Cohomology of Moduli Spaces of Curves

Intercontinental Zoominar

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$H^*(\overline{\mathcal{M}}_{g,n})$ $\qquad$ $H^*(\mathcal{M}_{g,n})$

<u>Q1</u>: What is $\dim_{\mathbb{Q}} H^k(\overline{\mathcal{M}}_{g,n})$, $\dim_{\mathbb{Q}} H^k(\mathcal{M}_g)$

<u>Q2</u>: Which motivic structures appear in $H^k(\overline{\mathcal{M}}_{g,n})$ and in the semi-simplification of $H^k(\mathcal{M}_g)$?

Hodge structures, ℓ-adic Galois representations

$\bar{M}_{g,n}$: H^k pure wt k.

$$H^k(\bar{M}_{g,n}, \mathbb{C}) \cong \bigoplus_{p+q=k} \underline{H^{p,q}}$$

$$H^{p,q} \cong H^q(\bar{M}_{g,n}, \Omega^p),$$

$$H^{q,p} = \overline{H^{p,q}}$$

Small k :

$H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \rightsquigarrow$ L-function.

Thm (Chenevier-Lannes 2019)

Assume L-functions attached to irreducible subreps of $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}_\ell)$ have analytic continuation and functional equation. Then for $k \le 22$, $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}_\ell)$ is a sum of Tate twists of:

$$1,\ \Delta_{11},\ \Delta_{15},\ \Delta_{17},\ \Delta_{19},\ \Delta_{21}$$

$$\Delta_{19,7},\ \Delta_{21,5},\ \Delta_{21,9},\ \Delta_{21,13}$$

$$\mathrm{Sym}^2 \Delta_{11}.$$

Conditional Predictions:

- $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for odd $k < 11$

- $H^k(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus \mathbb{L}$ for even $k < 22$

  $\left(H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) = H^{k/2,k/2} \right)$.

- $H^{11}(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus H^{11}(\overline{\mathcal{M}}_{1,11})$

  $= 0$ when $\overline{\mathcal{M}}_{g,n}$ is unirational.

- $\# \overline{\mathcal{M}}_{g,n}(\mathbb{F}_q)$, $\# \mathcal{M}_{g,n}(\mathbb{F}_q)$ polynomial in q

  for $\begin{cases} 3g-3+n < 11 \\ 3g-3+n \leq 12 \text{ and } \overline{\mathcal{M}}_{g,n} \text{ unirational} \end{cases}$

Thm (Bergström Faber P 2022)

- $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for $k \in \{7, 9\}$
- $\# \overline{\mathcal{M}}_{4,n}(\mathbb{F}_q)$, $\# \mathcal{M}_{4,n}(\mathbb{F}_q)$ polynomial in q for $n \leq 3$.

$$\# \mathcal{M}_{4,3}(\mathbb{F}_q) = q^{12} + 4q^{11} + 7q^{10} - 4q^9 - 13q^8 + 4q^7 - q^6 - 11q^3 + 2q^2 + 2q - 1$$

eg. $\# \mathcal{M}_{4,3}(\mathbb{F}_3) = 1\,497\,092$ (confirmed)

Thm: (Canning Larson P 2022)

- $H^{11}(\overline{\mathcal{M}}_{g,n}) \cong \begin{cases} V_{n-10,1^{10}} \otimes H^{11}(\overline{\mathcal{M}}_{1,11}) & \text{for } g=1, n\geq 11 \\ 0 & \text{otherwise.} \end{cases}$

- $H^k(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus \mathbb{L}^{k/2}$ even $k<14$.

- For $g\geq 2$, $d=\dim \overline{\mathcal{M}}_{g,n}$

  $\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q) = \text{polynomial}(q) + O(q^{d-13/2})$

  $= \text{polynomial}(q)$ if $d\leq 12$.

$$H^*(\mathcal{M}_g) \cong H^*(\mathrm{Mod}(S_g)) \cong H^*(\mathrm{BDiff}^+(S_g)) \subseteq H^*(\sigma_\infty)$$

Thm (Harer 1980s)

$\mathrm{vcd}(\mathcal{M}_g) = 4g-5$ and $H^k(\mathcal{M}_g)$ stabilizes as $g \to \infty$.

