{\mathcal{O}_{\mathbb{P}^2}(-1)} = \Omega_{M^L_{2d-1,d}}$$

To get a recursion for $\Omega_{M^L_{2d-1,d}}$, we need the following key geometric properties of quiver moduli:

$$M^{L,\mathrm{fr}}_{2d,d} \simeq M^L_{2d+1,d} \simeq M^L_{2d+1,d+1}, \quad \Omega_{M^L_{2d,d}} = P_{\mathbb{P}^{d-1}} \Omega_{M^L_{2d-1,d}}$$

The second isomorphism is induced by the reflection functor in $\mathrm{Rep}_{\mathbb{C}}\, Q$. Then the above formula becomes

$$z^L_{d+1} = \sum_{\substack{a_1+a_2\cdots+a_d=d \\ a_i \geq 0}} \frac{(2d)!}{\prod_{k=1}^d \left((2k)!\right)^{a_k} (a_k)!} \prod_{k=1}^d \left(\left(P_{\mathbb{P}^{k-1}}\right)^2 z^L_k\right)^{a_k}$$

with $z^L_d(q) = \Omega_{M^L_{2d-1,d}}(q)$.

So by summing over d, we have

$$1+\sum_{d>0}\frac{z^L_{d+1}}{(2d)!}x^d=\exp\left(\sum_{k>0}\frac{\left(P_{\mathbb{P}^{k-1}}\right)^2 z^L_k}{(2k)!}x^k\right)$$

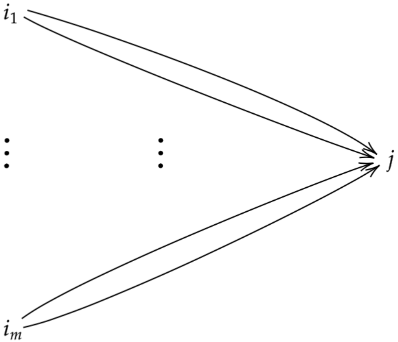
By further taking a derivative $2x\frac{d}{dx}$ on both sides, we have

$$\sum_{d>0}\frac{z^L_{d+1}}{(2d-1)!}x^d=\left(\sum_{k>0}\frac{\left(P_{\mathbb{P}^{k-1}}\right)^2 z^L_k}{(2k-1)!}x^k\right)\left(\sum_{d\geq 0}\frac{z^L_{d+1}}{(2d)!}x^d\right)$$

So by taking the coefficients of x^{d-1} on both sides, we get the recursion

$$z^L_d=\sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} z^L_{d_1}z^L_{d_2}\left(P_{\mathbb{P}^{d_1-1}}\right)^2\binom{2d-3}{2d_1-1}$$

For the quivers associated to $(\mathbb{P}^2, \text{conic})$:



We use $M^C_{m,d}$ to denote the corresponding quiver moduli, and use $M^{C,\text{fr}}_{m,d}$ to denote the corresponding framed moduli.

The key geometric properties are

$$M_{d,d}^{C,\mathrm{fr}} \simeq M_{d+1,d}^{C} \simeq M_{d+1,d+2}^{C}, \quad \Omega_{M_{d,d}^{C}} = P_{\mathbb{P}^{2d-1}} \Omega_{M_{d-1,d}^{C}}$$

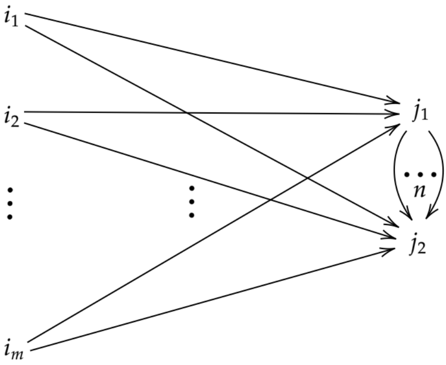
Still the second isomorphism is induced by reflection functor and can be proven in a similar way. It yields

$$1 + \sum_{d>0} \frac{z_{d+2}^{C}}{d!} x^{d} = \exp\left(\sum_{k>0} \frac{\left(P_{\mathbb{P}^{2k-1}}\right)^{2} z_{k}^{C}}{(k)!} x^{k} \right)$$

with $z_d^C = \Omega_{M_{d-1,d}^{C}}$. By taking a derivative $x\frac{d}{dx}$ on both sides, we get the recursion

$$z_d^C = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} z_{d_1}^C z_{d_2}^C P_{\mathbb{P}^{2d_1-1}} \binom{d-3}{d_1-1}$$

For the quivers associated to $(F_n, C_n + f)$:



We use $M^{F_n}_{m,d_1,d_2}$ to denote the corresponding quiver moduli, and use $M^{F_n,\mathrm{fr}}_{m,d_1,d_2}$ to denote the corresponding framed moduli.

