The cohomology ring of moduli spaces of 1-dimensional sheaves on $\mathbb{P}^2$

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Let $X$ be the quintic 3-fold.

$$GW^X_{g=0,d=1} = 2875$$

$$GW^X_{g=0,d=2} = \frac{4876875}{8} = 609250 + \frac{2875}{2^3}.$$  

**Problem**

*How to define the “true curve counts” $n^X_{g=0,d=1} = 2875$, $n^X_{g=0,d=2} = 609250$ intrinsically, in a way that they are obviously integers?*

Moduli stacks of 1-dimensional sheaves

Given a sheaf $F$ on $\mathbb{P}^2$ with 1-dimensional support, its slope is

$$\mu(F) = \frac{\chi(F)}{d(F)} \in \mathbb{Q}$$

where $d(F) = c_1(F) \in H^2(\mathbb{P}^2) \cong \mathbb{Z}$ is the degree of the curve where $F$ is supported. A sheaf is (semi)stable if

$$\mu(G)(\leq)\mu(F) \text{ for every } G \subsetneq F.$$ 

Let

$$\mathcal{M}_{d,\chi} \longrightarrow M_{d,\chi}$$

be the moduli stack and coarse moduli space of semistable sheaves on $\mathbb{P}^2$ with $d(F) = d$, $\chi(F) = \chi$. 
Basic properties

1. The coarse moduli space \( M_{d,\chi} \) is projective and irreducible, of dimension

\[
\dim M_{d,\chi} = d^2 + 1.
\]

2. For any \( d, \chi \), the stack \( \mathcal{M}_{d,\chi} \) is smooth of dimension

\[
\dim \mathcal{M}_{d,\chi} = d^2.
\]

3. When \( \gcd(d, \chi) = 1 \), all the semistable sheaves parametrized by \( \mathcal{M}_{d,\chi} \) or \( M_{d,\chi} \) are automatically stable, the coarse moduli space \( M_{d,\chi} \) is smooth, and

\[
\mathcal{M}_{d,\chi} \simeq M_{d,\chi} \times B\mathbb{G}_m.
\]

4. Twisting by line bundles and duality produces isomorphisms

\[
\begin{align*}
\mathcal{M}_{d,\chi} & \overset{\sim}{\rightarrow} \mathcal{M}_{d,\chi + kd} \\
M_{d,\chi} & \overset{\sim}{\rightarrow} M_{d,\chi + kd} \\
\mathcal{M}_{d,\chi} & \overset{\sim}{\rightarrow} \mathcal{M}_{d,-\chi} \\
M_{d,\chi} & \overset{\sim}{\rightarrow} M_{d,-\chi}
\end{align*}
\]
Hilbert–Chow morphism

There is a morphism

$$h : M_{d, \chi} \to |\mathcal{O}_{\mathbb{P}^2}(d)|$$

that sends a 1-dimensional sheaf to its (fitting) support. A generic point $[C] \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ corresponds to a smooth curve $C$ of genus

$$g = \frac{(d - 1)(d - 2)}{2},$$

and the fiber over $[C]$ is

$$h^{-1}([C]) \simeq \text{Jac}(C).$$

Note:

$$|\mathcal{O}_{\mathbb{P}^2}(d)| \simeq \mathbb{P}^{\frac{d(d+3)}{2}}.$$
Examples

1. For $d = 1, 2$ and any $\chi$, the Hilbert–Chow morphism is an isomorphism

\[ M_{1,\chi} \sim |O_{P^2}(1)| \simeq P^2 \]
\[ M_{2,\chi} \sim |O_{P^2}(2)| \simeq P^5 \]

2. For $d = 3$, the Hilbert–Chow morphism identifies $M_{3,1}$ with the universal cubic:

\[ |O_{P^2}(3)| \times P^2 \supseteq C_3 \simeq M_{3,1} \xrightarrow{h} |O_{P^2}(3)| \simeq P^9. \]

In other words,

\[ h^{-1}([E]) \simeq E. \]
Examples

$M_{2,0} \cong \mathbb{G}_m \cong \mathbb{P}^5$

$\{x,y = 0\}/\mathbb{G}_m \times \mathbb{G}_m$

$\{A, B, A, B, AB\}/\text{nilpotent} \cong GL_2$
Theorem (Toda+Konishi, conjectured by Katz)

We have \( n_{g=0,d}^{K\mathbb{P}^2} = (-1)^{d^2+1} e(M_{d,\chi}) \) for any \( \chi \) coprime with \( d \), i.e.

\[
GW_{g=0,d}^{K\mathbb{P}^2} = \sum_{d=kd'} \frac{(-1)^{(d')^2+1}}{k^3} e(M_{d',\chi}).
\]

In particular, \( e(M_{d,\chi}) \) does not depend on \( \chi \).

