Castelnuovo bound and Gromov–Witten invariants of the quintic 3-fold

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Introduction

- BCOV conjectures and Castelnuovo bound.
- Results.
- Idea of proof.

Based on the joint work with Yongbin Ruan, arXiv: 2210.13411

Review: Gromov–Witten invariants

Let X be a compact Calabi–Yau (CY) 3-fold and $\beta \in H_2(X, \mathbb{Z})$ be a homology class. We are interested in counting algebraic curves in X of class β.

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- Modern approach: Gromov–Witten (GW) invariants (symplectic: Ruan–Tian; algebraic: Li–Tian, Behrend–Fantechi)
$$\overline{M}_g(X, \beta) = \{\text{stable maps } f : C \to X \mid f_*[C] = \beta\}$$
$$N_{g,\beta} := \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

- DT/PT/GV-invariants...

GW-invariants and physics

For simplicity, we restrict ourselves to the quintic 3-fold $X \subset \mathbb{P}^4$, one of the most famous compact CY 3-folds.

We define the genus g GW potential of X as

$$F_g(t) := \sum_{d \geq 0} N_{g,d} t^d.$$

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We consider two different string theories: type IIA (A-model) and type IIB (B-model). The A-model corresponds to the geometry on X while the B-model corresponds to the mirror quintic. In A-model, F_g is the genus g topological string amplitude.

$$\text{B-modle} \overset{\textit{Mirror Symmetry}}{\Longleftrightarrow} \text{A-model}.$$

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- $g = 3$: predicted by Katz-Klemm-Vafa, using the reformulation of topological string amplitudes as a computation of BPS states in M-theory compactifications.
- Higher genus: predicted by Huang-Klemm-Quackenbush for all $F_g(t)$ for $g \leq 53$! Using four B-model conjectures and one A-model conjecture.

Four B-model conjectures

- **Yamaguchi–Yau's finite generation** asserts that

$$F_g(t) = Y^{-(g-1)} P_g(Z_1, Z_2, Z_3, Z_4, Y)$$

for a polynomial P_g where Z_i and Y are explicit generating series in t constructed from genus zero data.

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- **Holomorphic anomaly equation** implies that the monomials of P_g containing Z_i are determined by lower genus data. Moreover,

$$f_g := P_g(0, 0, 0, 0, Y) = \sum_{i=0}^{3g-3} a_{i,g} Y^i$$

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- **The orbifold regularity** asserts that $a_{i,g} = 0$ for $i \leq \lceil \frac{3g-3}{5} \rceil$.
- **The conifold gap condition** determines $a_{i,g}$ for $i \geq g$ recursively from lower genus data.

Four B-model conjectures

Using these four B-model conjectures, we only need to determine $\{a_{i,g}\}_{i=\lceil \frac{3g-3}{5} \rceil+1}^{g-1}$ to derive $F_g(t)$. In other words, we need to fix $\lfloor \frac{2g-2}{5} \rfloor$ many initial conditions for each genus g.

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- Yamaguchi–Yau's finite generation and holomorphic anomaly equation are proved by Guo-Janda-Ruan and Chang-Guo-Li independently.
- The orbifold regularity is proved by Guo-Janda-Ruan.
- The conifold gap condition is proved by Guo-Janda-Ruan for $g \leq 5$.

The A-model conjecture: Castelnuovo bound

Let

$$Z_{GW}(\lambda, t) = \sum_{g\geq 0}(F_g(t) - N_{g,0})\lambda^{2g-2} = \sum_{d>0}\sum_{g\geq 0} N_{g,d}\lambda^{2g-2}t^d.$$

We expand it in terms of the Fourier series and obtain

$$Z_{GW}(\lambda, t) = \sum_{g\geq 0}\sum_{d>0}\sum_{r\geq 1}\frac{n_{g,d}}{r}\cdot\left(2\sin\left(\frac{r\lambda}{2}\right)\right)^{2g-2}\cdot t^{rd}.$$

The coefficients $n_{g,d}$ are called **Gopakumar–Vafa (GV) invariants**. In physics, these invariants count the number of BPS states in the associated M-theory.

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- (Ionel–Parker) $n_{g,d}\in\mathbb{Z}$.
- (Doan–Ionel–Walpuski) $n_{g,d}=0$ for $g\gg 0$.

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Conjecture (Castelnuovo bound)

We have $n_{g,d} = 0$ for any

$$g > \frac{d^2 + 5d + 10}{10}.$$

Castelnuovo bound

Theorem

The Castelnuovo bound conjecture holds.

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Corollary

1. If $g \leq 53$ and $g \neq 51$, then
$$n_{g,d} = 0$$
for $d \leq \lfloor \frac{2g-2}{5} \rfloor$.
2. If $g = 51$, then
$$n_{51,d} = 0$$
for $d \leq \lfloor \frac{2g-2}{5} \rfloor - 1 = 19$.