Also: $H^{4g-5}(\mathcal{M}_g) = 0$ (Church Farb Putman 2012, Morita Sakasai Suzuki 2013)

Thm (Tillman 1997)

$H^*(\mathcal{M}_g)$ stabilizes to $H^*(\Omega^\infty X)$.

Thm (Madsen Weiss 2007) (Mumford's Conjecture)

Stabilizes to $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$

Stable range: $k \leq \frac{2g-2}{3}$ (Wahl 2010)

Thm (Harer Zagier 86)

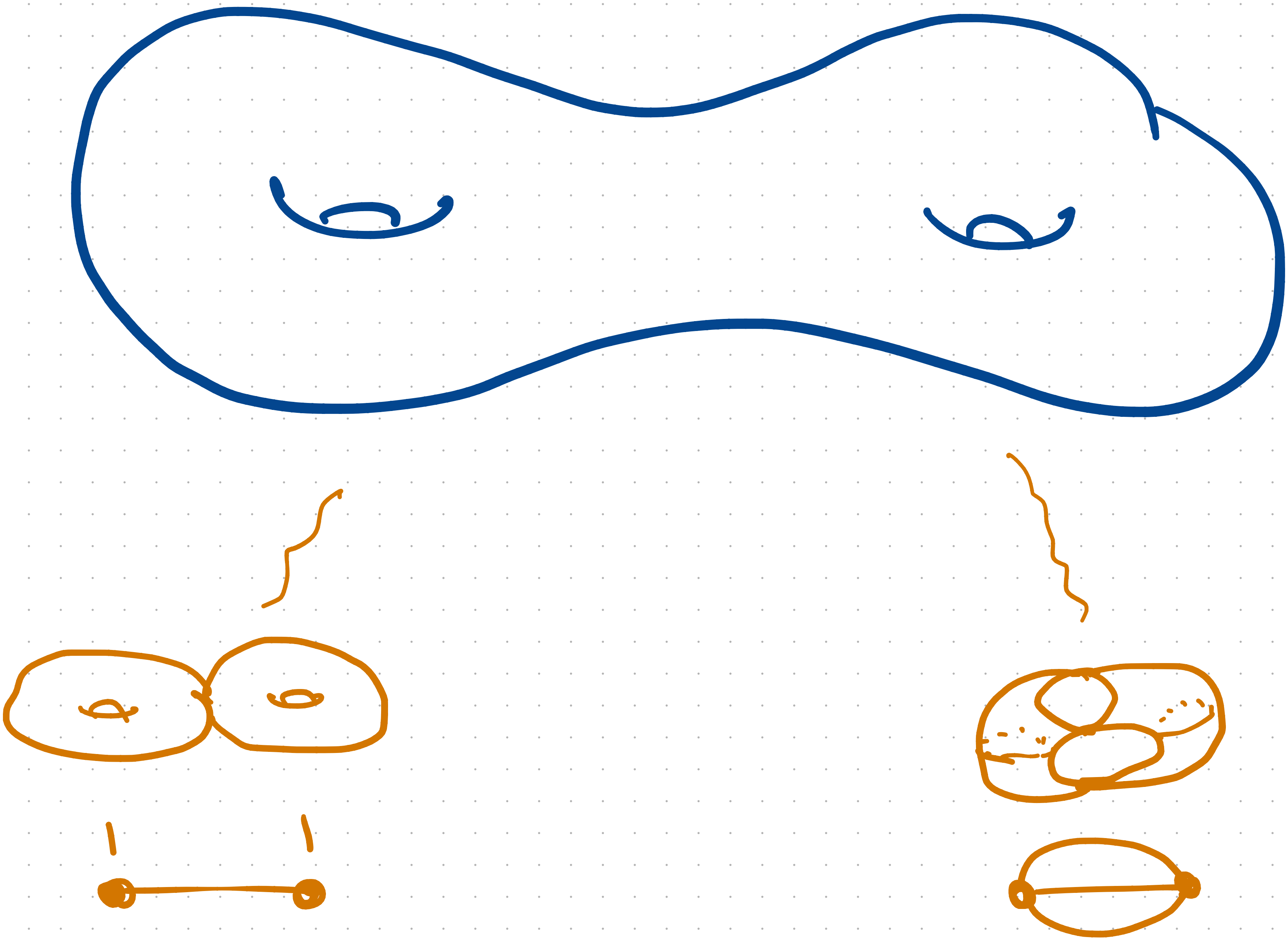
$$(-1)^{g+1}\chi(\mathcal{M}_g) \sim \sqrt{\frac{\pi}{g}} \cdot \left(\frac{g}{\pi e}\right)^{2g} \quad (!!)$$

Q: Where is the unstable cohomology?