- The same holds for an arbitrary \( \chi \) (not necessarily coprime with \( d \)) if we replace the Euler characteristic \( e(M_{d,\chi}) \) by the intersection Euler characteristic.

- To get higher genus Gopakumar–Vafa/Gromov–Witten invariants we need to introduce the perverse filtration.
Perverse filtration

There is a filtration on the intersection cohomology of $M_{d,\chi}$ associated to the Hilbert–Chow morphism $h: M_{d,\chi} \to |\mathcal{O}_{\mathbb{P}^2}(d)|$:

$$P_0 IH^*(M_{d,\chi}) \subseteq P_1 IH^*(M_{d,\chi}) \subseteq \ldots \subseteq P_{2g} IH^*(M_{d,\chi}) = IH^*(M_{d,\chi}).$$

Theorem (de Cataldo–Migliorini)

Let $f: X \to Y$ be a morphism between smooth and proper varieties with equidimensional fibers. Denote by $L: H^*(X) \to H^{*+2}(X)$ the operator of multiplication by $f^*\eta$ where $\eta$ is an ample class in $Y$. Then the perverse filtration associated to $f$ is

$$P_k H^m(X) = \sum_{i \geq 1} \ker(L^{\dim Y+k+i-m}) \cap \im(L^{i-1}) \cap H^m(X).$$
\( \chi \)-independence

**Theorem (Maulik–Shen)**

Given any \( \chi, \chi' \), there is a natural isomorphism of graded vector spaces

\[
IH^*(M_d, \chi) \simeq IH^*(M_d, \chi')
\]

that respects the perverse filtrations.

Let

\[
\Omega_d(q, t) = (-1)^{d^2+1} q^{-g} t^{-b} \sum_{i,j \geq 0} \dim \text{Gr}_i^P IH^{i+j}(M_d, \chi) q^i t^j
\]

This encodes (refined) Gopakumar–Vafa invariants of \( K\mathbb{P}^2 \), as defined by Maulik–Toda. It is a Laurent polynomial symmetric under \( q \leftrightarrow q^{-1} \) and \( t \leftrightarrow t^{-1} \) by Hard Lefschetz symmetries.

Notation: \( \text{Gr}_i^P = P_i/P_{i-1} \).
Gopakumar–Vafa/Gromov–Witten correspondence

Conjecture (GV/GW)

We have

\[
\exp \left( \sum_{g,d} \text{GW}^{K_{\mathbb{P}^2}}_{g,d} u^{2g-2} Q^d \right) = \text{PE} \left( -\frac{q}{(1-q t)(1-q/t)} \sum_{d \geq 1} \Omega_d(q,t) Q^d \right)
\]

after setting \( t = 1, q = e^{i u} \).

Above, \( \text{PE} \) is the plethystic exponential:

\[
\text{PE}(f(q,t,Q)) = \exp \left( \sum_{k \geq 1} \frac{1}{k} f(q^k, t^k, Q^k) \right)
\]
Tautological classes

Goal: describe the rings in terms of generators and relations.

Definition

Let $\mathcal{F}$ be the universal sheaf on $\mathcal{M}_{d,\chi} \times \mathbb{P}^2$. Let $p, q$ be the projections of $\mathcal{M}_{d,\chi} \times \mathbb{P}^2$ onto $\mathcal{M}_{d,\chi}$ and $\mathbb{P}^2$, respectively. We define for $k \geq 0, j = 0, 1, 2$

$$c_k(j) = p_*(\text{ch}_{k+1}(\mathcal{F}) q^* H^j) \in H^{2k+2j-2}(\mathcal{M}_{d,\chi}).$$

If $\gcd(d, \chi) = 1$, the coarse moduli space $\mathcal{M}_{d,\chi}$ also has a universal sheaf, but it is not unique! We choose a normalized universal sheaf by requiring that $c_1(1) = 0$ in $H^2(\mathcal{M}_{d,\chi})$.

