THE PROPER LG SUPERPOTENTIAL IS THE OPEN MIRROR MAP

w/ GRÄFNITZ, RUDDAT

This talk is about

$$M(Q) = 1 + 2Q + 5Q^2 + 32Q^3 + 286Q^4 + \dots$$

$$\left(= \left(\frac{Q}{z(Q)}\right)^{1/3}_{*} \text{, where } Q(z) = z \exp\left(\sum_{k \geq 1} \frac{(3k)!}{k(k!)^3} z^k \right) \right)$$

It has appeared in many contexts

- Thm: (a) Superpotential of mirror of $(\mathbb{P}^2, E_{\text{smth}})$ [GRZ]

- Open mirror map for AV branes in $K_{\mathbb{P}^2}$ [AKV, GZ]
- Open/closed Picard-Fuchs equations for MS [M, L-M]

$K_{\mathbb{P}^2}$ mir: closed mirror map, open mirror map M

$$uv = 1 + x + y + \frac{z}{xy} \sim z^{-1/3}_{*} + x + y + \frac{1}{xy}$$

- Slab function in G-S construction for mirror of $K_{\mathbb{P}^2}$ [G-S]
- Open invts of $K_{\mathbb{P}^2}$ and closed invts of $K_{\widehat{\mathbb{P}^2}}$ [C, CLL, CLT, CCLT]

$$M(Q) = \sum_d n_{dL-C} (-Q)^d$$

Road Map

[CPS] : Tropical def. of W

Then : Tropical - Relative

Relative - Local (Local - Open given by Mayr-Lerche)

ALSO: W from wall crossing. Rmk: Also higher genus (Bousseau)

$\mathbb{P}$ toric Fano $\qquad W : (\mathbb{C}^\times)^2 \to \mathbb{C}$ $\qquad$ Hori-Vafa mirror (toric)

$\mathbb{P} = \mathbb{P}^2$ $\qquad W = x + y + \frac{1}{xy}$ $\qquad$ HV, Cho-Oh

$(\mathbb{P}, E)$, E smooth $\qquad W = \sum_\beta n_\beta Q^\beta$ $\qquad$ Fukaya, FOOO

Gross fibration

Auroux, Seidel : W jumps across walls

Toric degeneration of pair $\Rightarrow$ Tropical : (Scattering, broken lines)

$\mathcal{X} = \{xyz - t(V + f_3)\} \subset \mathbb{P}(1113) \times \mathbb{A}^1$

$\cup$

$\mathcal{D} = \{V = 0\}$

Carl-Pumperla-Siebert

Gross-Pandharipande-Siebert

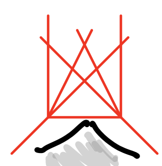
van Garrel-Graber-Ruddat

Gräfnitz

Bousseau $(g > 0)$

Scattering Diagram

(dual $\cap$ cpx)



Thm $\qquad (\mathbb{P}^2, E)$

$Q^\beta = x^a y^b$

Unbounded chamber $(\Rightarrow a = 0)$

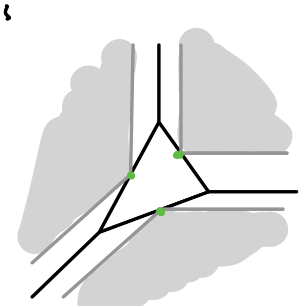
$$W = y M(Q), \quad Q = (t/y)^3$$

$$= y(1 + 2Q + 5Q^2 + 32Q^3 + \dots)$$

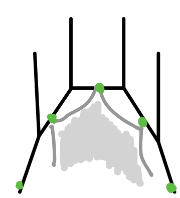
$Q^d \longleftrightarrow y^{-(3d-1)}$

Computing $(\mathbb{P}^2, E)$:

Toric degeneration $\xrightarrow{GS}$ Dual intersection Cpx $\xrightarrow{GPS}$ Consistent Scattering



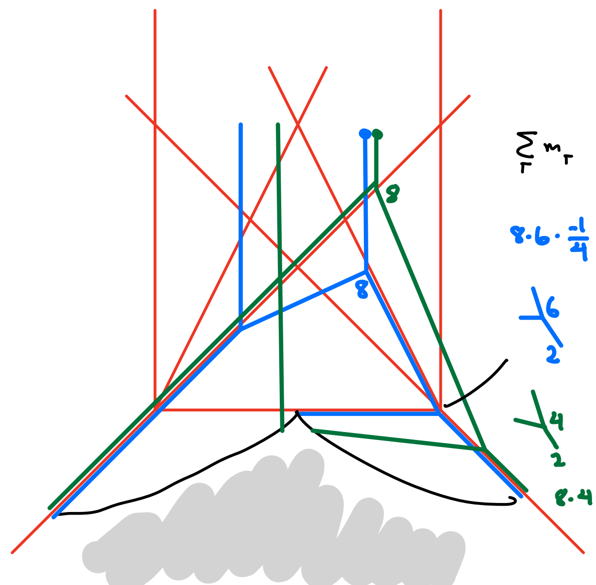
unwrap [CPS, G] $\longrightarrow$



COMPUTE $32Q^3$

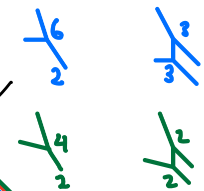
$(8 = 3\cdot 3 - 1)$

$m_\Gamma = \frac{1}{|\mathrm{Aut}\,\Gamma|}\cdot m_V \cdot m_E$



$\sum_\Gamma m_\Gamma =$

$8\cdot 6\cdot \frac{-1}{4} + \frac{1}{2}\cdot 8\cdot 3\cdot 3 = 24$



$8\cdot 4\cdot \frac{-1}{4} + \frac{1}{2}\cdot 8\cdot 2\cdot 2 = 8$

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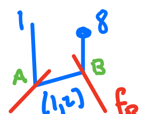
Or: Broken Lines



