Curves in Calabi-Yau threefolds and Topological Quantum Field Theory

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Abstract

We continue our study of the local Gromov-Witten invariants of curves in Calabi-Yau threefolds.

We define relative invariants for the local theory which give rise to a 1+1-dimensional TQFT taking values in the ring $\mathbb{Q}[[t]]$. The associated Frobenius algebra over $\mathbb{Q}[[t]]$ is semisimple. Consequently, we obtain a structure result for the local invariants. As an easy consequence of our structure formula, we recover the closed formulas for the local invariants in case either the target genus or the degree equals 1.

1 Notation, definitions and results

A central problem in Gromov-Witten theory is to determine the structure of the Gromov-Witten invariants. Of special interest is the case where the target manifold is a Calabi-Yau threefold. We prove a structure result for the local Gromov-Witten invariants of a curve in a Calabi-Yau threefold.

First, we define a relative version of the local invariants. In Theorem 1.2, we prove the new invariants determine a 1+1-dimensional semisimple TQFT taking values in the ring $\mathbb{Q}[[t]]$. The structure formula for the invariants, Theorem 1.3, is obtained from the semisimple TQFT and completely determines the dependence of the invariants on the target genus.

The local invariants are derived from considering rigid curves in Calabi-Yau threefolds. Let X be a non-singular curve of genus g in a Calabi-Yau threefold Y. Assuming certain rigidity conditions on $X \subset Y$, there are well defined local Gromov-Witten invariants of X in Y. That is, the contribution to the Gromov-Witten invariants of Y by maps with image X is well defined. These local invariants depend only on g, h, and d, which are respectively, the target genus, the domain genus, and the degree of the maps to X.

In [4], we defined an integral depending only on d, h, and g that gives the value of the local invariants whenever $X \subset Y$ satisfies the requisite rigidity. The integral is given by

$$N_d^{h-g}(g) = \int_{[\overline{M}_h(X,d[X])]^{vir}} c(I(X)),$$

see [4] (c.f. [3]) for details of this discussion. Here, $c(\cdot)$ denotes total Chern class and $I(X)$ is defined by

$$\begin{aligned} I(X) &= -R^{\bullet}\pi_* f^*(K_X \oplus \mathcal{O}_X) \\ &= R^1\pi_* f^*(K_X \oplus \mathcal{O}_X) - R^0\pi_* f^*(K_X \oplus \mathcal{O}_X) \end{aligned}$$

which is understood to be an element of K-theory.

It will be convenient to work with the version of the local invariants corresponding to stable maps with possibly disconnected domains:

Definition 1.1. *Let X be a non-singular curve of genus g. Let $\overline{M}^{\bullet}(X,d)$ be the moduli space of degree d stable maps*

$$f : C \to X$$

where C is a possibly disconnected curve. We require f to be nonconstant on each connected component of C. Let

$$[\overline{M}^{\bullet}(X,d)]^{vir} \in A_*(\overline{M}^{\bullet}(X,d);\mathbb{Q})$$

denote the virtual fundamental cycle of the moduli space.

Following [22], the superscript $^{\bullet}$ is used to denote the moduli space with possibly disconnected domain curves. The usual genus subscript is omitted as we consider all domain genera: the moduli space is a countable union of connected components with varying expected dimensions. The branch points of a stable map to X are well-defined by [10]. The number of branch points b of map f equals the expected dimension of the moduli space at the moduli point $[f]$.

We define the **(possibly disconnected) local Gromov-Witten invariants** to be

$$Z_d^b(g) = \int_{[\overline{M}^\bullet(X,d)]^{vir}} c_b(I(X)).$$

The relationship between the possibly disconnected invariants and the (connected) invariants $N_d^{h-g}(g)$ is easily seen to be

$$\sum_{d>0}\sum_{b\geq 0} Z_d^b(g)t^b q^d = \exp\left(\sum_{d>0}\sum_{b\geq 0} N_d^{h-g}(g)t^b q^d\right)$$

where

$$2h-2 = (2g-2)d + b.$$

The generating function for the degree d, local, disconnected invariants is

$$Z_d(g) = \sum_{b=0}^{\infty} Z_d^b(g)t^b.$$

The series $Z_d(g)$ is our basic object of study. Clearly, the disconnected invariants $Z_d^b(g)$ and the connected invariants $N_d^{h-g}(g)$ contain equivalent information.

In Section 3, we use J. Li's theory of relative stable maps [18, 19] to construct relative versions of the local invariants. These relative invariants obey a gluing law which allows us to construct a Topological Quantum Field Theory (TQFT):

Theorem 1.2. *There exists a $1+1$-dimensional TQFT, $Z_d(-)$, with the following three properties:*

(i) $Z_d(-)$ is semisimple,

(ii) $Z_d(-)$ takes values in $\mathbb{Q}[[t]]$,

(iii) $Z_d(-)$ applied to a genus g closed surface yields the value $Z_d(g)$, the generating series of the local invariants.

The $t=0$ specialization of $Z_d(-)$ is a well-known TQFT obtained from the gauge theory of the symmetric group S_d, see Lemma 4.3. The TQFT determined by S_d was studied by Dijkgraaf-Witten and Freed-Quinn [6, 11]. Our TQFT may be viewed as a 1-parameter deformation of the Dijkgraaf-Witten/Freed-Quinn theory.

Corresponding to any 1+1-dimensional TQFT is a Frobenius algebra. In our case, the dimension of the corresponding Frobenius algebra is $p(d)$, the number of partitions of d. As a corollary of Theorem 1.2, we deduce the following structure formula.

Theorem 1.3. *There exist universal power series $\lambda_\alpha \in \mathbb{Q}[[t]]$, labelled by partitions α of d (denoted $\alpha \vdash d$), which determine the local invariants by:*

$$Z_d(g) = \sum_{\alpha \vdash d} \lambda_\alpha^{g-1}.$$

Moreover, the constant term of the series λ_α is given by

$$\left(\frac{d!}{\dim R_\alpha}\right)^2$$

where R_α is the irreducible representation of the symmetric group associated to α.

The two main theorems of [23] (theorems 1 and 2) compute the local invariants in the case of degree 1 and the case of target genus 1. We recover these two results as immediate corollaries of the above structure theorem (Corollaries 1.5 and 1.4 below):

Corollary 1.4. *The series $Z_d(1)$ is the constant series $p(d)$. In particular, the genus two and higher multiple cover contributions of a super-rigid elliptic curve are all zero.*

By the localization calculation of Faber-Pandharipande [9], $Z_d(0)$ is given by

$$Z_d(0) = \sum_{\alpha \vdash d} \frac{t^{2d}}{\mathfrak{z}(\alpha)} \prod_{i=1}^{\ell(\alpha)} \left(2\sin(\frac{\alpha_i t}{2})\right)^{-2}$$

where the sum is over all partitions α of d. Here, $\ell(\alpha)$ is the length of the α, and $\mathfrak{z}(\alpha)$ is a combinatorial factor (see Definition 3.3). In particular, for $d = 1$, we have

$$Z_1(0) = \left(\frac{\sin(t/2)}{t/2}\right)^{-2}$$

which we combine with our structure formula to deduce the following:

Corollary 1.5. *The $d = 1$ local invariants are given by*

$$Z_1(g) = \left(\frac{\sin(t/2)}{t/2}\right)^{2g-2}.$$

Recent progress (to be explained in a forthcoming paper [2]) has allowed us to completely determine the TQFT $Z_d(-)$ for small values of d. In particular, for $d = 2$ we have

Theorem 1.6 ([2]). *The $d = 2$ local invariants are given by*

$$Z_2(g) = \left(\frac{\sin(t/2)}{t/2}\right)^{4g-4} \left\{(4 - 4\sin(t/2))^{g-1} + (4 + 4\sin(t/2))^{g-1}\right\}.$$

The above formula gives the double cover contributions of any smooth curve in a Calabi-Yau threefold with a generic normal bundle. The above formula also verifies the local Gopakumar-Vafa conjecture for degree 2 maps, i.e. the corresponding BPS invariants are integers. See [4] for a discussion of the BPS invariants and the local Gopakumar-Vafa conjecture.

2 Semisimple TQFTs over complete local rings

Let $(n+1)\mathbf{Cob}$ be the symmetric monoidal category with objects given by compact oriented n-manifolds and morphisms given by (diffeomorphism classes of) oriented cobordisms. An $(n+1)$**-dimensional TQFT with values in a commutative ring** R is a symmetric monoidal functor

$$Z : (n+1)\mathbf{Cob} \to R\mathbf{mod},$$

where $R\mathbf{mod}$ is the category of R-modules. The definition amounts to the following axioms for Z:

(i) To each compact oriented n-manifold Y, Z assigns an R-module $Z(Y)$.

