# CLASSIFICATION OF SINGULARITIES (WED. NOV 20) READ SECTION 5.6

THE LAURENT SERIES OF A FUNCTION $f(z)$ HAS THE FORM

$$(*) \quad f(z) = \left( a_0 + a_1(z-z_0) + \dots + a_n(z-z_0)^n + \dots \right) + \left( \frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \dots + \frac{a_{-n}}{(z-z_0)^n} + \dots \right)$$

FOR SOME COEFFICIENTS $a_n$ FOR $n = 0, \pm 1, \pm 2, \dots$.

LET'S ASSUME THAT $f(z)$ IS ANALYTIC IN $0 < |z - z_0| < R$ BUT HAS AN ISOLATED SINGULARITY AT $z = z_0$. THEN WE CLASSIFY THE SINGULARITY AT $z = z_0$ USING THE LAURENT SERIES $(*)$.

- $f(z)$ HAS A <u>POLE</u> OF ORDER $m > 0$ AT $z = z_0$ IF $a_{-m} \neq 0$ BUT $a_{-m-1} = a_{-m-2} = \dots$

  THE L. SERIES THEN HAS THE FORM

  terminates here.

  $$f(z) = \left( a_0 + a_1(z-z_0) + \dots \right) + \left( \frac{a_{-1}}{z-z_0} + \dots + \frac{a_{-m}}{(z-z_0)^m} \right)$$

  <u>EXAMPLE</u>:
  - $f(z) = \dfrac{z-1}{z^2}$ HAS A POLE OF ORDER 2 AT $z = 0$ SINCE WE HAVE L. SERIES $f(z) = \dfrac{1}{z^2} - \dfrac{1}{z}$ NEAR $z = 0$.
  - $f(z) = \dfrac{z}{(z+1)(z+2)}$ HAS A POLE OF ORDER 1 AT $z = -2$ SINCE WE CAN WRITE $f(z) = \dfrac{(z+2) - 2}{[(z+2) - 1]} \dfrac{1}{(z+2)} = \dfrac{2}{(z+2)} + 1 + (z+2) + \dots$ VALID FOR $0 < |z+2| < 1$.
  - $f(z) = \dfrac{z - \sin z}{z^5}$ HAS A POLE OF ORDER 2 AT $z = 0$ SINCE WE HAVE BY TAYLOR SERIES $f(z) = \dfrac{z - (z - z^3/3! + \dots)}{z^5} = \dfrac{1}{6z^2} + \dots$ AS $z \to$

  A POLE OF ORDER 1 IS CALLED A SIMPLE POLE.

- $f(z)$ HAS AN <u>ESSENTIAL SINGULARITY</u> AT $z = z_0$ IF THE LAURENT SERIES HAS AN <u>INFINITE NUMBER</u> OF TERMS IN POSITIVE POWERS OF $1/z - z_0$.

  <u>EX</u>: $f(z) = e^{1/z}$ $\quad f(z) = 1 + \dfrac{1}{z} + \dfrac{1}{2z^2} + \dfrac{1}{6z^3} + \dots + \dfrac{1}{n! z^n} + \dots$ HAS AN ESSENTIAL SINGULARITY AT $z = 0$.

  $f(z) = \sin(1/z) = 1/z - 1/6z^3 + \dots$ ALSO HAS AN ESSENTIAL SING. AT $z = 0$

- $f(z)$ HAS A <u>REMOVABLE SINGULARITY</u> AT $z = z_0$ IF $f(z)$ IS NOT DEFINED AT $z = z_0$ BUT THAT $\lim_{z \to z_0} f(z)$ EXISTS. IN THIS CASE $a_{-1} = a_{-2} = \ldots = 0$ IN THE L-SERIES.

<u>EX</u>: $f(z) = \dfrac{z - \sin z}{z^3}$ NOT DEFINED AT $z = 0$.

BUT USING TAYLOR SERIES $f(z) = \dfrac{z - (z - z^3/6 + \ldots)}{z^3} \to 1/6$ AS $z \to 0$.

THUS $a_{-1} = a_{-2} = \ldots = 0$.

$$f(z) = \begin{cases} (z - \sin z)/z^3 & z \neq 0 \\ 1/6 & z = 0 \end{cases}$$

IS AN ANALYTIC FUNCTION.