Remark

The class $c_2(0) \in H^2(\mathcal{M}_{d,\chi})$ is relatively ample with respect to the Hilbert–Chow map. The class $c_0(2) \in H^2(\mathcal{M}_{d,\chi})$ is the pullback of an ample class from $|\mathcal{O}_{\mathbb{P}^2}(d)|$. 
**Tautological generation**

We have algebra homomorphisms

\[
\mathbb{D} := \mathbb{Q}[c_0(2), c_1(1), c_2(0), c_1(2), c_2(1), c_3(0), \ldots] \to H^*(\mathcal{M}_d, \chi)
\]

\[
\widetilde{\mathbb{D}} := \mathbb{D}/\langle c_1(1) \rangle \to H^*(M_d, \chi)
\]

**Theorem (Pi–Shen, KLMP)**

The homomorphisms above are surjective, i.e. \( H^*(\mathcal{M}_d, \chi) \) and \( H^*(M_d, \chi) \) are generated as algebras by tautological classes. More precisely, \( H^*(\mathcal{M}_d, \chi) \) is generated by the tautological classes of (algebraic) degree \( \leq d \) and \( H^*(M_d, \chi) \) is generated by tautological classes of degree \( \leq d - 2 \).

**Problem**

Describe the ideal of relations, i.e.

\[
\ker \left( \mathbb{D} \to H^*(\mathcal{M}_d, \chi) \right), \quad \ker \left( \widetilde{\mathbb{D}} \to H^*(M_d, \chi) \right).
\]
Proposition

Let $F, F'$ be semistable sheaves of type $(d, \chi)$ and $(d', \chi')$, respectively. If

$$\frac{\chi'}{d'} < \frac{\chi}{d} < \frac{\chi'}{d'} + 3,$$

then

$$\text{Hom}(F, F') = \text{Ext}^2(F, F') = 0.$$

Proof.

A map from a semistable object to another semistable object with smaller slope is necessarily trivial, so $\text{Hom}(F, F') = 0$. By Serre duality,

$$\text{Ext}^2(F, F') = \text{Hom}(F', F(-3))^\vee = 0.$$
(Generalized) Mumford relations

1. This means that

\[ \mathcal{V} = R^p \mathcal{R} \text{Hom}(\mathcal{F}, \mathcal{F}')[1] \]

is a vector bundle on \( \mathcal{M}_{d, \chi} \times \mathcal{M}_{d', \chi'} \) of rank \( dd' \), with fibers

\[ \mathcal{V}|_{(F, F')} = \text{Ext}^1(F, F'). \]

2. Hence

\[ c_j(\mathcal{V}) = 0, \quad \text{for } j > dd'. \]

3. Using Grothendieck–Riemann–Roch and Newton’s identities, we express \( c_j(\mathcal{V}) \) in terms of tautological classes on \( \mathcal{M}_{d, \chi} \) and \( \mathcal{M}_{d', \chi'} \).

4. If we already understand \( H^*(\mathcal{M}_{d', \chi'}) \), we obtain relations on \( H^*(\mathcal{M}_{d, \chi}) \) by taking Kunneth components of \( c_j(\mathcal{V}) \).
Problem

How do we know when we found all the relations?

- If \( \text{gcd}(d, \chi) = 1 \), \( M_{d,\chi} \) satisfies Poincaré duality, so we can obtain all the relations once we cut the dimension of the top degree cohomology to 1.
- Alternatively: Betti numbers of \( M_{d,\chi} \) for small \( d \) are known (Choi–Chung, Bousseau,...).
- Betti numbers for the stacks \( \mathcal{M}_{d,\chi} \) can be obtained using BPS integrality + \( \chi \)-independence.
BPS integrality

Let

\[ \Omega_{M_d, \chi}(q) = (-q)^{-d^2 - 1} \sum_{i \geq 0} \dim IH^i(M_d, \chi) q^i = \Omega_d(q, q) \]

\[ \Omega_{\mathcal{M}_d, \chi}(q) = (-q)^{-d^2} \sum_{i \geq 0} \dim H^i(\mathcal{M}_d, \chi) q^i. \]

Theorem (Mozgovoy–Reineke)

Let \( \mu \in \mathbb{Q} \) be a fixed slope. We have

\[ \sum_{\substack{d \geq 0 \\ \chi = d \mu}} \Omega_{\mathcal{M}_d, \chi}(q) Q^d = \text{PE} \left( \frac{q}{q^2 - 1} \sum_{\substack{d \geq 1 \\ \chi = d \mu}} \Omega_{M_d, \chi}(q) Q^d \right). \]
BPS integrality: $\mathcal{M}_{2,0}$

