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Seiberg-Witten à la Furuta and genus bounds for classes with divisibility

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1. Introduction

An important classical question in 4-manifold topology asks for a lower bound on the genus of an embedded surface Σ in terms of its homology class $[\Sigma] \in H_2(X, \mathbf{Z})$. For example, the classical Thom conjecture (proved by Kronheimer and Mrowka [5]) states that if $X = \mathbf{CP}^2$ and $\Sigma \hookrightarrow \mathbf{CP}^2$ is a smooth embedding with $[\Sigma] = d[H]$ where $[H] \in H_2(\mathbf{CP}^2)$ is the hyperplane class and $d > 0$ then

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}.$$

A generalization of this inequality exists for manifolds with non-zero Seiberg-Witten invariants and gives bounds in terms of the self-intersection of Σ and the pairings of Σ with the Seiberg-Witten basic classes (see [7]).

These methods break down for manifolds with zero Seiberg-Witten invariants which includes those manifolds that decompose as connected sums $X = X_1 \# X_2$ with $b_+(X_i) > 0$. For example, an unknown question related to the $\frac{11}{8}$-th conjecture is

Question 1. *Does the connected sum of n copies of the $K3$ surface split off an $S^2 \times S^2$?*

If one had strong enough genus bounds one could potentially give a negative answer to the above question by using the embedded surfaces in the $S^2 \times S^2$ summand.

Another interesting example is $\mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2$. Mikhalkin [6] has shown that the genus-minimizing surfaces in $\mathbf{CP}^2$ can have their genus reduced further after direct sum with additional copies of $\mathbf{CP}^2$. With better bounds on manifolds such as $\mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2$ we could determine if Mikhalkin's examples are sharp.

Before gauge theoretic methods were introduced into 4-manifold topology, genus bounds were obtained by assuming divisibility conditions on the class of Σ and studying the branched cover ([9],[8]). This idea was combined with the gauge theoretical methods of the Yang-Mills equations by Kotschick and Matic [4].

In this short note we apply Furuta's "$\frac{10}{8}$th's theorem" to the classical techniques of branched covers to obtain genus bounds. We also outline a strategy for generalizing Furuta's technique in this setting to improve the genus bounds. We lay some groundwork for implementing that strategy.

Furuta's theorem is the following important non-existence result for spin four-manifolds:

Theorem 1.1 (Furuta [3]). *Let X be a smooth 4-manifold with intersection from*

$$Q_X = 2kE_8 \oplus m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $k > 0$. Then $2k + 1 \leq m$.

To obtain genus bounds on $\Sigma \hookrightarrow X$, we make homological assumptions on the class of Σ so that there is a q-fold cover $Y \to X$ branched along Σ and such that Y is spin. We get a genus bound on Σ by applying Theorem 1.1 to Y:

Proposition 1.2. *Let $\Sigma \hookrightarrow X$ be a smooth embedding of a surface of genus g and self-intersection $[\Sigma] \cdot [\Sigma] = n$. Let $q = p^r$ be a prime power such that $q|[\Sigma]$ and let $Y \to X$ be the q-fold branched cover branched along Σ. Further suppose if q is even that $PD([\Sigma]) \equiv w_2(X) \mod 2$ and if q is odd suppose that X is spin (these conditions guarantee that Y is spin). Then*

$$g(\Sigma) \geq 1 + \frac{5n(1+q)}{24q} + \frac{2}{q-1} - \frac{q}{2(q-1)}(e_X + \frac{5}{4}\sigma_X)$$

where e_X and σ_X are the Euler characteristic and signature of X respectively.

On a spin 4-manifold, the Seiberg-Witten moduli space for the trivial Spin$^{\mathbf{C}}$ has an action of $Pin(2)$. Furuta's technique produces $Pin(2)$ representations V and W such that the virtual representation $[V] - [W]$ is equivalent to $[\mathrm{Ker}\, D] - [\mathrm{Coker}\, D]$ where D is the operator obtained by linearizing the Seiberg-Witten equations at the trivial solution. Furthermore, his technique then produces a map

$$f : (B(V), S(V)) \to (B(W), S(W))$$

where $B(\cdot)$ and $S(\cdot)$ denote the unit ball and sphere respectively.

By applying the equivariant K-theory functor to the above map, and employing operations in K-theory, Furuta deduces Theorem 1.1.