(ii) To each oriented cobordism W from Y_1 to Y_2, Z assigns an R-module homomorphism $Z(W) : Z(Y_1) \to Z(Y_2)$.

(iii) If two oriented cobordisms are equivalent $W \cong W'$ by a boundary preserving diffeomorphism, then $Z(W) = Z(W')$.

(iv) The trivial oriented cobordism corresponds to the identity homomorphism, $Z(Y \times [0,1]) = \mathrm{Id}_{Z(Y)}$.

(v) The concatenation of cobordisms corresponds to the composition of the corresponding R-module homomorphisms.

(vi) The disjoint union of n-manifolds corresponds to the tensor product of R-modules, $Z(Y_1 \coprod Y_2) = Z(Y_1) \otimes Z(Y_2)$, and the disjoint union of cobordisms corresponds to the tensor product of homomorphisms, $Z(W_1 \coprod W_2) = Z(W_1) \otimes Z(W_2)$.

(vii) The empty n-manifold corresponds to the ground ring, $Z(\emptyset) = R$.

A compact oriented $(n+1)$-manifold W may be viewed as a oriented cobordism between empty manifolds. Then,

$$Z(W) \in Hom_R(R,R) \cong R.$$

The element $Z(W) \in R$ is a topological invariant of W.

TQFTs of dimension 1+1 are in bijective correspondence with commutative Frobenius algebras. The result goes back to Dijkgraaf's thesis, and has been proven in various contexts by Sawin [25], Abrams [1], and Quinn [24]. The form of the correspondence that we quote is due to Kock [16]:

Theorem 2.1. *The category of 1+1-dimensional TQFTs taking values in R is equivalent to the category of commutative Frobenius algebras over R.*

A **commutative Frobenius algebra over** R is a commutative R-algebra A equipped with a counit $\mu : A \to R$ and a coassociative, cocommutative, comultiplication $\triangle : A \to A \otimes A$ satisfying the Frobenius relation and the counit axiom:

$$(m \otimes \mathrm{Id})(a \otimes \triangle(b)) = (\mathrm{Id} \otimes m)(\triangle(a) \otimes b) = \triangle(m(a \otimes b))$$
$$(\mathrm{Id} \otimes \mu)(\triangle(a)) = (\mu \otimes \mathrm{Id})(\triangle(a)) = a$$

where $m : A \otimes A \to A$ is multiplication. The axioms imply that A is finitely generated as an R-module.

Given an invertible element $\lambda \in R$, we can give R the structure of a Frobenius algebra by setting $\triangle(1) = \lambda$ and (consequently) $\mu(1) = \lambda^{-1}$. We denote this Frobenius algebra by R_λ. A Frobenius algebra is **semisimple** if it is isomorphic to $R_{\lambda_1} \oplus \cdots \oplus R_{\lambda_n}$ for some $\lambda_1, \ldots, \lambda_n \in R$.

A 1+1-dimensional TQFT is semisimple if the corresponding Frobenius algebra is semisimple. If W_g is a closed surface of genus g, and Z is a semi-simple TQFT, then an elementary argument from the axioms yields:

$$Z(W_g) = \sum_{i=1}^{n} \lambda_i^{g-1}. \tag{1}$$

The following basic result is the key to proving the semisimplicity of the TQFT that we construct from local Gromov-Witten invariants.

Proposition 2.2. *Let R be a complete local ring. Let $m \subset R$ be the maximal ideal and let A be a Frobenius algebra over R. Suppose that A is free as an R-module and that A/mA is a semi-simple Frobenius algebra over R/m. Then A is semi-simple (over R).*

PROOF: Let $e_1, \dots, e_n \in A$ be representatives for an idempotent basis of A/mA, that is $e_i e_i - e_i \in m$ for all i and $e_i e_j \in m$ for all $i \neq j$. By Nakayama's lemma, $\{e_i\}$ is a basis of A, and we wish to construct a new basis which is idempotent in A. We begin by constructing an idempotent basis in A/m^2A.

It is easy to see that an element is invertible in A if and only if it is invertible in A/m. In particular, $1 - 2e_i$ is invertible since its square is 1 modulo m. Let $b_i = e_i e_i - e_i$ and set

$$e_i' = e_i + b_i(1 - 2e_i)^{-1}.$$

A short computation shows that

$$e_i' e_i' - e_i' = b_i^2 (1 - 2e_i)^{-2}$$

which is in m^2 for all i since $b_i \in m$. Then $e_i' e_j' \in m^2$ for $i \neq j$ follows from the facts that $(e_i' e_j')^2$, $e_i' e_i'(e_j' e_j' - e_j')$, and $e_j'(e_i' e_i' - e_i')$ are all in m^2. Thus $\{e_i'\}$ is an idempotent basis for A/m^2.

We construct a sequence of bases $\{e_i'\}, \{e_i''\}, \dots, \{e_i^{(k)}\}$ by setting

$$\begin{aligned} b_i^{(k)} &= e_i^{(k)} e_i^{(k)} - e_i^{(k)} \\ e_i^{(k+1)} &= e_i^{(k)} + b_i^{(k)}(1 - 2e_i^{(k)})^{-1}. \end{aligned}$$

The same argument as above shows that $\{e_i^{(k)}\}$ is an idempotent basis for $A/m^{k+1}A$. Since R is complete, there exists $\tilde{e}_i \in A$ such that $\tilde{e}_i = e_i^{(k)}$ mod

m^{k+1} for all k. By construction, $\tilde{e}_i$ is an idempotent basis for A. Let $\lambda_i = \mu(\tilde{e}_i)^{-1}$. Since A is free as an R-module, each $\tilde{e}_i$ generates an R summand of A and thus we have constructed the desired isomorphism of Frobenius algebras:

$$A \cong R_{\lambda_1} \oplus \cdots \oplus R_{\lambda_n}.$$

□

3 Relative local invariants and gluing

Motivated by the symplectic theory of A.-M. Li and Y. Ruan [17], J. Li has developed an algebraic theory of relative stable maps to a pair (X, B). This theory compactifies the moduli space of maps to X with prescribed ramification over a non-singular divisor $B \subset X$, [18, 19]. Li constructs a moduli stack of relative stable maps together with a virtual fundamental cycle and proves a gluing formula. Consider a degeneration of X to $X_1 \cup_B X_2$, the union of X_1 and X_2 along a smooth divisor B. The gluing formula expresses the virtual fundamental cycle of the usual stable map moduli space of X in terms of virtual cycles for relative stable maps of (X_1, B) and (X_2, B). The theory of relative stable maps has also been pursued in [13, 14], [7].

In our case, the target is a non-singular curve X of genus g, and the divisor B is a collection of points $b_1, \ldots, b_r \in X$.

Definition 3.1. *Let $(X, b_1, \ldots b_r)$ be a fixed non-singular genus g curve with r distinct marked points. Let $\alpha_1, \ldots, \alpha_r$ be partitions of d. Let*

$$\overline{M}^{\bullet}(X, (\alpha_1 \ldots \alpha_r))$$

be the moduli stack of relative stable maps (in the sense of Li)[1] *with target $(X, b_1, \ldots, b_r)$ satisfying the following:*

(i) The maps have degree d.

(ii) The maps are ramified over b_i with ramification type α_i.

(iii) The domain curves are possibly disconnected, but the map is not degree 0 on any connected component.

(iv) The domain curves are not *marked.*

[1] For a formal definition of relative stable maps, we refer to [18] Section 4.

The partition $\alpha_i \vdash d$ determines a ramification type over b_i by requiring the monodromy of the cover (considered as a conjugacy class of S_d) has cycle type α_i. We will use a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ to shorten the notation to $\overline{M}^\bullet(X, \boldsymbol{\alpha})$ when there is no risk of confusion.

Our moduli spaces of relative stable maps differ from Li's in a few minor ways:

(i) We do not require the domain to be connected.

(ii) We do not fix the domain genus, so $\overline{M}^\bullet(X, \boldsymbol{\alpha})$ is a countable union of components.

(iii) We do not mark the domain curve at all. Li's spaces include a marking of the ramification locus.

We make these modifications to Li's theory to simplify the combinatorics of the gluing theory. It is straightforward to express our moduli spaces in terms of unions, products, and finite quotients of Li's spaces.