The key geometric properties are

$$M^{F_n,\mathrm{fr}}_{m,d_1,d_2} \simeq M^{F_n}_{m+1,d_1+1,d_2+n+1},\ \Omega_{M^{F_n}_{m,d_1,d_2}} = P_{\mathbb{P}^{d_1+d_2-1}}\Omega_{M^{F_n}_{m-1,d_1,d_2}}$$

with $m=(1-n)d_1+d_2$. It yields

$$1+\sum_{\substack{(1-n)d_1+d_2\geq 0\\ d_1+d_2>0}} \frac{z^{F_n}_{d_1+1,d_2+n+1}}{((1-n)d_1+d_2)!}x_1^{d_1}x_2^{d_2} = G\exp\left(\sum_{(1-n)k_1+k_2>0}\frac{\left(P_{\mathbb{P}^{d_1+d_2-1}}\right)^2 z^{F_n}_{k_1,k_2}}{((1-n)k_1+k_2)!}x_1^{k_1}x_2^{k_2}\right)$$

with $z^{F_n}_{d_1,d_2}=\Omega_{M^{F_n}_{(1-n)d_1+d_2-1,d_1,d_2}}$ and

$$G=1+\sum_{\substack{(1-n)d_1+d_2=0\\ d_1+d_2>0}} z^{F_n}_{d_1+1,d_2+n+1}x_1^{d_1}x_2^{d_2}$$

By taking a derivative $(1-n)x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}$ on both sides, we get the recursion

$$z^{F_n}_{d_1,d_2} = \sum_{\substack{k_1+k_1'=d_1\\ k_2+k_2'=d_2}} z^{F_n}_{k_1,k_2} z^{F_n}_{k_1',k_2'}\left(P_{\mathbb{P}^{k_1+k_2-1}}\right)^2\binom{(1-n)d_1+d_2-3}{(1-n)k_1+k_2-1}$$

Numerical results

Table: GW-invariants of $\mathcal{O}_{\mathbb{P}^2}(-1)$

d	$F_d^{\mathcal{O}_{\mathbb{P}^2}(-1)}$
1	1
2	$(q-1)^2/q$
3	$(q-1)^4(q^2+5q+1)/q^3$
4	$(q-1)^6(q^6+7q^5+29q^4+64q^3+29q^2+7q+1)/q^6$
5	$(q-1)^8(q^{12}+9q^{11}+46q^{10}+175q^9+506q^8$ $+1138q^7+1727q^6+1138q^5+506q^4+175q^3+46q^2+9q+1)/q^{10}$
6	$(q-1)^{10}(q^{20}+11q^{19}+67q^{18}+298q^{17}+1080q^{16}+3313q^{15}+8770q^{14}$ $+20253q^{13}+40352q^{12}+67279q^{11}+84792q^{10}+67279q^9+40352q^8$ $+20253q^7+8770q^6+3313q^5+1080q^4+298q^3+67q^2+11q+1)/q^{15}$

Table: BPS states $n_{g,d}$ for $\mathcal{O}_{\mathbb{P}^2}(-1)$

$d \backslash g$	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	-1	0	0	0	0	0	0
3	7	-1	0	0	0	0	0
4	-138	66	-13	1	0	0	0
5	5477	-5734	3031	-970	190	-21	1

Table: GW-invariants of $\mathcal{O}_{\mathbb{P}^2}(-2)$

d	$F_d^{\mathcal{O}_{\mathbb{P}^2}(-2)}$
1	$-(-q)^{1/2}/(q-1)$
2	-1
3	$(q-1)(q^2+2q+1)/(-q)^{3/2}$
4	$(q-1)^2(q^6+3q^5+7q^4+10q^3+7q^2+3q+1)/q^4$
5	$-(q-1)^3(q^{12}+4q^{11}+11q^{10}+25q^9+46q^8$ $+71q^7+84q^6+71q^5+46q^4+25q^3+11q^2+4q+1)/(-q)^{15/2}$
6	$(q-1)^4(q^{20}+5q^{19}+16q^{18}+41q^{17}+92q^{16}+182q^{15}+323q^{14}+522q^{13}+759q^{12}+978q^{11}$ $+1074q^{10}+978q^9+759q^8+522q^7+323q^6+182q^5+92q^4+41q^3+16q^2+5q+1)/q^{12}$