HENCE WE HAVE 3-TYPES OF SINGULAR POINTS: POLES, ESSENTIAL SINGULARITIES, AND REMOVABLE SINGULARITIES.

<u>DEFINITION</u>: THE COEFFICIENT $a_{-1}$ IN LAURENT SERIES IS CALLED THE RESIDUE OF $f(z)$ AT $z = z_0$. WE NOW GIVE METHODS TO CALCULATE IT WHEN $f(z)$ HAS A POLE OF ORDER $m$ AND WHEN $f(z)$ HAS AN ESSENTIAL SINGULARITY AT $z = z_0$. $a_{-1} = \mathrm{RES}[f; z_0] \Leftarrow$ NOTATION

## CALCULATING THE RESIDUE $a_{-1}$ IN LAURENT SERIES

<u>CASE I</u> SUPPOSE $f(z)$ HAS AN <u>ESSENTIAL SINGULARITY</u> AT $z = z_0$. THEN THE ONLY WAY TO CALCULATE $a_{-1}$ IS TO DETERMINE THE LAURENT SERIES AND CALCULATE $a_{-1}$ EXPLICITLY.

<u>EXAMPLE</u> CALCULATE $a_{-1}$ FOR $f(z) = z e^{1/z}$, ABOUT $z = 0$.

THE L. SERIES FOR $f(z)$ IS $\quad e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!}$

$$f(z) = z\left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots\right]$$

$$f(z) = z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \dots$$ HENCE FROM TERM $a_{-1} = 1/2$.

<u>CASE 2</u> SUPPOSE $f(z)$ HAS A POLE OF ORDER $m$ AT $z = z_0$. THEN

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \quad \text{IN } 0<|z-z_0|<$$

NOW MULTIPLY BY $(z-z_0)^m$ TO GET

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots$$

Now to get $a_{-1}$ differentiate $m-1$ times and divide by $(m-1)!$ and evaluate the result at $z = z_0$. This isolates $a_{-1}$.

$$(*) \quad a_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$ See Formula 4 on p. 248.

<u>EXAMPLE:</u>

- $f(z) = \dfrac{z - \sin z}{z^5}$ HAS A POLE OF ORDER 2 AT $z=0$.

$$a_{-1} = \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0} \frac{d}{dz}\left[\frac{z^2\,(z-(z - z^3/3! + z^5/5! + \dots))}{z^5}\right] = 0$$

$$\longrightarrow a_{-1} = 0.$$

- $f(z) = \dfrac{(z+1)}{(z^2+9)}$ HAS A SIMPLE POLE AT $z = 3i$.

THE RESIDUE $a_{-1}$ IS

$$a_{-1} = \lim_{z\to 3i}\left[(z-3i) f(z)\right] = \lim_{z\to 3i}\left[\frac{(z-3i)(z+1)}{(z-3i)(z+3i)}\right] = \frac{3i+1}{6i}$$

$$\Longrightarrow \quad a_{-1} = \frac{3-i}{6}$$

- FIND RESIDUE FOR $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$ AT $z = -1$.

NOTICE $f(z) \sim \dfrac{3}{5(z+1)^2}$ AS $z \to -1$ SO WE HAVE A POLE OF ORDER 2 AT $z = -1$.

TO FIND $a_{-1}$ WE COULD WORK OUT L. SERIES DIRECTLY AND IDENTIFY $a_{-1}$ OR ELSE USE FORMULA (A) ON P. 3.

$$a_{-1} = \lim_{z \to -1} \frac{d}{dz} \left[ (z+1)^2 f(z) \right] = \lim_{z \to -1} \frac{d}{dz} \left[ \frac{z^2 - 2z}{z^2+4} \right]$$

OR $$a_{-1} = \lim_{z \to -1} \left[ \frac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} \right] = -\frac{14}{25}$$

<u>SPECIAL CASE</u> SUPPOSE $f(z)$ HAS A SIMPLE POLE (POLE OF ORDER 1) AT $z = z_0$. THEN WE CAN EXPRESS $f(z)$ AS

$$f(z) = \frac{P(z)}{Q(z)}$$ WHERE $P(z)$, $Q(z)$ ANALYTIC AT $z = z_0$

$P(z_0) \neq 0$

$Q(z_0) = 0$ <u>BUT</u> $Q'(z_0) \neq 0$.