Conj: (Kontsevich 1993, CFP 2014)

Fix $k \geq 6$. Then $H^{4g-k}(\mathcal{M}_g) = 0$ for k suff. large

$$\left(\dim_{\mathbb{Q}} H^k(\sigma_\infty) < \infty \right).$$



Deligne's weight spectral sequence for $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$

- Stratify by topological type: (dual graph G)

$$\mathcal{M}_G := \prod_{v \in V(G)} \mathcal{M}_{g_v, n_v} \; ; \; \overline{\mathcal{M}}_G := \prod_v \overline{\mathcal{M}}_{g_v, n_v}$$

$$\overline{\mathcal{M}}_g = \bigsqcup_G \mathcal{M}_G / \mathrm{Aut}(G)$$

$$E^1_{\bullet, k} : \quad 0 \to H^k(\overline{\mathcal{M}}_g) \to \bigoplus_{|E(G)|=1} H^k(\overline{\mathcal{M}}_G)^{\mathrm{Aut}(G)} \to \cdots$$

$$\to \bigoplus_{|E(G)|=2} \left(H^k(\overline{\mathcal{M}}(G)) \otimes \det E(G) \right)^{\mathrm{Aut}(G)} \to \cdots$$

Degenerates at E^2, abuts to $\mathrm{gr}^W_k H_c^{k+\bullet}(\mathcal{M}_g)$.

Thm: (Chan–Galatius–P 2021)

$$E^1_{\bullet,0} \xrightarrow[\text{q. isom}]{\sim} G^{(g)}$$ (Kontsevich's commutative graph complex in genus g)

Thm: $\mathfrak{grt}_1 \cong \bigoplus_g W_0 H_c^{2g}(\mathcal{M}_g) \supseteq \mathrm{Lie}\langle \sigma_3, \sigma_5, \sigma_7, \dots \rangle$

(Willwacher 2015, Brown 2012)

Cor: $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g) > \rho^g + c$

$\rho \approx 1.3247\ldots$ real root of $t^3 - t - 1$

Let $V_\bullet^{*} = \Lambda^2\left(\bigoplus_g W_0 H_c^{*}(\mathcal{M}_g)\right)$