Table: BPS states $n_{g,d}$ for $\mathcal{O}_{\mathbb{P}^2}(-2)$

d \ g	0	1	2	3	4	5	6
1	-1	0	0	0	0	0	0
2	-1	0	0	0	0	0	0
3	-4	1	0	0	0	0	0
4	-32	28	-9	1	0	0	0
5	-400	792	-721	365	-105	16	-1

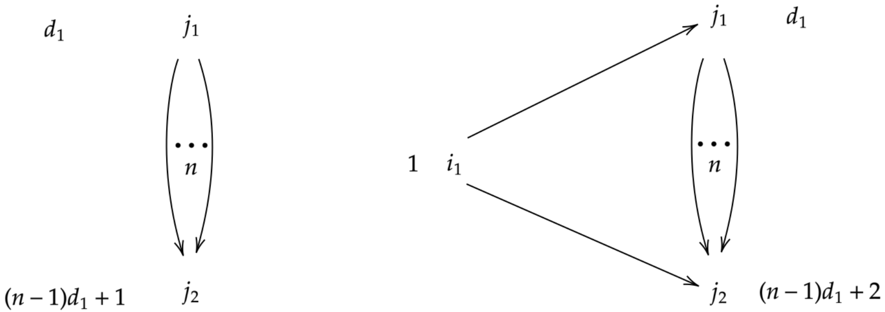
Table: GW-invariants of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$

(d_1, d_2)	$F_{d_1,d_2}^{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)}$
$(1,0)$	$(-q)^{1/2}/(q-1)$
$(2,0)$	0
$(3,0)$	0
$(1,1)$	-1
$(2,1)$	$(q-1)/(-q)^{1/2}$
$(3,1)$	$(q-1)^2/q$
$(2,2)$	$(q-1)^2(q^2+4q+1)/q^2$
$(3,2)$	$(q-1)^3(q^4+5q^3+12q^2+5q+1)/(-q)^{7/2}$
$(3,3)$	$-(q-1)^4(q^8+6q^7+23q^6+58q^5+94q^4+58q^3+23q^2+6q+1)/q^6$

Table: BPS states $n_{g,(d_1,d_2)}$ for $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$

(d_1, d_2) \ g	0	1	2	3	4
$(1,0)$	1	0	0	0	0
$(2,0)$	0	0	0	0	0
$(3,0)$	0	0	0	0	0
$(1,1)$	-1	0	0	0	0
$(2,1)$	1	0	0	0	0
$(3,1)$	-1	0	0	0	0
$(2,2)$	-6	1	0	0	0
$(3,2)$	24	-9	1	0	0
$(3,3)$	-270	220	-79	14	-1

For (F_n, C_n+f) with $n\geq 1$ and $\beta=d_1C_{-n}+d_2f$, we need to determine those $F_{d_1,d_2}^{\mathcal{O}_{F_n}(-C_n-f)}$ such that $T_{\log}\cdot\beta=(1-n)d_1+d_2<3$. It corresponds to determine Donaldson-Thomas invariants for the following quivers:



When $n=1,2$, these initial $F_{d_1,d_2}^{\mathcal{O}_{F_n}(-C_n-f)}$ can be explicitly determined.

Table: GW invariants of $\mathcal{O}_{F_1}(-C_1 - f)$

(d_1, d_2)	$F_{d_1,d_2}^{\mathcal{O}_{F_1}(-C_1-f)}$
$(0,1)$	$(-q)^{1/2}/(q-1)$
$(1,1)$	$-(-q)^{1/2}/(q-1)$
$(1,2)$	1
$(1,3)$	$-(q-1)/(-q)^{1/2}$
$(1,4)$	$-(q-1)^2/q$
$(2,2)$	-1
$(2,3)$	$-(q-1)(q^2+3q+1)/(-q)^{3/2}$
$(2,4)$	$(q-1)^2(q^4+4q^3+8q^2+4q+1)/q^3$
$(3,3)$	$(q-1)(q^2+2q+1)/(-q)^{3/2}$
$(3,4)$	$-(q-1)^2(q^6+4q^5+11q^4+17q^3+11q^2+4q+1)/q^4$
$(4,4)$	$(q-1)^2(q^6+3q^5+7q^4+10q^3+7q^2+3q+1)/q^4$