To improve the bound of Theorem 1.2, we outline the following strategy: Choose a $\mathbf{Z}/q$-invariant metric on Y, and try to lift the action of $\mathbf{Z}/q$ to the spin bundle. In this note, we show that when q is 2^r the action lifts to a $\mathbf{Z}/2q$ action on the spin bundle. Consequently, one obtains a Seiberg-Witten moduli space with a $\mathbf{Z}/q\tilde{\times}Pin(2)$-action where $\tilde{\times}$ denotes a twisted product (see Section 3).

Furuta's technique in this setting should give a $\mathbf{Z}/q\tilde{\times}Pin(2)$ equivariant map $f : (B(V), S(V)) \to (B(W), S(W))$ where again $[V] - [W]$ represents the K-theoretic index of the linearized Seiberg-Witten equations. The characters of $[V]-[W]$ can be determined by the G-index theorem and computations show that the virtual representation $[V]-[W]$ can be completely determined and depends in on the genus of Σ and its self-intersection $[\Sigma] \cdot [\Sigma]$.

Equivariant K-theory methods applied to the map f should then give a refinement of Proposition 1.2.

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2. Branched covers

Let $\Sigma \hookrightarrow X$ be an embedding of an oriented surface of genus $g = g(\Sigma)$ into an oriented, simply-connected smooth four manifold X. We assume that the class $[\Sigma] \in H_2(X, \mathbf{Z})$ is divisible by a prime power $q = p^r$ so we can consider the ramified cover

$$f : Y \to X$$

that is generically q-to-one and branched along Σ. Y is constructed as follows:

Let ν denote a tubular neighborhood of Σ and $\partial\nu$ its boundary; let $W = X \backslash \Sigma$. It is shown in [9] and [8] that $H_1(W, \mathbf{Z}) = \mathbf{Z}/d$ where $d = \max(a \in \mathbf{N} : a|[\Sigma])$ is the divisibility of $[\Sigma]$. Let $\tilde{W} \to W$ be the regular covering associated to the homomorphism

$$\pi_1(W) \to H_1(W) \to \mathbf{Z}/q.$$

Since the normal bundle N_Σ has degree $[\Sigma]^2 = n$ and $q|n$ there is a q-fold cover of N_Σ by the line bundle of degree n/q given by $(v, x) \mapsto (v^q, x)$. Using the identification of ν with N_Σ one thus gets a q-cover $\tilde{\nu} \to \nu$ ramified along Σ. Using the Mayer-Vietoris sequence it is easy to see that over $\nu \cap W$ $\tilde{W}$ and $\tilde{\nu}$ agree and so we can form $Y = \tilde{W} \cup \tilde{\nu}$. It turns out that the assumption that q is a prime power implies that $H_1(Y)$ is finite [9].

A manifold Y admits a spin structure if and only if $w_2(Y) = 0$. Brand gives a general formula for the characteristic numbers of a general branched cover [2]. Let

$$\alpha = \frac{q-1}{q} PD(\Sigma) \mod 2.$$

In our case Brand's formula then becomes

$$w_2(Y) = f^*(w_2(X) + \alpha).$$

Note that if q is odd then $\alpha = 0$. To guarantee that Y is spin we will always make the following assumption:

Assumption 1. *If q is odd we assume that X is spin. If q is even, we assume that $PD(\Sigma)/q$ is characteristic.*

Since $H_1(Y)$ is torsion, $b_1 = b_3 = 0$ and so to determine the intersection form of Y it is sufficient to compute the signature σ_Y and Euler charateristic e_Y. The Euler characteristic can be computed by an elementary counting argument using a triangulation of X subordinate to $\Sigma \hookrightarrow X$. To compute the signature requires the G-signature theorem. The results are

$$\begin{aligned} e_Y &= q e_X + (q-1)(2g-2), \\ \sigma_Y &= q\sigma_X + \frac{1-q^2}{3q}(\Sigma \cdot \Sigma). \end{aligned}$$

Under Assumption 1, Y is spin and the number E_8's and hyperbolic pairs in the intersection form are determined by the above formulas and Furuta's inequality translates into Proposition 1.2.

3. Lifting the $\mathbf{Z}/q$-action to the spin bundle

In this section we show that when $q = 2^r$ the Seiberg-Witten moduli spaces have an action of $Pin(2)\tilde{\times}\mathbf{Z}/q$.