NOTICE $$a_{-1} = \lim_{z \to z_0} \left[ (z - z_0) f(z) \right] = \lim_{z \to z_0} \left[ (z - z_0) \frac{P(z)}{Q(z)} \right]$$

$$a_{-1} = \lim_{z \to z_0} \left[ \frac{P(z)}{\frac{Q(z) - Q(z_0)}{z - z_0}} \right] = \frac{P(z_0)}{Q'(z_0)}$$ (NOTICE $Q(z_0) = 0$ SO WE ADDED AND SUBTRACTED 0)

<u>IMPORTANT</u> : SUPPOSE $f(z) = P(z)/Q(z)$ AND THERE IS A SIMPLE POLE AT $z = z_0$.

THEN $$a_{-1} = \frac{P(z_0)}{Q'(z_0)}.$$ (see Example 2 on P. 247)

This gives perhaps the easiest method to FIND the residue when we have a simple pole.

**EXAMPLE:** $f(z) = \dfrac{z+1}{z^2-2z}$ HAS A SIMPLE POLE AT $z=0$ AND $z=2$.

$P = z+1$, $Q = z^2-2z$ $\quad Q' = 2z-2$.

- RESIDUE OF $f$ at $z=0$ is $a_{-1} = \dfrac{P(0)}{Q'(0)} = \dfrac{1}{-2}$
- Residue of $f$ at $z=2$ is $a_{-1} = \dfrac{P(2)}{Q'(2)} = \dfrac{3}{2}$

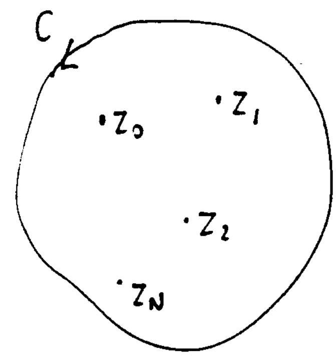
## RESIDUE THEOREM

LET $f(z)$ HAVE ISOLATED SINGULARITIES AT THE POINTS $z = z_0, z_1, z_2, \ldots, z_N$ inside a closed contour $C$ oriented counterclockwise. Assume $f(z)$ is analytic at all other points. then

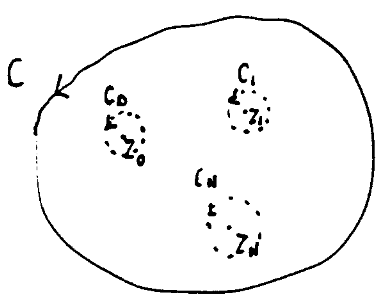
CAUCHY-RESIDUE THEOREM

$$\int_C f(z)\,dz = 2\pi i\left(\text{RES}[f; z_0] + \text{RES}[f; z_1] + \ldots + \text{RES}[f; z_N]\right)$$

WHERE $\text{RES}[f; z_j]$ is THE Residue of $f(z)$ at $z = z_j$ (i.e. the coefficient $a_{-1}$ IN Laurent series OF $f(z)$ at $z = z_j$).




**PROOF** LET $C_0, \ldots, C_N$ be counterclockwise oriented circles enclosing $z_0, \ldots, z_N$ and centered at these points as shown below. THEN SINCE $f$ is analytic in region inside $C$ but outside the union of the $C_j$ we have by CAuchy Integral FORMULA




$$\int_C f\,dz - \int_{C_0} f\,dz - \ldots - \int_{C_N} f\,dz = 0$$

(Recall our earlier derivation in class with cut out regions.

HENCE $\displaystyle\int_C f\,dz = \int_{C_0} f\,dz + \ldots + \int_{C_N} f\,dz$.

NOW WE CAN DEVELOP $f$ IN L.SERIES NEAR EACH $z_0$ AND EXTRACT THE RESIDUE FROM EACH TERM $\longrightarrow$ $\displaystyle\int_C f\,dz = 2\pi i\left[\text{RES}[f; z_0] + \ldots + \text{RES}[f; z_N]\right]$.

<u>EXAMPLE</u> EVALUATE $\int_C e^{1/z}\,dz$ C CLOSED CONTOUR $|z|=1$ ORIENTED IN COUNTER-CLOCKWISE SENSE.

LET $f(z) = e^{1/z}$. THEN THE L. SERIES IS

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots + \frac{1}{N!\,z^N} + \cdots$$

THUS $z=0$ IS AN ESSENTIAL SINGULARITY.