Table: BPS states $n_{g,(d_1,d_2)}$ for $\mathcal{O}_{F_1}(-C_1 - f)$

(d_1, d_2) \ g	0	1	2	3
$(0,1)$	1	0	0	0
$(1,1)$	-1	0	0	0
$(1,2)$	1	0	0	0
$(1,3)$	-1	0	0	0
$(1,4)$	1	0	0	0
$(2,2)$	-1	0	0	0
$(2,3)$	5	-1	0	0
$(2,4)$	-18	8	-1	0
$(3,3)$	-4	1	0	0
$(3,4)$	49	-36	10	-1
$(4,4)$	-32	28	-9	1

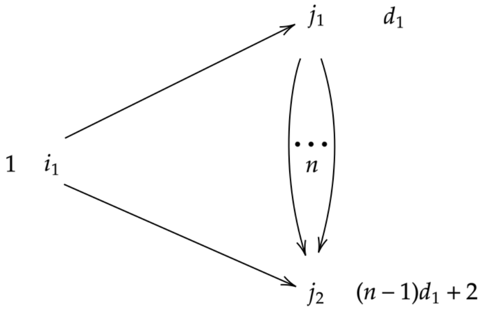
Table: GW invariants of $\mathcal{O}_{F_2}(-C_2 - f)$

(d_1, d_2)	$F_{d_1,d_2}^{\mathcal{O}_{F_2}(-C_2-f)}$
$(0,1)$	$(-q)^{1/2}/(q-1)$
$(1,2)$	$(-q)^{1/2}/(q-1)$
$(1,3)$	-1
$(1,4)$	$(q-1)/(-q)^{1/2}$
$(1,5)$	$(q-1)^2/q$
$(1,6)$	$(q-1)^3/(-q)^{3/2}$
$(2,3)$	$(-q)^{1/2}/(q-1)$
$(2,4)$	$-(q^2+2q+1)/q$
$(2,5)$	$(q-1)(q^4+3q^3+5q^2+3q+1)/(-q)^{5/2}$
$(2,6)$	$(q-1)^2(q^6+4q^5+8q^4+12q^3+8q^2+4q+1)/q^4$
$(3,4)$	$(-q)^{1/2}/(q-1)$
$(3,5)$	$-(q^4+2q^3+5q^2+2q+1)/q^2$
$(3,6)$	$(q-1)(q^8+3q^7+8q^6+14q^5+20q^4+14q^3+8q^2+3q+1)/(-q)^{9/2}$

Table: BPS states $n_{g,(d_1,d_2)}$ for $\mathcal{O}_{F_2}(-C_2 - f)$

(d_1, d_2) \ g	0	1	2	3	4
$(0,1)$	1	0	0	0	0
$(1,2)$	1	0	0	0	0
$(1,3)$	-1	0	0	0	0
$(1,4)$	1	0	0	0	0
$(1,5)$	-1	0	0	0	0
$(1,6)$	1	0	0	0	0
$(2,3)$	1	0	0	0	0
$(2,4)$	-4	1	0	0	0
$(2,5)$	13	-7	1	0	0
$(2,6)$	-38	33	-10	1	0
$(3,4)$	1	0	0	0	0
$(3,5)$	-11	6	-1	0	0
$(3,6)$	72	-89	46	-11	1

But when $n > 2$, we need to determine the Donaldson-Thomas invariants for the following quivers:



No explicit closed formulas are known to us.

Comparison with the recursion from Virasoro constraints

By embedding $\mathcal{O}_{\mathbb{P}^2}(-1)$ into $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2})$. GW invariants of $\mathcal{O}_{\mathbb{P}^2}(-1)$ equal to the corresponding invariants of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2})$. We can apply the Virasoro constraints and get another recursion:

$$N_{1,d} = -\frac{d(d-1)}{24}N_{0,d} - \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \frac{(d-1)(2d-1)}{2}\binom{2d-3}{2d_1-2}N_{0,d_1}N_{1,d_2}$$

This recursion is different from the recursion coming from the all-genus WDVV recursion:

$$N_{1,d} = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(N_{0,d_1}N_{0,d_2}\frac{d_1^4}{12} - \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} (N_{0,d_1}N_{1,d_2}+N_{0,d_2}N_{1,d_1})d_1^2 \right)\binom{2d-3}{2d_1-1}$$

Using computer, we check that up to degree 19, the two recursions give the same answer. But a proof for the equivalence between these two recursions is still missing.

Thank you!