$1+t^3x^{-1}y^{-3} = f_A$ $\qquad$ $f_B = 1+3t^6xy^{-6}$

$$y^{-1} \underset{A}{\rightsquigarrow} y^{-1} f_A^{(0,1)\wedge(1,2)} = t^3x^{-1}y^{-2} \underset{B}{\rightsquigarrow} t^3x^{-1}y^{-2} f_B^{(1,2)\wedge(0,8)} \Rightarrow 24\,t^9y^{-8}$$

By extending end upwards $\rightsquigarrow \uparrow$, can relate broken lines / tropical disks to 2-pointed relative invts.

DEFS:

- $R_{p,q} = \sum_{\beta\cdot E = p+q} R_{p,q}(\beta)\, t^\beta$

class $\beta \in H_2(\mathbb{P})$

fixed pt mult p along E

moving pt mult q along E

$$= \sum_\beta \sum_g R^g_{p,q}(\beta)\, t^\beta \hbar^{2g}$$

$$R^g_{p,q}(\beta) = \int_{[\mathcal{M}^{rel}_{g,2}(X,\beta)]} (-1)^g \lambda_g\, ev_1^*(pt)$$

- $R^{trop}_{p,q} = \sum_{T \text{ trop curves}} m_T(e^{i\hbar})\, t^{\beta(T)}$

Relations

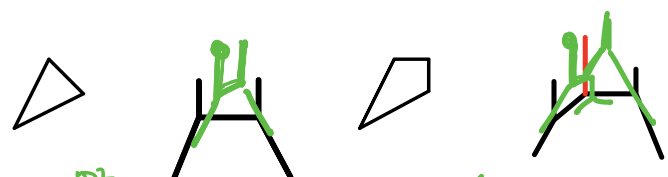
Tropical-relative and blow-up relations

$$\hat\beta = \beta - C$$

$$\begin{array}{ccccc}
 & R^{trop}_{1,n}(X,\beta) & \overset{[GRZ]^*}{=} & R^{trop}_n(\hat X,\hat\beta) & \\
[Grä] & \Vert & & \Vert & [Grä] \\
 & R_{1,n}(X,\beta) & & R_n(\hat X,\hat\beta) &
\end{array}$$

tangency c $\hat E$

* CORRESPONDENCE BET. $\mathbb{P}^2$ TROP DISKS & F_1



$\mathbb{P}^2$ $F_1 = \hat{\mathbb{P}}^2$

Gathmann-Chen $R_{1,n} = n^2 R_{n,1}$

vGGR $R_n(\hat X) = (-1)^{n+1} n N(K_{\hat X})$

$\Downarrow$

Relative - Local : $N(K_{\hat X},\hat\beta) = (-1)^{n+1} n R_{n,1}(X,\beta)$

Open - Open : Inv'ts of AV branes $\underset{\mu=0}{}$ = Inv'ts of T^3 fibers $\underset{\mu=2}{}$

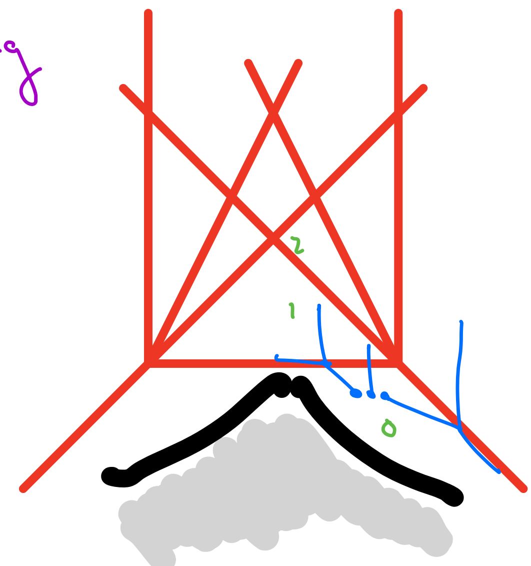
Open - Local [C, CLL]: $M(Q) = 1 + \sum_{d\geq 1} N^0(K_{F_1}, dL-C)(-Q)^d$

CONCLUSION :

$$\begin{cases} = 1 + \sum n R_{n,1}(\mathbb{P}^2; dL)\, Q^d \\ = 1 + \sum R^{trop}_{n,1}(\mathbb{P}^2)\, Q^d \\ = y^{-1} W(Q), \quad Q = (t/y)^3 \end{cases}$$

Rmk: $y^{-(3d-1)}$ ✓

Wall crossing



$y + x + \frac{t^3}{xy}$

$\int$

$W_0 = t\left(y + x^{-1}y + \frac{x}{y^2}\right)$

W_1: (0,-1) $f = t(1+x^{-1})$

$y \to y f^{(0,-1)\cdot(0,1)}$

$x^{-1}y \to x^{-1}y f^{(0,-1)\cdot(-1,1)}$

$xy^{-2} \to xy^{-2} f^{(0,-1)\cdot(1,-2)}$

$t(1+x^{-1})y \to t(1+x^{-1})y f^{-1} = y$

$txy^{-2} \to txy^{-2} f^{2} = t^3 xy^{-2}(1 + 2x^{-1} + x^{-2})$
$= t^3 2y^{-2} + \dots$

$W_1 = 1y + 2t^3y^{-2} + \dots$
$= y(1 + 2Q + \dots)$