

Virasoro constraints in sheaf theory

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Vertex algebras

Joint work w/ A. Bojko, M. Moreira

Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$.

- I. Virasoro constraints
- II. Vertex algebras
- III. Wall-crossing formulas

I. Virasoro constraints

- X smooth projective variety / \mathbb{C}
- $\alpha \in H^*(X, \mathbb{Q})$
- H ample line bundle

* $\mathcal{M}_X^{\text{H-ss}}(\alpha) = M$: moduli of H -semistable sheaves F
 s.t. $\text{ch}(F) = \alpha$

very sensitive to X, α

"less sensitive" to $H \implies$ wall-crossing formulas

Assumptions ① \nexists strictly H -semistable sheaf in M

② $\text{Ext}^i(F, F) = 0 \quad \forall F \in M, \forall i \geq 3$

③ \exists universal sheaf \mathbb{F} on $M \times X$

① + ② \implies virtual fundamental class

$$[M]^{\text{vir}} \in A_{\text{vdim}}(M) \xrightarrow{\text{cl}} H_{2\text{vdim}}(M, \mathbb{Z}) \dashrightarrow H_{2\text{vdim}}(\mathcal{M}_X^{\text{vir}})$$

\uparrow
 $1 - \chi(\mathbb{F}, \mathbb{F})$

③ \implies natural cohomology classes $\sum_{\mathbb{F}} : \text{ID}^X \longrightarrow H^*(M, \mathbb{C})$

Def. Descendent algebra

$$ID^X := \left\langle ch_i^H(\sigma) \mid i \geq 0, \sigma \in H^*(X, \mathbb{C}) \right\rangle$$

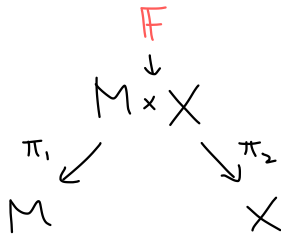
↑
formal symbols

$$* ch_i^H(\lambda_1 \sigma_1 + \lambda_2 \sigma_2) = \lambda_1 \cdot ch_i^H(\sigma_1) + \lambda_2 \cdot ch_i^H(\sigma_2)$$

Realization homomorphism

$$\Sigma_{\mathbb{F}} : ID^X \rightarrow H^*(M, \mathbb{C})$$

$$ch_i^H(\sigma) \mapsto \pi_{1,*} \left(ch_{\mathbb{F}} \cdot \pi_2^* \sigma \right)$$



$$i + \dim X - p \text{ if } \sigma \in H^{p,*}(X, \mathbb{C})$$

$$\Rightarrow \Sigma_{\mathbb{F}}(ch_i^H(\sigma)) \in H^{i+X}(M, \mathbb{C})$$

Virasoro operators

$$L_n = R_n + T_n \hookrightarrow ID^X, \quad n \geq -1$$

* R_n : derivation s.t.

$$R_n(ch_i^H(\sigma)) = i(i+1)\dots(i+n) ch_{i+n}^H(\sigma)$$

* T_n : multiplication by

$$T_n = \sum_{i+j=n} i! j! \sum_s (-1)^{\dim X - p_s^L} ch_i^H(\sigma_s^L) ch_j^H(\sigma_s^P)$$

$$\text{where } \Delta_{*+d}(X) = \sum_s \sigma_s^L \otimes \sigma_s^P$$

$$\text{Indeed, } [L_n, L_m] = (m-n) L_{n+m}$$

Wrong guess $\int_{[M]^{vir}} \sum_{\mathbb{F}} (L_n(D)) = 0, \quad \forall n \geq -1, \quad \forall D \in \mathbb{D}^X$

1) introduce "correction operator"

Conjecture ([Morozov-Oblomkov-Okounkov-Pandharipande], [M], [van Bree])

$$\int_{[M]^{vir}} \sum_{\mathbb{F}} \left((L_n + \frac{1}{r} S_n)(D) \right) = 0, \quad \forall n \geq -1, \quad \forall D \in \mathbb{D}^X$$

↑ ... ?

* [MOOP, M] : $M = PT_n(x, \beta), S^{[n]}$

* [vB] : $M = M_S^{H-ss}(r, c_1, c_2)$ where $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$

2) combine $\{L_n\}_{n \geq -1}$ to define

$$L_{wt_0} := \sum_{n=-1}^{\infty} \frac{(-1)^n}{(n+1)!} L_n \circ R_{-1}^{n+1} : \mathbb{D}^X \rightarrow \mathbb{D}_{wt_0}^X$$

$:= \ker(R_{-1})$

* $\Sigma = \sum_{\mathbb{F}} : \mathbb{D}_{wt_0}^X \rightarrow H^*(M, \mathbb{C})$

Conjecture (BLM) $\int_{[M]^{vir}} L_{wt_0}(D) = 0, \quad \forall D \in \mathbb{D}^X$

Proposition (BLM) 1) \iff 2) for old cases.