Using $\mu = 0$ and taking the $Q^2$ coefficient on both sides:

$$
\sum_{i \geq 0} \dim H^i(\mathcal{M}_{2,0}) q^i = (-q)^4 \left( \frac{q}{q^2 - 1} \Omega_2(q) + \frac{q^2}{2(q^4 - 1)} \Omega_1(q^2) \right)
+ \frac{q^2}{2(q^2 - 1)^2 \Omega_1(q^2)}
$$

$$
= 1 + q^2 + 2q^4 + 2q^6 + 3q^8 + q^{10} - q^{14}
\frac{1}{(1 - q^2)(1 - q^4)}
$$

$$
= 1 + 2q^2 + 5q^4 + 8q^6 + 14q^8 + 18q^{10} + \cdots
$$
Virasoro representation

**Definition**

Let $R_n : \mathbb{D} \to \mathbb{D}$, $n \geq -1$, be a derivation of the algebra

$$\mathbb{D} = \mathbb{Q}[c_0(2), c_1(1), c_2(0), c_1(2), \ldots]$$

defined on generators by

$$R_n(c_k(j)) = \frac{(k + j + n - 1)!}{(k + j - 2)!} c_{k+n}(j).$$

These operators define a representation of $\text{Vir}_{\geq -1}$ on $\mathbb{D}$, i.e.

$$[R_n, R_m] = (m - n)R_{n+m}.$$
Virasoro geometricity

Theorem (Lim–M)

The action of $R_n$, $n \geq -1$, descends to $H^*(\mathcal{M}_{d,\chi})$ via the realization morphism $\mathbb{D} \to H^*(\mathcal{M}_{d,\chi})$.

In other words, $R_n$ preserves the ideal of tautological relations.

Theorem (KLPM)

The derivations $R_n$ preserve each of the ideals of (generalized) Mumford relations obtained from $\mathcal{M}_{d',\chi'}$ for each fixed $d', \chi'$. 
Virasoro representation

Theoretically we do not get new relations, but very useful in practice for implementation on the computer:

**Theorem (KLPM)**

*The ideal of Mumford relations is the smallest ideal containing the relations obtained from the vanishing*

\[ c_{dd'+1}(\mathcal{V}) = c_{dd'+2}(\mathcal{V}) = 0 \]

*which is closed under \( R_n \).*
Calculations

**Theorem (KLMP)**

The procedure explained determines completely the cohomology rings $H^\ast(M_{d,\chi})$ for $d \leq 5$ and $\gcd(d, \chi) = 1$ and $H^\ast(\mathcal{M}_{d,\chi})$ for $d \leq 4$.

All the rings are available online.

**Corollary**

The GV/GW (and refined GV/PT) correspondence holds up to degree 5.

**Remark**

It turns out that the relations proven for $H^\ast(M_{5,1})$ are not all of them (but for $H^\ast(M_{5,2})$ they are). However, they cut down the top degree of the ring to dimension 1, so we can recover the missing relations by Poincaré duality.
Example: $H^*(M_{5,1}), H^*(M_{5,2})$

Both are generated by

$$c_0(2), c_2(0), c_1(2), c_2(1), c_3(0), c_2(2), c_3(1), c_4(0)$$

Relations for $M_{5,1}$:

<table>
<thead>
<tr>
<th>degrees</th>
<th>$H^{10}$</th>
<th>$H^{12}$</th>
<th>$H^{14}$</th>
<th>$H^{30}$</th>
<th>$H^{34}$</th>
<th>$H^{36}$</th>
<th>$H^{38}$</th>
<th>$H^{40}$</th>
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<tbody>
<tr>
<td># of rel.</td>
<td>3</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>1</td>
<td>$1^\times$</td>
<td>$2^\times$</td>
<td>1</td>
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Relations for $M_{5,2}$:

<table>
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<th>degrees</th>
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<th>$H^{12}$</th>
<th>$H^{14}$</th>
<th>$H^{16}$</th>
<th>$H^{18}$</th>
<th>$H^{28}$</th>
<th>$H^{30}$</th>
<th>$H^{32}$</th>
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<td>2</td>
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Example: \( H^*(\mathcal{M}_{2,0}) \)

\[
H^*(\mathcal{M}_{2,0}) \cong \mathbb{D}/(U(Vir_{\geq -1}) \cdot I_1)
\]

where

\[
I_1 = \langle 8c_2(0) - c_0(2),
\]

\[
c_2(2) + 2c_1(1)^2 c_2(0) + \frac{32}{3} c_2(0)^3 - 4c_1(1)c_3(0) - 4c_2(0)c_2(1),
\]

\[
c_3(1) + \frac{1}{12} c_1(1)^3 + 4c_1(1)c_2(0)^2 - 8c_2(0)c_3(0) - \frac{1}{2} c_1(1)c_2(1) \rangle
\]

