# From K3 surfaces to Hilbert schemes and back.

$S$ K3 surface $\rightsquigarrow$ 3 counting theories.

① $F$ Gieseker stable sheaf on $S$ (fixed)

$rk(F) > 0$.

$$\mathrm{Quot}(F, u) = \{ F \twoheadrightarrow Q \mid v(Q) = u \}$$

Mukai vector:

$$v(Q) = ch(Q) \cdot \sqrt{td_S}$$

$$= ch(Q) \cdot (1 + p).$$

Lemma: $\mathrm{Quot}(F, u)$ has a perfect obstr. theory with virtual tangent bdle $T^{vir} = R\mathcal{H}om_S(K, Q)$

$$= R\pi_* R\mathcal{H}om(K,Q)$$

$$0 \to K \to \pi_S^*(F) \to Q \to 0$$

$$\begin{tikzcd} \mathrm{Quot}(F,u)\times S \arrow[r, "\pi_S"] \arrow[d, "\pi"] & S \\ \mathrm{Quot}(F,u) & \end{tikzcd}$$

Proof:

$$0 \to K \to F \to Q \to 0 \qquad /\mathcal{H}om(-,Q)$$

$$\mathrm{Ext}^1(K,Q) \to$$

$$\begin{tikzcd} \to \mathrm{Ext}^2(Q,Q) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \mathrm{Ext}^2(F,Q) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Ext}^2(K,Q) \arrow[r] & 0 \\ \mathbb{E}x & \mathrm{Hom}(Q,F)^\vee & & \end{tikzcd}$$

$\mathrm{Ext}^2(F,Q) = 0$, $\mathrm{Hom}(Q,F)^\vee = 0$ $\Rightarrow$ $\mathrm{Ext}^2(K,Q) = 0$.

$$\begin{tikzcd} F \arrow[r, two heads] & Q \arrow[r] & F \end{tikzcd}$$

stable — $F \to Q \to F$ is $0$ by stab.

$\square$

Trace map:

$$\mathrm{Ext}^1(K,Q) \longrightarrow \mathrm{Ext}^2(Q,Q) \xrightarrow{\ tr\ } \mathrm{Ext}^2(\mathcal{O}_S,\mathcal{O}_S) = H^2(S,\mathcal{O}_S) = \mathbb{C}$$

Kiem-Li: $\exists$ reduced virtual class $[\mathrm{Quot}(F,n)]^{\cancel{\text{red}}\,\text{vir}} \longleftarrow \text{red.}$

with virtual tangent bundle $T^{\text{vir}} = R\mathcal{H}om(K,Q) + \mathcal{O}$.

<u>Defn</u> (Fantechi–Göttsche).

$$e^{\text{vir}}(\mathrm{Quot}(F,n)) = \int_{[\mathrm{Quot}(F,n)]^{\text{vir}}} e(T^{\text{vir}}).$$

**Def**: $F = I_\eta$ for $\eta$ length $n$ subscheme.

$u = (0, \beta, m)$

$$Q_{m,(\beta,n)} := e^{vir}\left(\mathrm{Quot}(F,u)\right).$$

Rmk:

If $F = I_\eta(D)$

$I_\eta(D) \to Q \quad / \otimes \mathcal{O}(-D)$

$I_\eta \to Q(-D).$

(2) $E$ elliptic curve.

$X = S \times E$.

$$DT^{S\times E}_{m,(\beta,n)} = \int_{[\mathrm{Hilb}_m(S\times E,(\beta,n))/E]^{vir}} 1$$

in $H_2(S\times E, \mathbb{Z})$

$\cong H_2(S,\mathbb{Z}) \oplus \mathbb{Z}[E]$

③ $S^{[n]}$ Hilbert scheme of $n$ points on $S$.