There is a universal diagram (of stacks):

$$\begin{tikzcd}
R \arrow[r, hook] & U \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{X} \arrow[dl, "p"] & B \arrow[l, hook'] \\
 & \overline{M}^\bullet(X, \boldsymbol{\alpha}) & &
\end{tikzcd}$$

where U is the universal domain curve, $\mathcal{X}$ is the universal target curve, f is the universal map, B is the universal prescribed branch divisor, and R is the universal prescribed ramification divisor. The divisors B and R are taken with reduced structure. Note that the map f can also be ramified away from R, but at R the map f has the ramification type prescribed by the data $\boldsymbol{\alpha}$.

One of the salient features of the relative theory is that the target curve may "bubble" off rational components meeting the original X in nodes. That is to say, the family

$$p : \mathcal{X} \to \overline{M}^\bullet(X, \boldsymbol{\alpha})$$

is nontrivial: special fibers have chains of rational curves attached to X at the points b_i. However, the universal prescribed branch locus B lies in the non-singular locus of the fibers of p.

Let $[\overline{M}^\bullet(X,\boldsymbol{\alpha})]^{vir}$ denote the virtual fundamental class. With our conventions, the virtual class is a countable sum of cycle classes with degree b part supported on the components of expected dimension b. The expected dimension of a component is given by

$$b = -\chi + \ell(\boldsymbol{\alpha}) - d(2g-2+r)$$

where $\ell(\boldsymbol{\alpha}) = \ell(\alpha_1) + \cdots + \ell(\alpha_r)$ is the sum of the lengths of the partitions and χ is the domain Euler characteristic.

Let ω be the relative dualizing sheaf of p. We define an element $I(X,\boldsymbol{\alpha})$ of the K-theory of coherent sheaves on $\overline{M}^\bullet(X,\boldsymbol{\alpha})$ by

$$\begin{aligned} I(X,\boldsymbol{\alpha}) &= -R^\bullet\pi_*(f^*(\omega(B)) \oplus \mathcal{O}(-R)) \\ &= R^1\pi_*(f^*(\omega(B)) \oplus \mathcal{O}(-R)) - R^0\pi_*(f^*(\omega(B)) \oplus \mathcal{O}(-R)). \end{aligned}$$

We define the **relative local invariants** by

$$Z_d^b(g)_{\boldsymbol{\alpha}} = \int_{[\overline{M}^\bullet(X,\boldsymbol{\alpha})]^{vir}} c_b(I(X,\boldsymbol{\alpha}))$$

and their corresponding generating series $Z_d(g)_{\boldsymbol{\alpha}} \in \mathbb{Q}[[t]]$ by

$$Z_d(g)_{\boldsymbol{\alpha}} = \sum_{b=0}^{\infty} Z_d^b(g)_{\boldsymbol{\alpha}}\, t^b.$$

The element $I(X,\boldsymbol{\alpha})$ has rank b when restricted to the components of the moduli space $\overline{M}^\bullet(X,\boldsymbol{\alpha})$ with expected dimension b. We regard $I(X,\boldsymbol{\alpha})$ as a (virtual) obstruction bundle and $c_b(I(X,\boldsymbol{\alpha}))$ as the associated Euler class.

Remark 3.2. In case $g=0$ and $r=1$, the above Euler class was previously defined by J. Li and Y. Song [20]. Li and Song were modeling "open string" Gromov-Witten theory using relative stable maps. They argued that the integral $Z_d^b(0)_\alpha$ should compute the multiple cover formula for maps to a disk in a Calabi-Yau threefold where the boundary of the disk lies on a fixed Lagrangian 3-manifold (see also Katz-Liu [15]). They arrived at the integrand by considering the obstruction theory of maps to the disk with the Lagrangian boundary conditions. It is plausible that such an open-string theory interpretation of the relative local invariants exists in general.

We will use Li's theory to obtain "gluing relations" among the relative (and usual) local invariants. The following combinatorial quantity arises frequently:

Definition 3.3. *Let $\gamma \vdash d$ be a partition of d and let $\gamma(k)$ be the number of parts of size k in γ, so $d = \sum_{k=1}^{\infty} \gamma(k)k$. We define:*

$$\mathfrak{z}(\gamma) = \prod_{k=1}^{\infty} k^{\gamma(k)} \gamma(k)!.$$

If $c(\gamma) \subset S_d$ denotes the conjugacy class in the symmetric group consisting of elements having cycle type γ, then $\mathfrak{z}(\gamma)$ is the order of the centralizer of $c(\gamma)$.

The basic gluing laws are given by the following:

Theorem 3.4. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$. For any choice $g_1 + g_2 = g$ and any splitting*

$$\{\alpha_1, \ldots, \alpha_r\} = \{\alpha_1, \ldots, \alpha_k\} \cup \{\alpha_{k+1}, \ldots, \alpha_r\},$$

we have:

$$Z_d(g)_{\boldsymbol{\alpha}} = \sum_{\gamma \vdash d} \mathfrak{z}(\gamma) Z_d(g_1)_{\alpha_1, \ldots, \alpha_k, \gamma} \, Z_d(g_2)_{\alpha_{k+1}, \ldots, \alpha_r, \gamma}. \tag{2}$$

We also have

$$Z_d(g+1)_{\boldsymbol{\alpha}} = \sum_{\gamma \vdash d} \mathfrak{z}(\gamma) Z_d(g)_{\alpha_1, \ldots, \alpha_r, \gamma, \gamma}. \tag{3}$$

The first formula corresponds to splitting a genus g surface with r boundaries along a separating curve to obtain two surfaces of genus g_1 and g_2 with $(k+1)$ and $(r-k+1)$ boundaries. The second formula corresponds to cutting a genus $g+1$ surface with r boundaries along a non-separating curve to obtain a genus g surface with $(r+2)$ boundaries. We defer the proofs of these formulas to Appendix A.

4 The TQFT

We will show the gluing formulas of Theorem 3.4 allow us to organize the invariants $Z_d(g)_{\boldsymbol{\alpha}}$ into a 1+1-dimensional TQFT over $\mathbb{Q}[[t]]$. Throughout Section 4, R will denote the ring of formal power series in t over $\mathbb{Q}$:

$$R = \mathbb{Q}[[t]].$$

To define our TQFT, $Z_d(-)$, we need an R-module $H = Z_d(S^1)$ associated to the circle (H is the "Hilbert space" of the theory). We define

$$Z_d(S^1) = H = \bigoplus_{\alpha \vdash d} R \; e_\alpha$$

to be a free R-module with a basis $\{e_\alpha\}_{\alpha\vdash d}$ labeled by partitions of d.

Using the given basis for H, we can express any module homomorphism

$$f : H^{\otimes r} \to H^{\otimes s}$$

in tensor notation $f^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r} \in R$ by

$$f(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r}) = \sum_{\beta_1,\ldots,\beta_s} f^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}\; e_{\beta_1} \otimes \cdots \otimes e_{\beta_s}.$$

Using multi-index notation and the Einstein summation convention, we simply write:

$$f : e_{\boldsymbol{\alpha}} \mapsto f^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} e_{\boldsymbol{\beta}}$$

We raise indices by the following formula:

$$Z_d(g)^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r} = \mathfrak{z}(\beta_1)\cdots\mathfrak{z}(\beta_s) Z_d(g)_{\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s}.$$

Then the gluing laws can be written succinctly:

$$Z_d(g_1+g_2)^{\boldsymbol{\beta},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\eta}} = Z_d(g_1)^{\boldsymbol{\beta},\gamma}_{\boldsymbol{\alpha}} Z_d(g_2)^{\boldsymbol{\delta}}_{\boldsymbol{\eta},\gamma}$$
$$Z_d(g+1)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = Z_d(g)^{\boldsymbol{\beta},\gamma}_{\boldsymbol{\alpha},\gamma}$$

Note that in the above equation, γ it is a single index, while the boldface indices ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, etc) stand for multi-indices. Since the γ is repeated on the left hand side, it is summed over by convention.

Let $W^s_r(g)$ be the connected, oriented, genus g, cobordism from a disjoint union of r boundary circles to s boundary circles. We define

$$Z_d(W^s_r(g)) : H^{\otimes r} \to H^{\otimes s}$$

by

$$e_{\boldsymbol{\alpha}} \mapsto Z_d(g)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} e_{\boldsymbol{\beta}}$$

where $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_r$ and $\boldsymbol{\beta} = \beta_1, \ldots, \beta_s$.