New cases : $M = M_c(r, d), M_S^{H-ss}(r, \beta, X), M_X^{H-ss}(r, c_1, c_2, c_3)$

← Favu 3-fold

Variation for pairs

$P_x^\sigma(V, \alpha) = P$: moduli of " σ -semistable" pairs

$$[V \xrightarrow{\varphi} F] \text{ st. } \text{ch}(F) = \alpha$$

Assumptions ① \nexists strictly σ -semistable pair in P

② $\text{Ext}^i([V \rightarrow F], F) = 0$ unless $i = 0, 1$.

③ $\exists!$ universal pair $[\pi_2^* V \xrightarrow{\psi} F]$ on $P \times X$

$$\Rightarrow [P]^{\text{vir}}, \sum_{(\pi_2^* V, F)} : \mathbb{D}^X \otimes \mathbb{D}^X \rightarrow H^*(P, \mathbb{C})$$

$\begin{matrix} \curvearrowright \\ \text{pair} \\ L_n \end{matrix}$

Conjecture (BLM) $\int_{[P]^{\text{vir}}} \sum_{(\pi_2^* V, F)} (L_n^{\text{pair}}(D)) = 0 \quad \forall n \geq 0, \forall D$

e.g. $C^{[n]}$, $S_\beta^{[0, m]}$, $\text{Quot}_S(V, \beta, n)$, $P_s^{t-\text{ss}}(\alpha)$

$\rightarrow [\mathcal{O}_S \rightarrow F]$ is μ_t -semistable if $t \in (0, \theta)$

* $0 \neq G \subsetneq F$, $\mu(G) \leq \mu(F) + \frac{t}{r(F)}$

* $s: \mathcal{O}_S \rightarrow G \subsetneq F$, $\mu(G) + \frac{t}{r(G)} \leq \mu(F) + \frac{t}{r(F)}$

II. Vertex algebras

$$\left(V, |0\rangle, T, Y(-, z) \right)$$

\mathbb{Z} -graded vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

vacuum vector
 $|0\rangle \in V_0$

state-field correspondence

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \text{End}(V) \llbracket z, z^{-1} \rrbracket$$

translation map

$$T: V_i \rightarrow V_{i+2}$$

Axioms for these data are related to 2d conformal field theory.

e.g. Locality: $\forall a, b \in V. \exists N \text{ s.t. } (z-w)^N [Y(a, z), Y(b, w)] = 0$

Example (free boson) * \mathfrak{h} \mathbb{C} -vector space

* \langle, \rangle symmetric non-degenerate form

1) $V := \text{Sym} \left[\bigoplus_{k > 0} \mathbb{C} t^{-k} \right]$ where $\deg(t) = -2$.

2) $|0\rangle = 1$ write $a \cdot t^k := a_{-k}$.

3) $T = \text{derivation s.t. } T(a_{-k}) = k \cdot a_{-k-1}$

4) $\forall a \in \mathfrak{h}, Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \text{End}(V) \llbracket z, z^{-1} \rrbracket$

* $[a_{(n)}, b_{(m)}] = \delta_{n+m, 0} n \cdot \langle a, b \rangle \quad \forall n, m \in \mathbb{Z}$

* $a_{(-k-1)} = 0 \cdot a_{-k-1}, a_{(k)} |0\rangle = 0 \quad \forall k \geq 0$

Virasoro constraints
in sheaf theory



Vertex algebras

* In many examples, \exists Virasoro algebra $\hookrightarrow V$.

↑ Lie algebra spanned by $\{L_n\}_{n \in \mathbb{Z}}$ & c .

* Recently, D. Joyce introduced sheaf theoretic vertex algebra.

$$(\mathcal{M}_X, \mathcal{O}, \Sigma, \Psi, \Theta) \rightsquigarrow V_{\bullet}^{\text{Joyce}} = H_*(\mathcal{M}_X, \mathbb{C})$$

$$\text{Crucially, } [M]^{\text{vir}} \in H_*(\mathcal{M}_X^{\text{rig}}, \mathbb{C}) \simeq \underbrace{V_{\bullet}^{\text{Joyce}} := V_{\bullet+2} / T(V_{\bullet})}_{\exists \text{ Lie bracket operation } \nabla}$$

↪ Lie bracket operation ∇

Def. $w \in V_{\bullet}$ is a *conformal element* if

$$Y(w, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfies 1) $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0} \cdot \frac{n^3-n}{12} \cdot c$

2) $L_0 \in V_{\bullet}$ diagonalizable. 3) $L_{-1} = T$.