The above corollary together with B-model conjectures allows us to compute $F_g(t)$ for $g \leq 50$. For $51 \leq g \leq 53$, we need another theorem.

Non-vanishing of GV-invariants

Theorem

Let $m \geq 2$ and $d = 5m$ be integers. Then we have

$$n_{\frac{d^2+5d+10}{10},d} = (-1)^{\binom{m+3}{3}-\binom{m-2}{3}+3} \cdot 5\left(\binom{m+3}{3} - \binom{m-2}{3}\right).$$

In particular, we have $n_{51,20} = 175$.

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Since four B-model conjectures above have been proved except the conifold gap condition, combined with the results above we have:

Corollary

Let $G \leq 53$ be a positive integer. Assume that the conifold gap condition holds for $F_g(t)$ and any $g \leq G$. Then we can compute $F_g(t)$ effectively up to genus G.

GV-invariants via PT-invariants

Let $P_{n,d}$ be the **Pandharipande–Thomas (PT) invariants** which counts stable pairs

$$[\mathcal{O}_X \xrightarrow{s} F] \in \mathrm{D}^b(X), \quad \chi(F) = n, \quad [F] = d \in H_2(X, \mathbb{Z}) = \mathbb{Z}$$

with F pure 1-dim and $\dim \operatorname{cok}(s) = 0$. The generating series of PT-invariants is defined as

$$Z_{PT}(q, t) := 1 + \sum_{d>0} \sum_{n \in \mathbb{Z}} P_{n,d} q^n t^d.$$

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We have the GW/PT correspondence proved by Pandharipande-Pixton:

$$Z_{GW}(\lambda, t) = F_{PT}(q, t) := \log Z_{PT}(q, t)$$

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Then combined with the finiteness result of Doan–Ionel–Walpuski, we obtain

$$F_{PT}(q,t) = \sum_{g \geq 0} \sum_{d>0} \sum_{r \geq 1} n_{g,d} \frac{(-1)^{g-1}}{r} \left((-q)^{\frac{r}{2}} - (-q)^{-\frac{r}{2}}\right)^{2g-2} t^{rd}.$$

GV-invariants via PT-invariants

To prove the Castelnuovo bound conjecture, it is sufficient to bound the genus of 1-dim subschemes:

Conjecture

For any 1-dim closed subscheme $C \subset X$ of degree d and (arithmetic) genus g, we have

$$g \leq \frac{d^2 + 5d + 10}{10}.$$

If this conjecture is true, then by the emptiness of moduli spaces we have $P_{n,d} = 0$ for any $1 - n > \frac{d^2+5d+10}{10}$. Then a calculation of the generating series implies the vanishing of connected PT and GV-invariants.

Bound for the genus

Theorem (Hartshorne)

Let $C \subset \mathbb{P}^3$ be a 1-dim closed subscheme of degree d and genus g. Then we have $g \leq \frac{(d-1)(d-2)}{2}$. Moreover, if C is not contained in $\mathbb{P}^2$, then we have $g \leq \frac{(d-2)(d-3)}{2}$.

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Theorem

Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree $n \leq 5$. Then for any one-dimensional closed subscheme $C \subset X$ of degree d and genus g, we have

$$g \leq \frac{1}{2n}d^2 + \frac{n-4}{2}d + 1.$$

Moreover, when C is not contained in any hyperplane section of X, we have

$$g \leq \frac{1}{2n}d^2 + (\frac{n-4}{2} - \frac{1}{n})d + 2 + \frac{1}{n}.$$

Extremal curves

Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree $n \leq 5$.

Theorem

Let C be a one-dimensional closed subscheme $C \subset X$ of degree $d > n$ and genus g. Then

$$g = \frac{1}{2n}d^2 + \frac{n-4}{2}d + 1$$

if and only if C is a complete intersection of a hyperplane section and a degree $\frac{d}{n}$ hypersurface section of X.

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Thus, the Hilbert scheme $\mathrm{Hilb}_X^{dt - \frac{1}{2n}d^2 - \frac{n-4}{2}d}$ is a projective bundle over $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_X(1)))$.

Then the non-vanishing result for GV-invariants of quintic follows from a a computation of DT-invariants and DT/PT correspondence (Bridgeland, Toda).