For a disconnected cobordism $W = W_1 \sqcup \cdots \sqcup W_n$, we define

$$Z_d(W) = Z_d(W_1) \otimes \cdots \otimes Z_d(W_n).$$

Proposition 4.1. *The functor $Z_d(-)$ defined above is a $(1+1)$-dimensional TQFT over R.*

PROOF: To show $Z_d(-)$ is indeed a functor, we must prove that $Z_d(-)$ takes the concatenation of cobordisms to the composition of R-module homomorphisms. The composition

$$Z_d(W_s^t(g_2)) \circ Z_d(W_r^s(g_1)) : H^{\otimes r} \to H^{\otimes s} \to H^{\otimes t}$$

determined by connected cobordisms is given by

$$e_{\boldsymbol{\alpha}} \mapsto Z_d(g_1)_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} e_{\boldsymbol{\beta}} \mapsto Z_d(g_1)_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} Z_d(g_2)_{\boldsymbol{\beta}}^{\boldsymbol{\gamma}} e_{\boldsymbol{\gamma}}$$

for

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r), \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_s), \quad \boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_t).$$

Applying the gluing laws we obtain

$$\begin{aligned} Z_d(g_1)_{\boldsymbol{\alpha}}^{\beta_1 \ldots \beta_s} Z_d(g_2)_{\beta_1 \ldots \beta_s}^{\boldsymbol{\gamma}} &= Z_d(g_1 + g_2)_{\boldsymbol{\alpha}\beta_2 \ldots \beta_s}^{\boldsymbol{\gamma}\beta_2 \ldots \beta_s} \\ &= Z_d(g_1 + g_2 + 1)_{\boldsymbol{\alpha}\beta_3 \ldots \beta_s}^{\boldsymbol{\gamma}\beta_3 \ldots \beta_s} \\ &\;\vdots \\ &= Z_d(g_1 + g_2 + s - 1)_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} . \end{aligned} \tag{4}$$

We have proven

$$Z_d(W_s^t(g_2)) \circ Z_d(W_r^s(g_1)) = Z_d(W_r^t(g_1 + g_2 + s - 1)).$$

Since the concatenation of $W_r^s(g_1)$ followed by $W_s^t(g_2)$ is $W_r^t(g_1 + g_2 + s - 1)$ (see the Figure), we have shown that $Z_d(-)$ is a functor, at least when applied to the subcategory of 2**Cob** consisting of connected cobordisms.

Similar computations apply to concatenations of disconnected cobordisms. For example, the concatenation of the cobordism $W_r^1(g_1) \sqcup W_s^1(g_2)$ followed by $W_2^t(g_3)$ yields the cobordism $W_{r+s}^t(g_1 + g_2 + g_3)$. Correspondingly, for

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r), \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_s), \quad \boldsymbol{\delta} = (\delta_1, \ldots, \delta_t),$$

the composition

$$Z_d(W_2^t(g_3)) \circ Z_d(W_r^1(g_1) \sqcup W_s^1(g_2))$$

is given by

$$\begin{aligned} e_{\boldsymbol{\alpha}} \otimes e_{\boldsymbol{\beta}} &\mapsto Z_d(g_1)_{\boldsymbol{\alpha}}^{\gamma_1} Z_d(g_2)_{\boldsymbol{\beta}}^{\gamma_2} e_{\gamma_1} \otimes e_{\gamma_2} \\ &\mapsto Z_d(g_1)_{\boldsymbol{\alpha}}^{\gamma_1} Z_d(g_2)_{\boldsymbol{\beta}}^{\gamma_2} Z_d(g_3)_{\gamma_1\gamma_2}^{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \\ &= Z_d(g_1)_{\boldsymbol{\alpha}}^{\gamma_1} Z_d(g_2 + g_3)_{\boldsymbol{\beta}\gamma_1}^{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \\ &= Z_d(g_1 + g_2 + g_3)_{\boldsymbol{\alpha}\boldsymbol{\beta}}^{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \end{aligned}$$

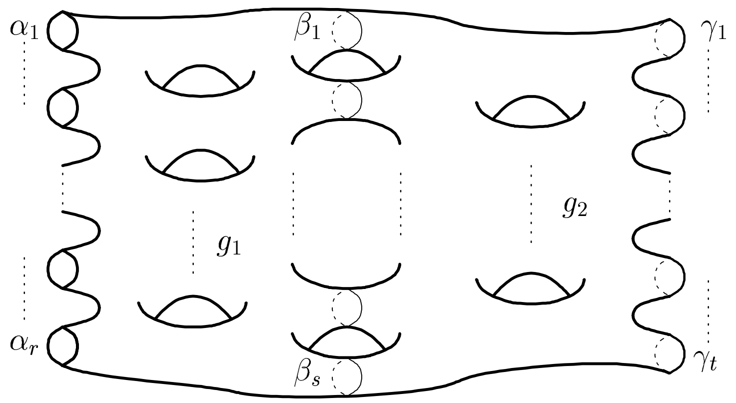

Figure 1: $W_r^s(g_1)$ concatenated with $W_s^t(g_2)$ is $W_r^t(g_1+g_2+s-1)$. The gluing formula expressed by this picture is given by Equation (4).

The general concatenation of cobordisms can be checked with similar computations.

To prove that $Z_d(-)$ is a symmetric monoidal functor, we must also check that $Z_d(-)$ takes the trivial cobordism $S^1 \times [0,1]$ to the identity. This is equivalent to the following lemma.

Lemma 4.2. *The local invariant of $\mathbb{P}^1$, relative 2 points is given by:*

$$Z_d(0)_{\alpha\beta} = \begin{cases} \frac{1}{\mathfrak{z}(\alpha)} & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

PROOF: The relative invariant $Z_d(0)_{\alpha\beta}$ can be computed by virtual localization since there is a $\mathbb{C}^\times$ action on $\mathbb{P}^1$ preserving the relative points $0, \infty \in \mathbb{P}^1$ and hence $\mathbb{C}^\times$ acts on $\overline{M}^\bullet(\mathbb{P}^1, (\alpha, \beta))$. However, we can compute $Z_d(0)_{\alpha\beta}$ more easily as follows.

The component of $\overline{M}^\bullet(\mathbb{P}^1, (\alpha, \beta))$ of virtual dimension 0 parameterizes stable maps $f : C \to \mathbb{P}^1$ that are unramified away from the prescribed ramification points $0, \infty \in \mathbb{P}^1$. Any such map must be of the form

$$f : \mathbb{P}^1 \sqcup \cdots \sqcup \mathbb{P}^1 \to \mathbb{P}^1$$

where on the ith component, f is of the form $z \mapsto z^{\alpha_i}$ for some $\alpha \vdash d$.

Therefore, if $\alpha \neq \beta$, then the virtual dimension 0 component of the moduli space $\overline{M}^\bullet(\mathbb{P}^1, (\alpha, \beta))$ is empty. If $\alpha = \beta$, then virtual dimension 0 component consists of a single moduli point $[f]$ corresponding to the above map. The map f has an automorphism group of order $\mathfrak{z}(\alpha)$, hence

$$Z_d^0(0)_{\alpha\beta} = \int_{[\overline{M}^{\bullet}(\mathbb{P}^1,(\alpha,\beta))]^{vir}} 1 = \begin{cases} \frac{1}{\mathfrak{z}(\alpha)} & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

We use the gluing law to derive the vanishing of all the terms of

$$Z_d(0)_{\alpha\beta} = \frac{1}{\mathfrak{z}(\beta)} Z_d(0)_\alpha^\beta$$

of higher degree in t:

$$Z_d(0)_\alpha^\beta = Z_d(0)_\alpha^\gamma Z_d(0)_\gamma^\beta.$$

For $b > 0$ we have

$$Z_d^b(0)_\alpha^\beta = \sum_{\gamma,\ b_1+b_2=b} Z_d^{b_1}(0)_\alpha^\gamma Z_d^{b_2}(0)_\gamma^\beta.$$

By induction, we may assume that $Z_d^{b'}(0)_\alpha^\gamma = 0$ for $0 < b' < b$. Then

$$\begin{aligned} Z_d^b(0)_\alpha^\beta &= \sum_\gamma \left(Z_d^b(0)_\alpha^\gamma Z_d^0(0)_\gamma^\beta + Z_d^0(0)_\alpha^\gamma Z_d^b(0)_\gamma^\beta \right) \\ &= 2 Z_d^b(0)_\alpha^\beta \end{aligned}$$

and so $Z_d^b(0)_{\alpha\beta} = 0$ for $b > 0$. This completes the proof of Lemma 4.2. $\square$

All the other axioms of a (1+1)-dimension TQFT given in Section 2 follow immediately from the definitions. The proof of Proposition 4.1 is complete. $\square$

To complete the proof of Theorem 1.2, we must prove $Z_d(-)$ is semisimple. By Proposition 2.2, it suffices to analyze $Z_d(-)$ at $t = 0$. Let

$$Z_d^0(-) : 2\mathbf{Cob} \to \mathbb{Q}\mathbf{mod}$$

denote composition of $Z_d(-)$ with the natural functor $R\mathbf{mod} \to \mathbb{Q}\mathbf{mod}$ obtained by setting $t = 0$.