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z} \cdot A$$

$$\beta_n + mA \longleftarrow\!\!| \ (\beta, m)$$

If $C \subset S$ with $\beta = [C]$, $p_2, \dots, p_{n-1} \in S$ distinct pts away from $C$,

$$\beta_n := \Big[ \{ \xi \in S^{[n]} \mid \operatorname{Supp}(\xi) = x + p_1 + \dots + p_{n-1} \}_{x \in C} \Big]$$

$$A = \Big[ \{ \xi \in S^{[n]} \mid \operatorname{Supp}(\xi) = 2p_1 + p_2 + \dots + p_{n-1} \} \Big]$$

$\rightsquigarrow$ $\overline{M}_g(S^{[n]}, \beta + kA)$ red. vdim

$(2n-3)(1-g) + 1$.

$$\overline{M}_E(S^{[n]}, \beta + kA) = \{ f : C \to S^{[n]} \mid C \simeq E + \text{rational tails} \}.$$

$$GW^{S^{[n]}}_{E,(\beta,m)} = \int_{[M_E(S^{[n]}, \beta+kA)]^{vir}} 1$$

$$= \int_{[M_{1,1}(S^{[n]}, \beta+kA)]^{vir}} \rho^*([E,0]) \cdot ev_1^*(\beta^\vee) \quad , \quad \beta^\vee \in H^2(S,\mathbb{Q}), \quad \beta \cdot \beta^\vee = 1.$$

**Thm A** (Nesterov + $\varepsilon \cdot \mathcal{O}$).

$$DT^{S\times E}_{m,(\beta,n)} = GW^{S^{(n)}}_{E,(\beta,m)} + e(S^{(n)}) \sum_{r \mid (\beta,m)} \frac{1}{r}(-1)^{m+\frac{m}{r}} Q_{n,\frac{(\beta,m)}{r}}$$

**Pf**:

Nesterov: Wallcrossing (hard part).

$\mathcal{O}$: Wallcrossing contribution.

$\rightsquigarrow$ Analysis of the cap $(K3\times \mathbb{P}^1, K3\times 0)$ $\square$

## Thm B (O.)

(a) $Q_{n,(\beta,m)}$ depends upon $\beta$ only through $\beta\cdot\beta = 2h-2$.

$$Q_{n,h,m} := Q_{n,(\beta,m)}$$

(b)

$$\sum_{m\in\mathbb{Z}}\sum_{h\geq 0} Q_{n,h,m}\, q^{h-1}(-p)^m = \frac{\Theta(p,q)^{2n-2}}{\Delta(q)}\cdot\left(-\wp(p,q)+\frac{1}{12}E_2(q)\right)^n$$

$$\Delta(q) = q\prod(1-q^n)^{24} \qquad E_2(q) = 1-24\sum_{n\geq 1}\sigma(n)q^n.$$

$$\Theta(p,q) = \left(p^{1/2}-p^{-1/2}\right)\prod_{n\geq 1}\frac{(1-pq^n)(1-p^{-1}q^n)}{(1-q^n)^2}$$

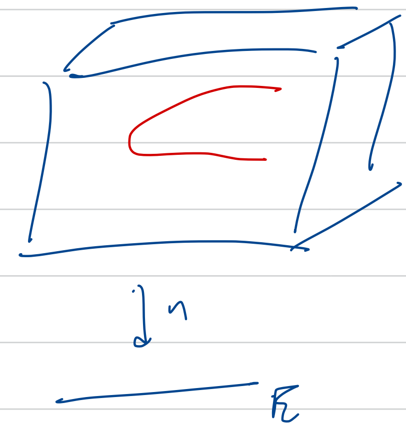
Rank Case $n=0$ : Oprea–Pandharipande. $\mathcal{O}_S \to Q$.

**Proof:**

(a) Write $Q_{n,(\beta,m)}$ as tautological integral over $S^{(n)}$.

(Gholampour–Thomas)

Ellingsrud–Göttsche–Lehn $\Rightarrow$ universal. //



(b)
$$Q_n(p,q) = \sum_{h,m} Q_{n,h,m}\, q^{h-1} (-p)^m$$

$\{ I_\eta \to Q \}$

$\ell(\eta) = n$ $\qquad$ $\mathrm{ch}(Q) = (0, \beta, m)$.