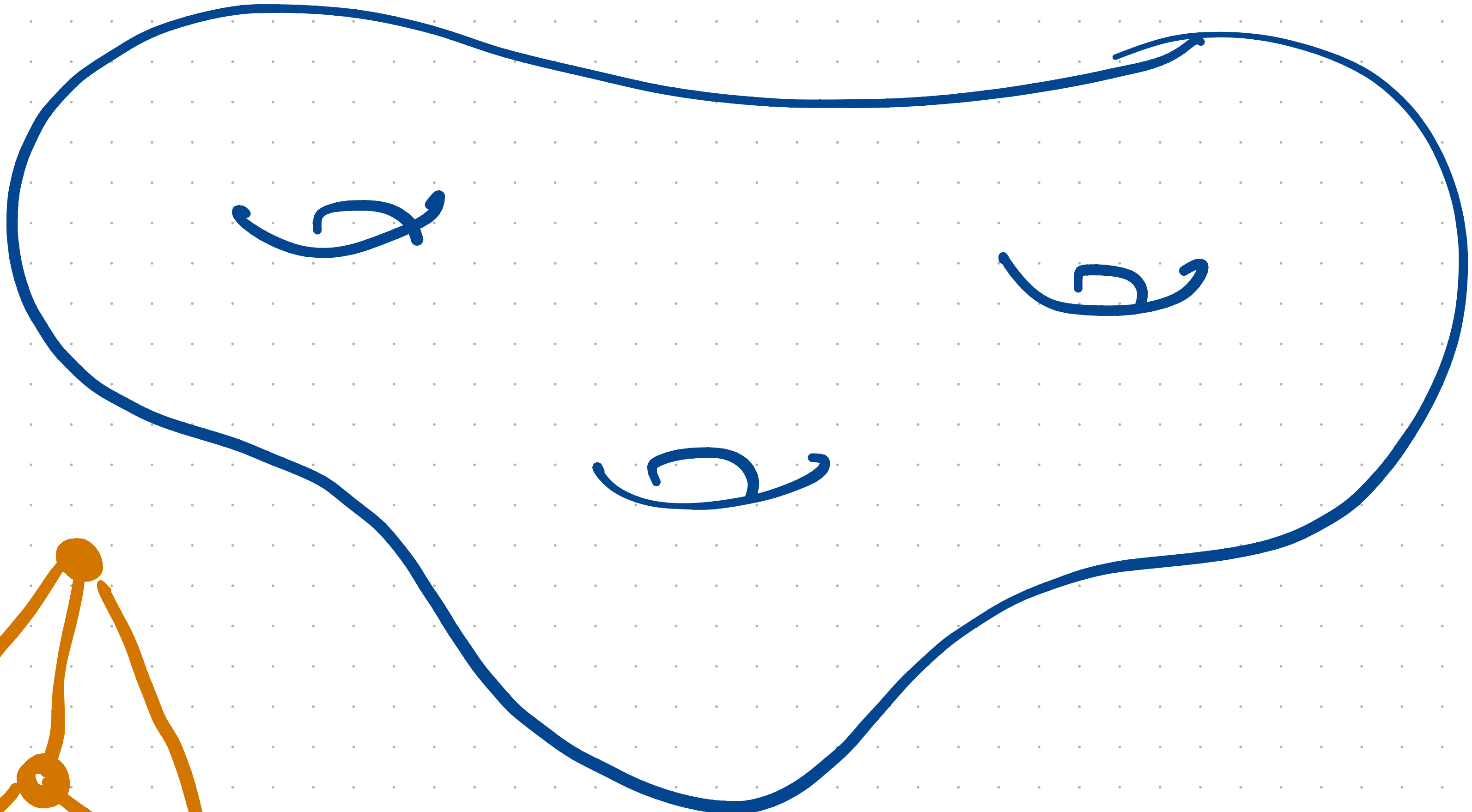
<u>Thm</u> (P Willwacher 2021)

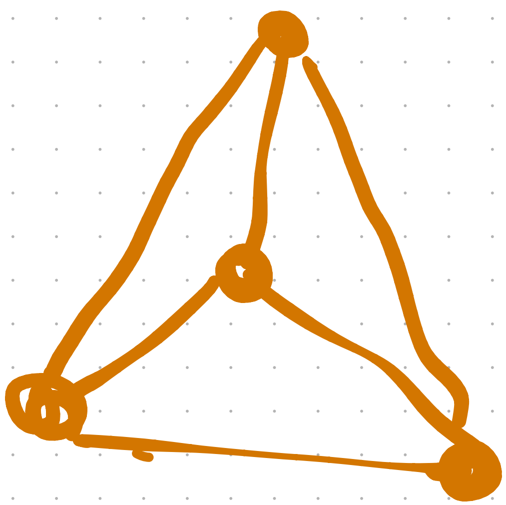
$$V_g^{\bullet-3} \oplus V_{g-1}^{\bullet-4} \subseteq \mathrm{gr}_2^W H_c^\bullet(\mathcal{M}_g).$$

Let $W_\bullet^{r,*} := \mathrm{Sym}^r\left(\bigoplus W_0 H_c^{*}(\mathcal{M}_g)\right)$

<u>Thm</u> (P Willwacher 2023)

$$V_{g-1}^{10,\bullet-21} \oplus V_{g-2}^{10,\bullet-22} \oplus V_{g-3}^{9,k-22} \oplus V_{g-5}^{6,k-22} \oplus V_{g-7}^{3,k-22} \subseteq \mathrm{gr}_{11}^W H_c^\bullet(\mathcal{M}_g)$$





THANK YOU