Lemma 4.3. *$Z_d^0(-)$ is the semisimple TQFT (over $\mathbb{Q}$) given by "finite gauge theory with gauge group S_d" studied by Dijkgraaf-Witten and Freed-Quinn [6, 11]. The corresponding Frobenius algebra is isomorphic to the center of the group algebra $\mathbb{Q}[S_d]$.*

PROOF: The invariant $Z_d^0(g)_{\boldsymbol{\alpha}}$ is, by definition, the degree of the virtual class of the expected dimension 0 components of $\overline{M}^{\bullet}(X, \boldsymbol{\alpha})$. Using the dimension formula, a relative stable map

$$[f : C \to (X, b_1, \ldots, b_r)]$$

is easily seen to lie in an expected dimension 0 component if and only if f is unramified away from the prescribed ramification points $b_1, \ldots, b_r$. Since such a map has no deformations, the expected dimension 0 components of $\overline{M}^{\bullet}(X, \boldsymbol{\alpha})$ have actual dimension 0. Hence, the invariant $Z_d^0(g)_{\boldsymbol{\alpha}}$ is equal to a weighted count of maps with only the prescribed ramification. Each map is weighted by the reciprocal of the number of automorphisms. The latter count is, by definition, the Hurwitz number.

Maps with only the prescribed ramification over $b_1, \ldots, b_r$ have a gauge theoretic interpretation in terms of principal S_d bundles over an r-punctured genus g surface. The Hurwitz numbers can be viewed as counting principal S_d-bundles over punctured surfaces. The associated TQFT has been studied in detail and is well-known to be semisimple [6], [11] (but see [5] for a short explanation). The proofs of both Lemma 4.3 and Theorem 1.2 are complete. $\square$

The Frobenius algebra $H = \oplus_\alpha \mathbb{Q} e_\alpha$ obtained from $Z_d^0(-)$ is isomorphic to the center of $\mathbb{Q}[S_d]$, the group ring of the symmetric group. The center has a Frobenius structure with counit

$$\mu\left(\sum_{g \in S_d} a_g g\right) = \frac{a_{\mathrm{Id}}}{d!}.$$

The isomorphism is given by

$$e_\alpha \mapsto \sum_{g \in c(\alpha)} g.$$

The basis $\{e_\alpha\}$ is not idempotent, but there is a natural idempotent basis $\{v_R\}$ labeled by irreducible representations R of S_d:

$$v_R = \frac{\dim R}{d!} \sum_{\alpha \vdash d} \chi_R(c(\alpha)) e_\alpha$$

where χ_R is the character of R. The calculation,

$$\mu(v_R) = \left(\frac{\dim R}{d!}\right)^2,$$

yields the following elegant formula counting unramified covers:

$$Z_d^0(g) = \sum_R \left(\frac{d!}{\dim R}\right)^{2g-2}.$$

Theorem 1.3 is an immediate consequence of Theorem 1.2 and the formula for closed surfaces (1).

Remark 4.4. We have proven the constant terms of the power series

$$\lambda_1, \ldots, \lambda_{p(d)}$$

appearing in our structure formula (Theorem 1.3) are exactly the numbers

$$\left(\frac{d!}{\dim R}\right)^2.$$

Hence, the series $\lambda_i \in \mathbb{Q}[[t]]$ have square roots in $\mathbb{Q}[[t]]$. The existence of these square roots may have deeper significance. TQFTs as defined in Section 2 are inherently "closed string" TQFTs. One can also axiomatize "open string" TQFTs (over a ring R) which include the closed theory as a subsector. Given a closed string TQFT, we may ask what are the possible open string TQFTs which contain it? If the closed string TQFT is semisimple and the corresponding λ_i's *have square roots* in R, then there is an elegant classification of the possible open string TQFT containing the given closed string TQFT. The open string TQFTs are completely determined by assigning a free R-module to each idempotent basis vector e_i and a choice of a sign for the square root of $\lambda_i^{-1} = \mu(e_i)$. See the lecture notes of G. Moore for a good discussion [21].

5 Analysis of the TQFT

In Section 4, we constructed $Z_d(-)$, a 1+1-dimensional TQFT over $R = \mathbb{Q}[[t]]$ from the local Gromov-Witten invariants. We now analyze the TQFT and the corresponding Frobenius algebra.

As before, let $H = \oplus_{\alpha \vdash d} R e_\alpha$ be $Z_d(S^1)$. The R-module H has the structure of a Frobenius algebra with multiplication $\cdot$ given by

$$e_\alpha \cdot e_\beta = Z_d(0)^\gamma_{\alpha\beta} e_\gamma,$$

unit **1** given by

$$\mathbf{1} = Z_d(0)^{\alpha} e_{\alpha},$$

comultiplication Δ given by

$$\Delta(e_\alpha) = Z_d(0)_{\alpha}^{\beta\gamma} e_\beta \otimes e_\gamma,$$

and counit μ given by

$$\mu(e_\alpha) = Z_d(0)_\alpha.$$

The relative local invariants $Z_d(0)_\alpha$ and $Z_d(0)_{\alpha\beta\gamma}$ therefore determine the whole TQFT and hence all the invariants (note that the invariant $Z_d(0)_{\alpha\beta}$ is given by Lemma 4.2). The invariants $Z_d(0)_{\alpha\beta\gamma}$, which correspond to the "pair of pants" are in general, difficult to compute.

On the other hand, the invariants $Z_d(0)_\alpha$ can be computed by localization.

Theorem 5.1. *The local relative invariant of* $\mathbb{P}^1$ *relative to one point is*

$$Z_d(0)_\alpha = (-1)^{d-l} \frac{t^d}{\mathfrak{z}(\alpha)} \prod_{i=1}^{l} \left(2 \sin(\frac{\alpha_i t}{2}) \right)^{-1} \tag{5}$$

where the parts of $\alpha \vdash d$ *are* $\alpha_1 + \cdots + \alpha_l = d$.

Warning 5.2. The local invariants of $\mathbb{P}^1$ relative to one point were previously studied (in the connected case) by J. Li and Y. Song [20]. However, the calculation of [20] was incomplete as most localization terms were left unanalyzed by the authors (with the hope that the contributions vanished). In fact, the omitted terms of [20] *do not* vanish, and the calculation there is wrong. Remarkably, the correct calculation differs only by the sign $(-1)^{d-l}$.

We will calculate the local invariants in the connected case

$$\int_{[\overline{M}_g(\mathbb{P}^1,\alpha)]^{vir}} c_b(I(\mathbb{P}^1, \alpha)), \tag{6}$$

where the relative point is taken to be ∞. The disconnected formula (5) will be obtained afterwards by exponentiation.

Lemma 5.3. *The connected local invariants (6) vanish if* $\ell(\alpha) > 1$.

PROOF: We analyze the K-theoretic element $I(\mathbb{P}^1, \alpha)$,

$$R^1\pi_*(f^*(\omega_{\mathcal{X}}(B)) \oplus \mathcal{O}(-R)) - R^0\pi_*(f^*(\omega_{\mathcal{X}}(B)) \oplus \mathcal{O}(-R)).$$

There is a canonical map, $\epsilon : \mathcal{X} \to \mathbb{P}^1$, obtained by contracting the destabilizations of the target. The basic isomorphism,

$$\omega_{\mathcal{X}}(B) \stackrel{\sim}{=} \epsilon^*(\omega_{\mathbb{P}^1}(\infty)),$$

is easily checked on the Artin stack of destabilizations of $\mathbb{P}^1$ at ∞. Of course,

$$\omega_{\mathbb{P}^1}(\infty) \stackrel{\sim}{=} \mathcal{O}_{\mathbb{P}^1}(-1).$$

Since both $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}(-R)$ are negative, $I(\mathbb{P}^1, \alpha)$ simplifies to

$$R^1\pi_*((\epsilon f)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus \mathcal{O}(-R)). \tag{7}$$

By Riemann-Roch, the rank of the bundle (7) equals the virtual dimension b of the moduli space $\overline{M}_g(\mathbb{P}^1, \alpha)$.