These 3 relations are obtained by using the (generalized) Mumford relations with \( d' = 1, \chi' = -1, -2 \).
Example: $H^*(\mathcal{M}_{2,0})$

Eliminating the redundant variables:

$$H^*(\mathcal{M}_{2,0}) \simeq \mathbb{Q}[c_1(1), c_2(0), c_2(1), c_3(0)]/l_2$$

where

$$l_2 = \langle c_1(1)c_2(0)^4 - 2c_2(0)^3c_3(0),$$

$$16c_2(0)^5 + c_1(1)^2c_2(0)^3 - 6c_1(1)c_2(0)^2c_3(0)$$

$$+ 6c_2(0)c_3(0)^2 + 2c_2(0)^3c_2(1),$$

$$c_1(1)^3c_2(0)^3 - 6c_1(1)c_2(0)c_3(0)^2 + 4c_3(0)^3$$

$$- 6c_1(1)c_2(0)^3c_2(1) + 12c_2(0)^2c_3(0)c_2(1),$$

$$2c_1(1)c_2(0)^3c_3(0) - 3c_2(0)^2c_3(0)^2 - c_2(0)^4c_2(1) \rangle$$
The $P = C$ conjecture

- Our moduli $M_{d,\chi}$ are analogues of moduli of Higgs bundles (replace $\mathbb{P}^2$ by $T^*C$).
- The $P$ filtration for Higgs bundles has been identified with a weight filtration on a character variety ($P = W$ conjecture, proofs by Maulik–Shen, Hausel–Mellit–Minets–Schiffmann, Maulik–Shen–Yin).
- We do not have a character variety side, but there is an intermediate filtration $C$ that is used in the proof ($P = C = W$) which makes sense for $M_{d,\chi}$.
- Non Calabi–Yau/compact setting changes many things, but $P = C$ seems to be true.
The $P = C$ conjecture

**Definition**

Let $C \cdot H^*(M_d, \chi)$ be the filtration defined by

$$C_k H^*(M_d, \chi) = \text{span}\{c_{k_1}(j_1) \cdots c_{k_l}(j_l) : k_1 + \ldots + k_l \leq k\}.$$

**Conjecture ($P = C$)**

$$P \cdot H^*(M_d, \chi) = C \cdot H^*(M_d, \chi).$$

**Corollary (KLMP)**

$P = C$ holds up to $d = 5$.

**Theorem (Maulik–Shen–Yin)**

$$P \cdot H^{\leq 2d-2}(M_d, \chi) \supseteq C \cdot H^{\leq 2d-2}(M_d, \chi).$$
Consequences of $P = C$

1. The $P$ filtration is multiplicative (but, unlike for Higgs bundles, it does not admit a multiplicative splitting).
2. The $C$ filtration is $\chi$-independent (for other del Pezzo surfaces, also polarization independent).
3. The $C$ filtration has “curious hard Lefschetz” symmetries.
4. $C_{2g} H^*(M_d, \chi) = H^*(M_d, \chi)$.
5. Vanishing of integrals:

$$\int_{M_d, \chi} c_{k_1}(j_1) \ldots c_{k_l}(j_l) = 0$$

for $k_1 + \ldots + k_l < 2g = (d - 1)(d - 2)$. 
Stacky $P = C$

There is a $P$ filtration on $H^*(\mathcal{M}_{d,\chi})$ defined by Ben Davison, which is compatible with the BPS integrality formula. The $C$ filtration can be defined on the stack easily.

**Conjecture (Stacky $P = C$)**

$$P \cdot H^*(\mathcal{M}_{d,\chi}) = C \cdot H^*(\mathcal{M}_{d,\chi}).$$

**Corollary (KLMP)**

The numerical stacky $P = C$ holds up to $d = 4$, i.e.

$$\dim \text{Gr}_i^P H^{i+j}(\mathcal{M}_{d,\chi}) = \dim \text{Gr}_i^C H^{i+j}(\mathcal{M}_{d,\chi})$$

for every $i, j \geq 0$ and $d \leq 4$. 
Thanks you!