**Step 1:** There exist power series $F(p,q)$, $G(p,q)$ s.t.
$$Q_n = F(p,q) \cdot G(p,q)^n.$$

Pf: $S$ Bryan-Leung K3

$$\begin{tikzcd} S \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[u, bend left] \end{tikzcd}$$

$$S \rightsquigarrow S \cup \underset{\substack{\uparrow \\ x_1}}{(\mathbb{P}^1 \times E)} \cup \cdots \cup \underset{\substack{\uparrow \\ x_n}}{(\mathbb{P}^1 \times E)} \cup (\mathbb{P}^1 \times E)$$

$$\eta = x_1 + \cdots + x_n \rightsquigarrow x_1 \quad \cdots \quad x_n$$

Apply Degeneration formula of Li-Wu. $\square$

Step 2: Idea Use Thm A.

$S \supset B$ section class

$F$ fiber class

$$DT_n(p,q) = \sum DT_{m,(B+hF,n)}\, q^{h-1} (-p)^m$$

$$H_n(p,q) = \sum GW^{(n)}_{E,(B+hF,m)}\, q^{h-1} (-p)^m.$$

Thm A:

$$DT_n(p,q) = H_n(p,q) + e(S^{(n)}) \cdot Q_n(p,q).$$

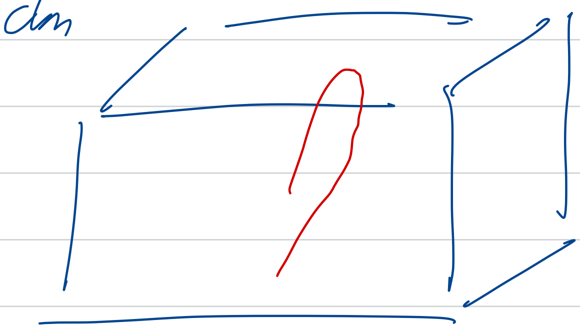
$\uparrow$ $F \cdot G^n$.

$\boxed{n=0}$ $S^{(0)} = pt.$

$\Rightarrow H_0(p,q) = 0.$

Take $\tilde{\beta}_h$ ihed clm

Curves in degm $(\tilde{\beta}_h, 0)$



E

DT side:

[MPT]

$$DT_0(p,q) = \sum e(P_m(S,\tilde{\beta}_h))\, q^{h-1} (-p)^m.$$

Kawai $=$ Yoshioka

$$\frac{1}{\Theta^2(p,q)\cdot\Delta(q)}.$$

$$\Rightarrow \frac{1}{\Theta^2 \Delta} = 0 + 1 \cdot F$$

$n=1$ $\quad S^{(1)} = S$

$[d, 0]$.

$$GW^{S}_{E, B+hF} = \int_{[\overline{M}_{1,1}(S, B+hF)]^{vir}} \varphi^*([E, 0]) \cdot ev_1^*(\beta^\vee)$$

$$= \int_{[M_{0,3}(S, B+hF)]^{vir}} ev_{12}^*(\Delta_S) \cdot ev_3^*(\beta^\vee)$$

$$= (2h-2) \int_{[\overline{M}_{0,0}(S, B+hF)]^{vir}} 1 = \left[ 2 \, D_\tau \left( \frac{1}{\Delta(q)} \right) \right]_{q^{h-1}}$$

$$D_\tau\left(\frac{1}{\Delta(q)}\right) = -\frac{E_2(q)}{\Delta} = \left[-2\,\frac{E_2(q)}{\Delta(q)}\right]_{q^{h-1}}$$

$$\Rightarrow H_1(p,q) = -2\,\frac{E_2(q)}{\Delta(q)}.$$

DT side:

$$DT_1 = \sum DT^{S\times E}_{m,(\beta_h,1)}\, q^{h-1}(-p)^n.$$

Thm (Bryan)

$$DT_1 = -24\,\frac{\wp(p,q)}{\Delta(q)}.$$

$$-24\,\frac{\wp(p,q)}{\Delta(q)} = -2\,\frac{E_2(q)}{\Delta(q)} + 24 \cdot \frac{1}{\Theta^2\Delta} \cdot G(p,q)$$

$$\parallel$$

$$DT_1 = H_1 + 24\, Q_1$$

$$G(p,q) = \Theta^2\left(-\wp + \frac{1}{12}E_2(q)\right). \qquad \square$$

<u>Upshot</u> (Nesterov, O., Pixton, Shen, ... ) Conjectured by O.–Pandharipande.