The sheaf $R^0\pi_*(\mathcal{O}_R)$ is a rank $\ell(\alpha)$ trivial bundle[2] on the moduli space $\overline{M}_g(\mathbb{P}^1, \alpha)$. The exact sequence

$$0 \to \mathcal{O}^{\ell(\alpha)-1} \to R^1\pi_*(\mathcal{O}(-R)) \to R^1\pi_*(\mathcal{O}) \to 0, \tag{8}$$

is easily obtained from the ideal sequence

$$0 \to \mathcal{O}(-R) \to \mathcal{O} \to \mathcal{O}_R \to 0.$$

If $\ell(\alpha) > 1$, then the bundle $R^1\pi_*(\mathcal{O}(-R))$ contains a trivial subfactor. Hence, the local invariant (6) vanishes unless α consists of only a single part. $\square$

PROOF OF THEOREM 5.1: Let α consist of the single part a. Writing $\mathcal{O}$ as $(\epsilon f)^*(\mathcal{O}_{\mathbb{P}^1})$ and using (8), we obtain

$$I(\mathbb{P}^1, (a)) = R^1\pi_*((\epsilon f)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})). \tag{9}$$

We will use the above form of $I(\mathbb{P}^1, (a))$ in the localization analysis below.

[2] To be precise, $R^0\pi_*(O_R)$ is trivial when pulled back to the finite étale cover of $\overline{M}_g(\mathbb{P}^1, \alpha)$ given by marking the ramification divisor. It is therefore trivial in K-theory over $\mathbb{Q}$ and so the following argument holds.

We now define the appropriate torus actions. Let $\mathbb{P}^1 = \mathbb{P}(V)$ where

$$V = \mathbb{C} \oplus \mathbb{C}.$$

Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{10}$$

Let $0, \infty$ be the fixed points $(1:0), (0:1)$ of the corresponding action on $\mathbb{P}^1$.

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ canonically lifts to a $\mathbb{C}^*$-action on the moduli space of maps $\overline{M}_g(\mathbb{P}^1, (a))$ relative to ∞. Discussions of the virtual localization formula in the relative context can be found in [20], [8], [12].

An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbb{P}^1$ is uniquely determined by the weights $[l_0, l_\infty]$ of the fiber representations at the fixed points L_0, L_∞. The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbb{P}^1}$ has weights $[1, -1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$ with weights $[-1, 0]$ and $[0, 0]$ respectively. An equivariant lift of $I(\mathbb{P}^1, (a))$ is canonically induced by (9).

The integral (6) may now be calculated via the virtual localization formula. The answer is expressed as a sum over all localization graphs, see [8]. Fortunately, our choice of equivariant liftings leads to a complete collapse of the sum.

Localization graphs are in bijective correspondence to the $\mathbb{C}^*$-fixed loci of $\overline{M}_g(\mathbb{P}^1, (a))$. Let $[f]$ be a $\mathbb{C}^*$-fixed point. The vertices of the associated graph Γ correspond to the connected components of the set

$$(\epsilon f)^{-1}(\{0, \infty\}). \tag{11}$$

The edges of Γ correspond to the components of the domain of f which map dominantly to $\mathbb{P}^1$ under ϵf.

Let Γ be a graph with a nonvanishing contribution to the integral

$$\int_{[\overline{M}_g(\mathbb{P}^1,(a))]^{vir}} c_b(I(\mathbb{P}^1, (a))), \tag{12}$$

Then,

(i) Since the monodromy condition (a) over ∞ is transitive, there can be only 1 connected component of (11) over ∞. Hence, there is a unique vertex v of Γ over ∞.

(ii) The weight 0 linearization of $\mathcal{O}_{\mathbb{P}^1}(-1)$ over ∞ implies v has valence 1, see [9].

(iii) The vertex v carries the class $c_{g(v)}(\mathbb{E}^*)^2$ obtained from the weight 0 linearizations of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$ over ∞. As the class vanishes for $g(v) > 0$ by Mumford's relation,

$$c(\mathbb{E}) \cdot c(\mathbb{E}^*) = 1,$$

v must be of genus 0.

(iv) By (i-iii), Γ has a single vertex v of genus 0 over ∞. The fixed moduli spaces over ∞ consist of relative maps to an *unparameterized* bubble with two relative points: the attaching point of the bubble and the bubbled ∞. The relative condition over the attaching point is (a) by the valence 1 restriction. The relative condition over the bubbled ∞ is (a) by assumption. The resulting unparameterized moduli space is degenerate, see [8]. The graph Γ must therefore have *no contracted components* over the point ∞ of the original $\mathbb{P}^1$.

Therefore, a *unique* graph Γ, consisting of a genus g vertex over 0, a unique edge of degree a, and a degenerate genus 0 vertex over ∞, contributes to (12).

A straightforward calculation using the virtual localization formula yields the formula:

$$\int_{[\overline{M}_g(\mathbb{P}^1,(a))]^{vir}} c_b(I(\mathbb{P}^1,(a))) = (-1)^{a-1} a^{2g-2} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} c_g(\mathbb{E}). \qquad (13)$$

We use the terminology of [8] for the unique contributing graph Γ:

- The automorphism factor $|\mathbf{A}_\Gamma|$ is a obtained from the Galois group of the dominant component.
- The vertex contribution is

$$\int_{\overline{M}_{g,1}} \frac{c(\mathbb{E}^*)\ (-1)^g c(\mathbb{E})\ (-1)^g c_g(\mathbb{E})}{\frac{1}{a} - \psi},$$

where the last two factors in the numerator are obtained from the integrand.

- The edge contribution is

$$\frac{(-1)^{a-1}\frac{(a-1)!}{a^{a-1}}}{\frac{a!}{a^a}},$$

  where the numerator is obtained from the integrand.

The contribution (13) of Γ is given by the product

$$\frac{1}{|\mathbf{A}_\Gamma|}\int_{\overline{M}_{g,1}}\frac{c(\mathbb{E}^*)(-1)^g c(\mathbb{E})(-1)^g c_g(\mathbb{E})}{\frac{1}{a}-\psi}\ \frac{(-1)^{a-1}\frac{(a-1)!}{a^{a-1}}}{\frac{a!}{a^a}}$$

after simplification using Mumford's relation.

Using the Hodge integral computation of [9], we find

$$\sum_{g\geq 0} t^b \int_{[\overline{M}_g(\mathbb{P}^1,(a))]^{vir}} c_b(I(\mathbb{P}^1,(a))) = (-1)^{a-1}\frac{t^a}{a}\left(2\sin(\frac{at}{2})\right)^{-1}.$$

The proof of Theorem 5.1 is completed by taking the associated disconnected integrals. □

As a corollary to Theorem 5.1 we can completely determine the TQFT $Z_1(-)$ and hence all the $d=1$ relative invariants.

Corollary 5.4. *The TQFT $Z_1(-)$ is isomorphic to R_λ where*

$$\lambda = \left(\frac{\sin(t/2)}{t/2}\right)^2$$

via the isomorphism

$$e_{[1]} \mapsto \left(\frac{\sin(t/2)}{t/2}\right)^{-1}.$$

Consequently, the $d=1$ relative invariants are given by

$$Z_1(g)_{\alpha_1,\ldots,\alpha_r} = \left(\frac{\sin(t/2)}{t/2}\right)^{2g-2+r}.$$

A The proof of the gluing formulas

To prove the gluing formulas in Theorem 3.4, we will consider algebraic degenerations corresponding to the (topological) splittings of the TQFT. To simplify the exposition, we first derive Equation (2) of Theorem 3.4 with $r = 0$.

Consider the nodal curve

$$W_0 = X_1 \bigcup_{b_1 = b_2} X_2$$

obtained by joining non-singular curves X_1 and X_2 of genus g_1 and g_2 at points $b_i \in X_i$. Let $W \to \mathbb{A}^1$ be a generic, 1-parameter deformation of W_0 for which the fibers W_t for $t \neq 0 \in \mathbb{A}^1$ are nonsingular curves of genus $g = g_1 + g_2$. (The base of the degeneration can be any smooth curve. For simplicity, we take it to be $\mathbb{A}^1$.)