We choose an invariant metric on Y so that the action of $\mathbf{Z}/q$ is an isometry. Let $g : Y \to Y$ generate the $\mathbf{Z}/q$-action. Then dg induces a $\mathbf{Z}/q$-action on the frame bundle $P \to Y$ that covers the action on Y. Since Y is spin, there is a double cover $\widehat{P}$ of P that restricts to each fiber as $Spin(4) \to SO(4)$. We utilize the following lemma (c.f. [1]):

Lemma 3.1. *An isometry $g : Y \to Y$ will have a lift $\widehat{dg}$ of dg to $\widehat{P}$ if $g^* : H^1(Y, \mathbf{Z}/2) \to H^1(Y, \mathbf{Z}/2)$ is the identity map. There are exactly two such lifts which we will denote $\pm\widehat{dg}$ and if $u : \widehat{P} \to \widehat{P}$ denotes the deck transformation then $u \circ \widehat{dg} = -\widehat{dg}$.*

To answer the question of whether a lift exists we solicit the help of a proposition in Kotschick and Matíc ([4] Prop. 2.1 and note the remark following the proof):

Proposition 3.2. *If $q = p^r$ is a power of a prime p, then $H_1(Y)$ has no p-torsion.*

Thus when $p = 2$, dg automatically lifts; however, $\widehat{dg}$ may have order $2q$ rather than q since $(\widehat{dg})^q$ is either u or the identity. The case of lifting involutions is considered in [1] and they show (Proposition 8.46):

Proposition 3.3. *Let Y be a spin manifold, $f : Y \to Y$ an involution preserving the orientation and spin structure, and let σ_i be the connected components of the fixed point set of f. Then*

$$\begin{aligned} \operatorname{codim} \Sigma_i &\equiv 0 \mod 4 \text{ if } \widehat{df} \text{ is order } 2 \\ \operatorname{codim} \Sigma_i &\equiv 2 \mod 4 \text{ if } \widehat{df} \text{ is order } 4. \end{aligned}$$

Applying the propostion to $f = g^{q/2}$ in our case, we see that $\pm\widehat{dg}$ has order $2q$ and $\widehat{dg}^q = u$.

The Seiberg-Witten equations for the trivial spin$_{\mathbf{C}}$ structure on Y can be written as equations for the pair

$$(A, \phi) \in \Omega^1(Y, \mathbf{R}) \times \Gamma(S^+).$$

They are (c.f. [3]):

$$\begin{aligned} \not\partial\phi + ia \cdot \phi &= 0, \\ \rho(id^+a) - (\phi \otimes \phi^*)_0 &= 0, \\ d^*a &= 0. \end{aligned}$$

The final equation is a gauge fixing condition. Consider $Pin(2)$ as the subgroup of the unit quaternions given by elements of the form $e^{i\theta}$ or $e^{i\theta}j$. Since S^+ is a quaternionic bundle, $Pin(2)$ naturally acts on $\Gamma(S^+)$ and we define the action of $Pin(2)$ on $\Omega^1(Y, \mathbf{R})$ to be multiplication by 1 or -1 for $e^{i\theta}$ or $e^{i\theta}j$ respectively. The above action can be seen to preserve the solution space of the Seiberg-Witten equations. The action of the $e^{i\theta}$ subgroup is just the usual action of the constant gauge transformations.

Since g is an isometry, $\widehat{dg}$ also preserves the solution space and so we get a natural action of $Pin(2) \times \mathbf{Z}/2q$ on the solution space. We wish to show that $(-1, q) \in Pin(2) \times \mathbf{Z}/2q$ acts trivially so that we have an action of

$$\frac{(Pin(2) \times \mathbf{Z}/2q)}{\mathbf{Z}/2} = Pin(2)\tilde{\times}\mathbf{Z}/q.$$

From Proposition 3.3, we know that $\widehat{dg}^q$ is the deck transformation u. Fiberwise, u acts by -1 on $Spin(4) = SU(2) \times SU(2)$ and since S^+ is the bundle associated to the standard respresentation of the first $SU(2)$ factor, u acts by -1 on sections of S^+. This is just the action of the constant gauge transformation $-1 \in Pin(2)$ thus $(-1, q)$ acts as $u^2 = 1$ on configurations.

In summary we have

Theorem 3.4. *Let Y and q be as in Proposition 1.2 and further assume that $q = 2^r$. Then the solution space to Seiberg-Witten equations for the trivial spin$_{\mathbf{C}}$ structure has an action of $Pin(2)\tilde{\times}\mathbf{Z}/q$.*

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