Slope stability

Let (X, H) be a polarised smooth projective variety of dimension n. For any sheaf $E \in \mathrm{Coh}(X)$, we define

$$\mu_H(E) := \begin{cases} \frac{H^{n-1}\mathrm{ch}_1(E)}{H^n\mathrm{ch}_0(E)}, & \text{if } \mathrm{ch}_0(E) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

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- (Bogomolov–Gieseker) For any μ_H-semistable object $E \in \mathrm{Coh}(X)$, we have

$$\Delta_H(E) = \left(H^{n-1}\mathrm{ch}_1(E)\right)^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E) \geq 0.$$

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$$\mu_{a,b}(E) := \begin{cases} \frac{-\frac{1}{2}a^2 H^n \mathrm{ch}_0^{bH}(E) + H^{n-2}\mathrm{ch}_2^{bH}(E)}{H^{n-1}\mathrm{ch}_1^{bH}(E)}, & \text{if } \mathrm{ch}_1^b(E) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

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- Any object $E \in \mathcal{A}^b$ has a HN filtration in terms of $\mu_{a,b}$-semistability defined above.
- For any $\mu_{a,b}$-semistable object $E \in \mathcal{A}^b$, we have $\Delta_H(E) \geq 0$.

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- For any $\mu_{a,b}$-semistable object $E \in \mathcal{A}^b$, we have $\Delta_H(E) \geq 0$.

In other words, we have a family of "stability conditions" parametrized by $\mathbb{R}_{>0} \times \mathbb{R}$

$$(a, b) \mapsto (\mathcal{A}^b, \mu_{a,b}).$$

Bayer–Macrì–Toda's generalized Bogomolov–Gieseker inequality

BMT Conjecture (Bayer–Macrì–Toda)

Let (X, H) be a polarised smooth projective 3-fold. Assume that $(a, b) \in \mathbb{R}_{>0} \times \mathbb{R}$ and E is any $\mu_{a,b}$-semistable object. Then

$$Q_{a,b}(E) := a^2\Delta_H(E) + 4\left(H\mathrm{ch}_2^{bH}(E)\right)^2 - 6\left(H^2\mathrm{ch}_1^{bH}(E)\right)\mathrm{ch}_3^{bH}(E) \geq 0.$$

Bayer–Macrì–Toda's generalized Bogomolov–Gieseker inequality

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Theorem (Chunyi Li)

Let X be the quintic 3-fold. Then the BMT conjecture holds for any

$$(a, b) \in U := \{(a, b) \in \mathbb{R}_{>0} \times \mathbb{R} \mid a^2 > (b - \lfloor b \rfloor)(\lfloor b \rfloor + 1 - b)\}.$$

Sketch of proof

We fix X to be the quintic 3-fold from now on.

- **A simple observation**: applying Li's theorem to $\mathcal{I}_C$ at $(a, b) = (\sqrt{\frac{2}{9}}, -\frac{4}{3})$, we obtain the desired bound.

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- **Bad news**: we can not do this in general, since $\mathcal{I}_C$ are only $\mu_{a,b}$-semistable for $b < 0$ and $a \gg 0$ in general. When a goes small, $\mathcal{I}_C$ will meet many walls, which destabilize $\mathcal{I}_C$.

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- **Good news**: there is a nice subset $\mathcal{V} \subset \mathbb{R}_{>0} \times \mathbb{R}_{<0}$ such that $(a, -\frac{4}{3}) \in \mathcal{V}$ for any $a > \sqrt{\frac{2}{9}}$ and the walls (HN factors) for $\mathcal{I}_C$ in $\mathcal{V}$ can be controlled.

Sketch of proof

We only need to consider pure 1-dim subschemes (curves). We have the following types of curves.

- **(Type I)**: $\mathcal{I}_C$ has no wall in $\mathcal{V}$.
- **(Type II)**: the upper-most wall is given by $\mathcal{O}_X(-H)$. In other words, $C \subset H$.
- **(Type III)**: other cases.

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- Bound for type I: Applying BMT's inequality on the boundary of $\mathcal{V}$, we can get a better bound than Castelnuovo bound.
- Bound for type II: Applying wall-crossing and BMT's inequality to the torsion sheaf $\mathcal{I}_{C/H}$.
- Bound for type III: $C = C_1 \cup C_2$, such that $\mathrm{length}(C_1 \cap C_2) = d_1$ and $d_1 < E(d)$. Then we can do induction and use

$$g(C) = g(C_1) + g(C_2) + \mathrm{length}(C_1 \cap C_2) - 1.$$

In this case, we can get a better bound than Castelnuovo bound.

Sketch of proof

- If $C \subset X$ has degree d and genus $g = \frac{1}{2n}d^2 + \frac{n-4}{2}d + 1$, then by the bound above, we know that C is of Type II (i.e. $C \subset H$).

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- Thus we can regard C as a curve in $H = X \cap \mathbb{P}^3 \subset \mathbb{P}^3$ and do wall-crossing on $\mathbb{P}^3$ for $\mathcal{I}_{C/H}$.
- Key point: if $g = \frac{1}{2n}d^2 + \frac{n-4}{2}d + 1$, then by BMT conjecture on $\mathbb{P}^3$ proved by Macrì, we can not cross the wall given by $\mathcal{O}_{\mathbb{P}^3}(\frac{d}{n})$. This means

$$\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_{C/H}(\frac{d}{n})) \neq 0.$$

Thanks!