$$-\frac{1}{\chi_{10}(p,q,\tilde q)} = \sum_{n\geq 1} DT_n(p,q)\,\tilde q^{\,n-1} = \sum_{n\geq 0} H_n(p,q)\,\tilde q^{\,n-1}$$

$$+\frac{1}{\Theta^2\Delta}\,\frac{1}{\tilde q}\prod_{n\geq 1}\frac{1}{\left(1-(\tilde q G)^n\right)^{24}}$$

## § Higher Rank

$GW_E^{M(v)}$

Old case: $v = (1, 0, 1-n)$
$w = v + u$
$u = (0, \beta, m)$

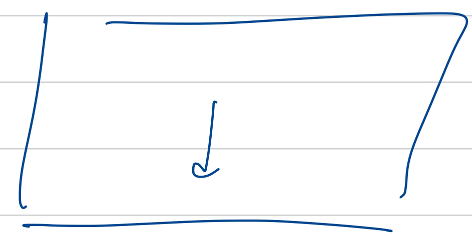
$M(v)$ fine* and proper moduli space of stable sheaves on K3 surface

$\rightsquigarrow$ $v$ primitive Mukai.

$M_{v,w}(S\times E)$ moduli space of generically stable sheaves $F$ on $S \times E$.

$v, w \in H^*(S,\mathbb{Z})$ inside the Mukai vector

$$ch(F)\cdot\sqrt{td_S} = v + pt_E \cdot w.$$



$E$

DT invs.

**Thm** (Nesterov + E. O.). (Assume that $div(v \wedge w) = div(w)$.)

Any fixed $F \in M(v)$.

$$DT^{S\times E}_{(v,w)} = GW^{M(v)}_{E,w'} + e(S^{(n)}) \sum_{r \mid w} \frac{1}{r}(-1)^{v\cdot w + \frac{v\cdot w}{r}} e^{vir}\left(Quot(F, u_r)\right).$$

$v \in H^*(S, \mathbb{Z}) \ni w$

$\theta : v^\perp \xrightarrow{\sim} H^2(M(v), \mathbb{Z})$.

$\langle w, - \rangle : v^\perp \longrightarrow \mathbb{Z}$

$\rightsquigarrow \quad w' := \langle w, - \rangle \in (v^\perp)^* \cong H^2(M(v), \mathbb{Z})^*$

$= H_2(M(v), \mathbb{Z})$.

where $v_r = -\frac{w}{r} + s_r v$

for $s_r \in \mathbb{Z}$ unique integer s.t.

$rk(v_r) \in \{0, \ldots, rk(v) - 1\}$.

$GW^{M(v)}_{E, w'}$ depends upon $v, w$ only via $\underline{v \cdot v, \; v \cdot w, \; w \cdot w}$ and $\underline{div(v \wedge w)}$.

(monodromy invariance).

$M(v) \sim S^{[n]}$.

$DT^{S \times E}_{(v, w)}$ " " (work in progress(*)).

Cor* For any $F \in M(v)$, $e^{vir}(\mathrm{Quot}(F, n))$

only depends on $v \cdot v$, $u \cdot v$, $u \cdot u$.

$\rightsquigarrow$ Evaluate all of them.

$$H^{2*}(S \times E, \mathbb{Z}) \cong H^{*}(S) \oplus H^{*}(S)$$

$$\underline{\pi_S^*(v) + \pi_S^*(w) \cdot pt_E} \longleftarrow (v, w)$$

$(1, 0, -\beta - nF, -m)$.

$v = (1, 0, 1-n)$

$w = (1, \beta, m)$.

$$\frac{1}{\Delta} \rightsquigarrow \frac{1}{\chi_{10}} \sim \frac{1}{\phi_{10}}$$

Thm 10. (Multiple cover for $K3 \times E$)

$$DT_{m,(\beta,n)} = \sum_{r \mid (\beta,n)} \frac{1}{r} (-1)^{r + \frac{n}{r}} DT_{\frac{m}{r}, (\mathrm{prim}(\frac{\beta}{r}), n)}$$