Universally counting curves in Calabi-Yau threefolds

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There are lots of ways of counting curves!

Most come from moduli spaces with virtual fundamental cycle lying over space of 1-cycles

(stable maps) $\mathcal{M}(X)$  (stable pairs) $\mathcal{P}(X)$

Functorial under open embeddings

Most are invariant under deformation

$\mathcal{Z}(X)_B = \bigcup_{b \in B} \mathcal{Z}(X_b)$

Universal invariant with these three properties!

(1-cycles)

(2-cycles)

(3-cycles)

(4-cycles)

(no compactness/properness assumption!)

(inspired by work of Ionel-Parker on Gopakumar-Vafa integrality conjecture)

$GW: H_0(CY^3, H^0_c(\mathcal{Z})) \longrightarrow \mathbb{Q}(\!(u)\!)$

$PT: \longrightarrow \mathbb{Z}(\!(q)\!)$

Recover usual invts by applying to $(X, \frac{1}{\beta} \in H^0_c(\mathcal{Z}(X)))$ for $X$ projective
Can also include "higher deformation invariants" wrt families over any simplex

\[
C_c^*(\text{CY3}, C^*_c(\mathbb{Z})) = \bigoplus_{X \in \text{CY3}/\Delta^k} C_c^{\ast+k}(\mathbb{Z}(X/\Delta^k))
\]

Theorem: This homology group (for complex CY3's) is supported in cohomological degree \(<=0\), and in degree 0 it is the free polynomial algebra on "equivariant local curve elements" \(x_{\{g,m\}}\).

Corollary: GW and PT are related by MNOP transformation on CY3's iff they are so on \(x_{\{g,m\}}\).

Prop: Eval on \(x_{\{g,m\}}\) coincides with localized equivariant count on local curve of genus g in class m.

Bryan--Pandharipande compute equivariant GW of local curves
Okounkov--Pandharipande compute equivariant DT of local curves

Conclude: MNOP correspondence on all CY3's

Generation statements is essentially a *transversality* assertion.

Almost complex geometry: transversality wrt generic almost complex structures

\[
\text{compute } \bar{H}_0(A_{\text{CY3}}, \bar{H}_*(\mathbb{Z}^{\text{CY}}))
\]

Complex geometry: transversality in total space after enlarging base, *locally* on cycle space

\[
\text{compute } \bar{H}_c^*(\mathbb{Z}^{\text{CY}}(C_{\text{CY3}}))
\]
Lemma: \(H^\bullet_c(Z^\text{cy}(\mathbb{P}x_3),\text{semi-reg})\) vanishes in degrees <0 and in degree 0 is freely generated by monomials in "geometric local curve elements".

Proposition: (Enough Divisors) Let \(X \to B\) be a family of threefolds, and let \(K \subseteq Z(X/B)\) be compact analytic set whose projection map \(K \to B\) is injective. After removing from \(X\) a closed set disjoint from the support of \(K\), there exist relative divisors \(D_i \subseteq X \times_B U_i\) (\(U_i \subseteq B\) open) which together "control" all cycles \(z\) in \(K\).

(A cycle \(z = \sum_i m_i C_i\) is *controlled* by a divisor when said divisor intersects all \(C_i\)).

Proof: Induct on dimension of base \(B\). Choosing divisors generically reduces to base of two real dimensions less. QED

Proposition: Comparison map \(H^\bullet_c(Z^\text{cy}(\mathbb{P}x_3),\text{semi-reg}) \to H^\bullet_c(Z^\text{cy}(\mathbb{P}x_3))\) is an isomorphism.

Proof: Use enough divisors and "generic transversality". QED
Generic transversality: Given a smooth divisor $D \subset X$, we can deform $X$ by any subspace of $\left| \mathcal{L}^0(D, \mathcal{T}_X(-\omega_D)) \right|_D$ and in the resulting family every connected curve intersecting $D$ can be made regular using a suitable subspace.

Equivariant local curve elements $\chi_{g,m}$

$E = \text{rank two vector bundle over curve } C \text{ of genus } g$

Fix weight $r_i$ maps $\lambda_i : Z(E,m) \rightarrow C$ with compact joint zero set. $\lambda_1, \ldots, \lambda_N$

$$\chi_{g,m} = \left( \frac{E^*(C^{N+1}-0)}{C-0}, 1 \prod \frac{1}{r_i} \right) \in H^2_c\left( \mathbb{Z}\left( \frac{E^*(C^{N+1}-0)}{C-0}/C\mathbb{P}^N \right) \right)$$  (independent of choice of $\lambda_i$'s)

Proposition: Monomials in equivariant and geometric local curve elements coincide modulo cycles of smaller covering multiplicity.

Question: $\chi_{g,m} = \gamma_{g,m}$?

Remark: Can use an algebraic trick to show that if $x_{g,m}$ generate then they necessarily freely generate. (analyze possible kernels and show they must be trivial)

Question: How to keep track of multiple covers in this framework?

Conjecture: For any complex CY3, the element $(X, f^*\mathcal{L}) \in H^0_c(Z^\mathcal{L}(C_{\mathcal{P}^3})(1-H^2_c))$ has the form

$$\prod \prod (\sum_{m \geq 0} x_{g,m} t^m)^{e_{g,m}(c)}$$  for integers $e_{g,m}(X)$ (compare Ionel--Parker).