In Li's theory, the moduli of relative stable maps spaces arise by constructing a good limit for the moduli spaces $\overline{M}^\bullet(W_t)$ as t approaches 0. Li's method involves a stack $\mathfrak{W}$ of expanded degenerations of W: an Artin stack over $\mathbb{A}^1$ which, in addition to W, includes degenerations to the curves $W[n]_0$ obtained by inserting a chains of $\mathbb{P}^1$'s between X_1 and X_2:

$$W[n]_0 = X_1 \cup \underbrace{\mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1}_{n-1} \cup X_2.$$

Following Li (but with our conventions (i-iii) of Definition 3.1 regarding the domain curve), we define $\overline{M}^\bullet(\mathfrak{W})$ to be the stack of non-degenerate, pre-deformable, degree d stable maps to $\mathfrak{W}$ (see [18] section 3 for the definitions of non-degenerate, pre-deformable, and $\mathfrak{W}$). Li proves that $\overline{M}^\bullet(\mathfrak{W})$ is a Deligne-Mumford stack (Theorem 0.1 of [18]). Each component of $\overline{M}^\bullet(\mathfrak{W})$ of given fixed expected dimension is proper and separated over $\mathbb{A}^1$ (also Theorem 0.1 of [18]).

$\overline{M}^\bullet(\mathfrak{W}_0)$, the central fiber of $\overline{M}^\bullet(\mathfrak{W})$, can be expressed, up to finite covers, as the union of products of relative stable map moduli spaces. Moreover, $\overline{M}^\bullet(\mathfrak{W})$ has a virtual fundamental class whose intersection with $\overline{M}^\bullet(\mathfrak{W}_t)$ for $t \neq 0$ is the usual virtual fundamental class $[\overline{M}^\bullet(W_t)]^{vir}$, and whose intersection with $\overline{M}^\bullet(\mathfrak{W}_0)$ is compatible with the decomposition into relative stable map spaces. To be precise, Li's virtual cycle formula (Theorem 3.15 of [19]), adapted to our setting and conventions is:

$$[\overline{M}^\bullet(\mathfrak{W}_0)]^{vir} = \sum_{\alpha \vdash d} \mathfrak{z}(\alpha)\, (\Phi_\alpha)_* \left([\overline{M}^\bullet(X_1, \alpha)]^{vir} \times [\overline{M}^\bullet(X_2, \alpha)]^{vir} \right). \tag{14}$$

Here the map

$$\Phi_\alpha : \overline{M}^\bullet(X_1,\alpha) \times \overline{M}^\bullet(X_2,\alpha) \to \overline{M}^\bullet(\mathfrak{W}_0)$$

is obtained by constructing a family of $\overline{M}^\bullet(\mathfrak{W}_0)$ maps from the universal maps over $\overline{M}^\bullet(X_i,\alpha)$ by gluing along the universal prescribed ramification and branch divisors:

$$\begin{tikzcd}
U_1 \bigcup\limits_{R_1=R_2} U_2 \arrow[rr, "f_1 \cup f_2"] \arrow[d, "\pi_1 \cup \pi_2"'] & & \mathcal{X}_1 \bigcup\limits_{B_1=B_2} \mathcal{X}_2 \arrow[dll, "\mathfrak{p}_1 \cup \mathfrak{p}_2"] \\
\overline{M}^\bullet(X_1,\alpha) \times \overline{M}^\bullet(X_2,\alpha) & &
\end{tikzcd} \tag{15}$$

Strictly speaking, we must to pass to the finite étale cover of

$$\overline{M}^\bullet(X_1,\alpha) \times \overline{M}^\bullet(X_2,\alpha)$$

obtained by marking the ramification divisor. The ordering is necessary to obtain the identification $R_1 = R_2$. However, the map $(\Phi_\alpha)_*$ on $\mathbb{Q}$-cycles is well defined. The degree of the finite étale map is included in our constant in the virtual cycle formula. Since our target is a curve and the branch divisor is a point, the diagonal constraint which occurs in Li's general cycle formula does not appear here.

In order to apply the virtual cycle formula to obtain our gluing formulas, we will define a K-theory class $I(\mathfrak{W})$ on $\overline{M}^\bullet(\mathfrak{W})$ with the following properties:

(i) For $t \neq 0$, $I(\mathfrak{W})$ restricts to $I(W_t)$ on $\overline{M}^\bullet(W_t)$.

(ii) For $t = 0$, $I(\mathfrak{W})$ restricts to a class that pulls back via Φ_α to

$$I(X_1,\alpha) \oplus I(X_2,\alpha)$$

on $\overline{M}^\bullet(X_1,\alpha) \times \overline{M}^\bullet(X_2,\alpha)$.

Proposition A.1. *Let $\pi : U \to \overline{M}^\bullet(\mathfrak{W})$, $p : \mathcal{X} \to \overline{M}^\bullet(\mathfrak{W})$, and $f : U \to \mathcal{X}$ be the universal domain, universal target, and the universal map for $\overline{M}^\bullet(\mathfrak{W})$. Let ω_p be the relative dualizing sheaf of the universal target. Define the K-theory class $I(\mathfrak{W})$ by*

$$I(\mathfrak{W}) = -R^\bullet \pi_* f^*(\omega_p \oplus \mathcal{O}_{\mathcal{X}}).$$

Let $I(\mathfrak{W})_t$ be the restriction of $I(\mathfrak{W})$ to $\overline{M}^\bullet(\mathfrak{W}_t)$. Then for $t \neq 0$,

$$I(\mathfrak{W})_t = I(W_t)$$

and for $t = 0$ we have

$$\Phi_\alpha^*(I(\mathfrak{W})_0) = I(X_1, \alpha) \oplus I(X_2, \alpha).$$

PROOF: First, the universal family $p : \mathcal{X} \to \overline{M}^\bullet(\mathfrak{W})$ is a flat family of prestable curves, so there exists a relative dualizing sheaf ω_p. Over $t \neq 0$, the family is constant with fiber W_t, so

$$\omega_{p_t} \cong K_{W_t},$$

and thus $I(\mathfrak{W})_t = I(W_t)$.

We now compute $\Phi_\alpha^*(I(\mathfrak{W})_0)$. Let $\pi_\alpha = \pi_1 \cup \pi_2$, $f_\alpha = f_1 \cup f_2$, and $p_\alpha = p_1 \cup p_2$ denote the maps in diagram (15), and let $U_\alpha = U_1 \cup U_2$ and $\mathcal{X}_\alpha = \mathcal{X}_1 \cup \mathcal{X}_2$. By the definition of Φ_α, we have

$$\Phi_\alpha^*(I(\mathfrak{W})_0) = -R^\bullet(\pi_\alpha)_* f_\alpha^*(\omega_{p_\alpha} \oplus \mathcal{O}_{\mathcal{X}_\alpha}).$$

Consider the following two short exact sequences of sheaves on U_α.

$$0 \to \mathcal{O}_{U_\alpha} \longrightarrow \mathcal{O}_{U_1} \oplus \mathcal{O}_{U_2} \longrightarrow \mathcal{O}_R \to 0$$
$$0 \to f_\alpha^*(\omega_{p_\alpha}) \to f_1^*(\omega_{p_1}(B_1)) \oplus f_2^*(\omega_{p_2}(B_2)) \to \mathcal{O}_R \to 0.$$

The first sequence is the usual normalization sequence. The second is obtained from standard facts about the dualizing sheaf ω of a nodal curve. We obtain the following equalities in K-theory:

$$f_\alpha^*(\mathcal{O}_{\mathcal{X}_\alpha}) = \mathcal{O}_{U_1} + \mathcal{O}_{U_2} - \mathcal{O}_R,$$

$$f_\alpha^*(\omega_{p_\alpha}) = f_1^*(\omega_{p_1}(B_1)) + f_2^*(\omega_{p_2}(B_2)) - \mathcal{O}_R,$$

where the first equation uses the isomorphism $\mathcal{O}_{U_\alpha} \cong f_\alpha^*(\mathcal{O}_{\mathcal{X}_\alpha})$.