Example. $V_{\bullet} = \text{Sym} \left[\bigoplus_{k \geq 0} \mathfrak{h} t^{-k} \right]$

$c = \dim(\mathfrak{h})$: central charge

$$w := \frac{1}{2} \sum_{a \in B} \hat{a}_{-1} \cdot a_{-1}$$

Assume X is either curve or surface w/ $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$

Thm (BLM) There is a **natural** conformal element

$$\omega \in V_{\bullet}^{\text{pair}} = H_{\star}(\mathcal{M}_X \times \mathcal{M}_X, \mathbb{C})$$

$$\text{s.t. } H^{\star}(\mathcal{M}_\alpha \times \mathcal{M}_\beta) \otimes H_{\star}(\mathcal{M}_\alpha \times \mathcal{M}_\beta) \longrightarrow \mathbb{C}$$

\uparrow
 L_n^{pair} : Virasoro operator
 on $\mathbb{D}^X \otimes \mathbb{D}^X$

\uparrow
 L_n^{pair} : Virasoro operator
 from $\underline{\omega}$

Remark We use the computation of $H_{\star}(\mathcal{M}_\alpha)$ by [Gross].

$$\text{In particular, } \mathbb{D}^X \otimes \mathbb{D}^X \xrightarrow{\cong} H^{\star}(\mathcal{M}_\alpha \times \mathcal{M}_\beta).$$

Corollary ① M satisfies Virasoro constraints

$$\Leftrightarrow [M]^{\text{vir}}, \omega = 0 \in V_{\bullet}^{\text{pair}}$$

$$\Leftrightarrow [M]^{\text{vir}} \in \check{P}_0 \subset (\check{V}_{\bullet}^{\text{pair}}, [\cdot, \cdot])$$

② P satisfies Virasoro constraints

$$\Leftrightarrow L_n^{\text{pair}}([P]^{\text{vir}}) = 0 \in V_{\bullet}^{\text{pair}}, \forall n \geq 0$$

$$\Leftrightarrow [P]^{\text{vir}} \in P_0 \subset V_{\bullet}^{\text{pair}}$$

\check{P}_0 : Lie subalgebra

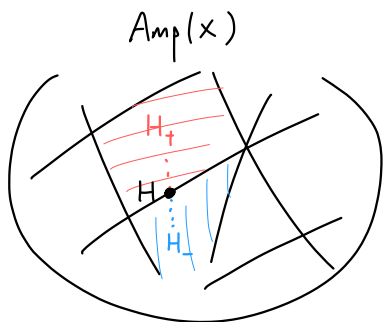
P_0 : Lie alg rep'n.

*: \check{P}_0, P_0 : space of primary/physical states.

III. Wall-crossing formulas

Thm (Joyce) " Wall-crossing formulas for $[M_x^{H-ss}(\alpha)]^{inv} \in (\check{V}_{\bullet}^{Joyce}, [,])$ are explicitly written in terms of $[,]$.

Example (Simple wall-crossing)



$$F \in M_x^{H_+ - ss} \setminus M_x^{H_- - ss}$$

iff $0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$, $F_i \in M_x^{H_{\pm} - ss}(\alpha_i)$
 * similarly for other complement with F_1, F_2 swapped
 ↑ irreducible

$$\Rightarrow [M_x^{H_+ - ss}(\alpha)]^{vir} - [M_x^{H_- - ss}(\alpha)]^{vir} = \left[[M_x^{H_{\pm} - ss}(\alpha_1)]^{vir}, [M_x^{H_{\pm} - ss}(\alpha_2)]^{vir} \right]$$

Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$.

key ideas of proof

* rank = 1 case : Virasoro constraints for $S^{[n]}$ [Moreira]

* wall-crossing [Joyce]

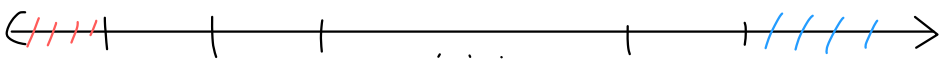
1) $P_S^{\infty}(\alpha) \leftrightarrow P_S^{0+}(\alpha), \quad t \in (0, \infty)$

2) $\gamma_\alpha = \chi(\alpha) \cdot [M_x^{H-ss}(\alpha)]^{inv} + (\text{low rank } \alpha\text{'s})$

* Relation to Virasoro constraints [BLM]

1) wall-crossing compatibility (\check{P}_0, P_0 : Lie algebra, rep'n)

2) projective bundle compatibility.



② $P_S^{0+}(\alpha) \xleftrightarrow{t \in \mathbb{R}_{>0}} \textcircled{1} P_S^{\infty}(\alpha) = \begin{cases} \emptyset & \text{if } rk\alpha > 1 \\ S_P^{[0,m]} & \text{if } rk\alpha = 1 \end{cases}$

projective bundle compatibility

③ $\gamma_\alpha \stackrel{\text{def}}{=} f_* \left(c_{top}(T_f) \cap [P_x^{0+}(\alpha)]^{vir} \right)$

$= \underbrace{\chi(\alpha)}_{>0} \underbrace{[M_x^{H-ss}(\alpha)]^{inv}}_{\textcircled{4}} + (\text{lower rank})$