The divisor sequence for $R_i \subset U_i$ yields

$$\mathcal{O}_{U_i} = \mathcal{O}_R + \mathcal{O}_{U_i}(-R_i)$$

in K-theory, so

$$f_\alpha^*(\omega_{p_\alpha} \oplus \mathcal{O}_{\mathcal{X}_\alpha}) = \sum_{i=1}^{2} f_i^*(\omega_{p_i}(B_i)) + \sum_{i=1}^{2} \mathcal{O}_{U_i}(-R_i).$$

After applying $-R^\bullet(\pi_\alpha)_*$ to both sides, we obtain

$$\begin{aligned}\Phi_\alpha^*(I(\mathfrak{W})_0) &= \sum_{i=1}^{2} -R^\bullet(\pi_i)_*\left(f_i^*(\omega_{p_i}(B_i)) + \mathcal{O}_{U_i}(-R_i)\right)\\ &= \sum_{i=1}^{2} I(X_i,\alpha).\end{aligned}$$

The proof of the Proposition is complete. $\square$

With the class $I(\mathfrak{W})$ in hand, the gluing formula is proven as follows. Using the fact that $\int_{[\overline{M}^\bullet(\mathfrak{W}_t)]^{vir}} c_b(I(\mathfrak{W}))$ is independent of t we compute:

$$\begin{aligned}
Z_d^b(g_1+g_2) &= \int_{[\overline{M}^\bullet(\mathfrak{W}_t)]^{vir}} c_b(I(\mathfrak{W})_t)\\
&= \int_{[\overline{M}^\bullet(\mathfrak{W}_0)]^{vir}} c_b(I(\mathfrak{W})_0)\\
&= \sum_{\alpha\vdash d} \mathfrak{z}(\alpha) \int_{[\overline{M}^\bullet(X_1,\alpha)]^{vir}\times[\overline{M}^\bullet(X_2,\alpha)]^{vir}} c_b\left(I(X_1,\alpha)\oplus I(X_2,\alpha)\right)\\
&= \sum_{\substack{\alpha\vdash d\\ b_1+b_2=b}} \mathfrak{z}(\alpha) \int_{[\overline{M}^\bullet(X_1,\alpha)]^{vir}} c_{b_1}(I(X_1,\alpha)) \int_{[\overline{M}^\bullet(X_2,\alpha)]^{vir}} c_{b_2}(I(X_2,\alpha))\\
&= \sum_{\substack{\alpha\vdash d\\ b_1+b_2=b}} \mathfrak{z}(\alpha)\, Z_d^{b_1}(g_1)_\alpha\, Z_d^{b_2}(g_2)_\alpha.
\end{aligned}$$

We have proven the gluing formula (2) in case $r=0$.

The proof of the second gluing formula (3) for $r=0$ is almost identical. We consider a degeneration

$$W \to \mathbb{A}^1$$

where W_t for $t\neq 0$ is a nonsingular genus $(g+1)$ curve and

$$W_0 = X/b_1 \sim b_2$$

is an irreducible nodal curve whose normalization (X,b_1,b_2) is a smooth genus g curve with marked points $b_1,b_2\in X$ lying over the node.

As in the previous case, we construct a stack $\mathfrak{W}$ of expanded degenerations of W and define $\overline{M}^\bullet(\mathfrak{W})$ to be the stack of non-degenerate, pre-deformable, stable maps to $\mathfrak{W}$. The cycle formula is now

$$[\overline{M}^\bullet(\mathfrak{W}_0)]^{vir} = \sum_{\alpha\vdash d} \mathfrak{z}(\alpha)(\Phi_{\alpha\alpha})_*([\overline{M}^\bullet(X,(\alpha,\alpha))]^{vir})$$

where

$$\Phi_{\alpha\alpha} : \overline{M}^\bullet(X,(\alpha,\alpha)) \to \overline{M}^\bullet(\mathfrak{W}_0) \tag{16}$$

is obtained by gluing together the two universal prescribed branched divisors and the two universal prescribed ramification divisors over $\overline{M}^\bullet(X,(\alpha,\alpha))$:

$$\begin{tikzcd}
R_1, R_2 \arrow[r, hook] & U_{\alpha\alpha} \arrow[r, "f_{\alpha\alpha}"] \arrow[d, "n"'] & \mathcal{X}_{\alpha\alpha} \arrow[d, "n"] & B_1, B_2 \arrow[l, hook'] \\
R \arrow[r, hook] & U \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{X} \arrow[dl, "\mathfrak{p}"] & B \arrow[l, hook'] \\
& \overline{M}^\bullet(X,(\alpha,\alpha)) & &
\end{tikzcd}$$

In the above diagram, $\pi_{\alpha\alpha} = \pi \circ n$ and $p_{\alpha\alpha} = p \circ n$ are the universal domain and universal range for $\overline{M}^\bullet(X,(\alpha,\alpha))$ respectively and the stacks

$$\mathcal{X} = \mathcal{X}_{\alpha\alpha}/B_1 \sim B_2 \qquad U = U_{\alpha\alpha}/R_1 \sim R_2$$

are obtained by gluing together the two universal prescribed branched divisors and the two universal prescribed ramification divisors respectively (after possibly passing to an étale cover). The $\overline{M}^\bullet(\mathfrak{W}_0)$ family given by (π, f, p) defines the map $\Phi_{\alpha\alpha}$.

As before, we define $I(\mathfrak{W}) = -R^\bullet\pi_* f^*(\omega_p \oplus \mathcal{O}_{\mathcal{X}})$. The analogue of Proposition A.1 is the assertion

$$\Phi^*_{\alpha\alpha}(I(\mathfrak{W})_0) = I(X,(\alpha,\alpha)). \tag{17}$$

which is proven in essentially the same way:

By applying $R^\bullet\pi_*(-)$ to the two exact sequences

$$0 \to \mathcal{O}_U \to n_*\mathcal{O}_{U_{\alpha\alpha}} \to \mathcal{O}_R \to 0$$

$$0 \to f^*\omega_p \to n_* f^*_{\alpha\alpha}(\omega_{p_{\alpha\alpha}}(B_1 + B_2)) \to \mathcal{O}_R \to 0$$

we obtain the following equalities in the K-theory of $\overline{M}^\bullet(X,(\alpha,\alpha))$:

$$R^\bullet\pi_*\mathcal{O}_U = R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{U_{\alpha\alpha}} - R^\bullet\pi_*\mathcal{O}_R$$
$$R^\bullet\pi_* f^*\omega_p = R^\bullet\pi_{\alpha\alpha *} f^*_{\alpha\alpha}(\omega_{p_{\alpha\alpha}}(B_1+B_2)) - R^\bullet\pi_*\mathcal{O}_R.$$

Therefore we get

$$\begin{aligned}\Phi^*_{\alpha\alpha}(I(\mathfrak{W})_0) &= -R^\bullet\pi_* f^*(\omega_p\oplus\mathcal{O}_{\mathcal{X}})\\ &= -R^\bullet\pi_{\alpha\alpha *} f^*_{\alpha\alpha}(\omega_{p_{\alpha\alpha}}(B_1+B_2)) + R^\bullet\pi_*\mathcal{O}_R\\ &\quad - R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{U_{\alpha\alpha}} \qquad\qquad + R^\bullet\pi_*\mathcal{O}_R.\end{aligned}\tag{18}$$

Since $n_*\mathcal{O}_{R_i} = \mathcal{O}_R$ we get the following equalities in K-theory:

$$\begin{aligned}2R^\bullet\pi_*\mathcal{O}_R - R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{U_{\alpha\alpha}} &= R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{R_1} + R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{R_2} - R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{U_{\alpha\alpha}}\\ &= -R^\bullet\pi_{\alpha\alpha *}\mathcal{O}_{U_{\alpha\alpha}}(-R_1-R_2)\end{aligned}$$

where the last equality comes from the divisor sequence for R_1+R_2.

Combining the above with equation 18, we get

$$\begin{aligned}\Phi^*_{\alpha\alpha}(I(\mathfrak{W})_0) &= -R^\bullet\pi_{\alpha\alpha *}(f^*_{\alpha\alpha}(\omega_{p_{\alpha\alpha}}(B_1+B_2))\oplus\mathcal{O}_{U_{\alpha\alpha}}(-R_1-R_2))\\ &= I(X,(\alpha,\alpha))\end{aligned}$$

which completes the proof of Equation (17).

The derivation of the gluing formula from Equation (17) is easily obtained:

$$\begin{aligned}Z^b_d(g+1) &= \int_{[\overline{M}^\bullet(\mathfrak{W}_t)]^{vir}} c_b(I(\mathfrak{W})_t)\\ &= \int_{[\overline{M}^\bullet(\mathfrak{W}_0)]^{vir}} c_b(I(\mathfrak{W})_0)\\ &= \sum_{\alpha\vdash d}\mathfrak{z}(\alpha)\int_{[\overline{M}^\bullet(X,(\alpha,\alpha))]^{vir}} c_b(I(X,(\alpha,\alpha))\\ &= \sum_{\alpha\vdash d}\mathfrak{z}(\alpha) Z^b_d(g)_{\alpha\alpha}.\end{aligned}$$

The proof of the gluing formulas in case $r>0$ is identical. Although Li does not explicitly state the virtual cycle formula necessary for the $r>0$ case, the techniques and results of [19] extend in a straightforward way. □

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