ALSO $\text{RES}[f;0] = 1$.

HENCE $\int_C e^{1/z}\,dz = 2\pi i\, \text{RES}[f;0] = 2\pi i$. by CAUCHY-RESIDUE THEOREM.

<u>EXAMPLE</u> EVALUATE $I = \int_C \frac{5z-2}{z(z-1)}\,dz$ C CLOSED CONTOUR $|z|=2$ ORIENTED COUNTERCLOCKWISE.

NOTICE THAT $f(z) = \frac{5z-2}{z(z-1)}$ HAS SIMPLE POLES AT $z=0$ AND $z=1$ AND THEY LIE WITHIN C.

THUS BY CAUCHY-RESIDUE-THEOREM (abbreviated by CRT) we have

$$(*) \quad I = 2\pi i \left[ \text{RES}[f;0] + \text{RES}[f;1] \right].$$

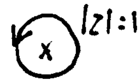
NOW To calculate residues we have

$$\text{RES}[f;0] = \lim_{z\to 0}\left[ z f \right] = \lim_{z\to 0}\left[ \frac{5z-2}{z-1} \right] = 2$$

$$\text{RES}[f;1] = \lim_{z\to 1}\left[ (z-1) f \right] = \lim_{z\to 1}\left[ \frac{5z-2}{z} \right] = 3$$

THEREFORE FROM $(*)$ $I = 2\pi i\,[2+3] = 10\pi i$.

**EXAMPLE** EVALUATE $I = \oint_{|z|=1} \frac{1}{z^2 \sin z}\, dz$




$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2\left(z - z^3/3! + \cdots\right)} = \frac{1}{z^3\left(1 - z^2/3! + \cdots\right)} \approx \frac{1}{z^3}\left(1 + z^2/3! + \cdots\right) \quad \text{AS } z \to 0$$

$$f(z) = \frac{1}{z^3} + \frac{1}{6z} + \cdots \quad \text{FOR } 0 < |z| < \pi \quad \text{SINCE } \sin \pi = 0$$

(i.e. distance to nearest sing. of $f(z)$).

NOTICE $a_{-1} = \text{RES}[f, 0] = \frac{1}{6}$ AND $z = 0$ is inside the contour $|z| = 1$.

WE HAVE THAT $f(z)$ HAS A POLE OF ORDER 3 AT $z = 0$ SINCE

$$\lim_{z \to 0} z^3 f(z) \text{ IS FINITE AND NONZERO.}$$

BY THE RESIDUE THEOREM $I = 2\pi i\, a_{-1} \longrightarrow I = \pi i/3$.

**EXAMPLE** EVALUATE $I = \oint_{|z|=8} \frac{1}{z^2 + z + 1}\, dz$ NOTICE $f(z) = \frac{1}{z^2 + z + 1}$

HAS POLES AT THE ZEROES OF THE DENOMINATOR. i.e. AT $z = \frac{-1 \pm \sqrt{1-4}}{2}$

$$z_\pm = \frac{-1 \pm i\sqrt{3}}{2} \qquad \text{BOTH POLES SATISFY } |z_\pm| < 8 \text{ AND SO}$$

LIE INSIDE THE CONTOUR $|z| = 8$. NOW BY RES. THEOREM

$$I = 2\pi i\left[\text{RES}[f; z_+] + \text{RES}[f; z_-]\right].$$

NOW TO CALCULATE THE RESIDUE WE WRITE

$$f(z) = \frac{1}{(z - z_+)(z - z_-)} \qquad \text{let } P(z) = 1$$

$$Q(z) = (z - z_+)(z - z_-).$$

THEN $\text{RES}[f; z_-] = \frac{P(z_-)}{Q'(z_-)}$ $\quad \text{RES}[f; z_+] = \frac{P(z_+)}{Q'(z_+)}$. (since $z_\pm$ are simple poles)

NOW $Q'(z_+) = z_+ - z_-$ AND $Q'(z_-) = (z_- - z_+)$. THUS

$$\text{RES}[f; z_-] = 1/(z_- - z_+) \qquad \text{RES}[f; z_+] = 1/(z_+ - z_-).$$

(adding residues gives 0)

THIS YIELDS UPON ADDING THAT $I \equiv 0$.

**EXAMPLE** EVALUATE $I = \oint_{|z|=1} e^{1/z} \sin(1/z)\, dz$

NOW $z=0$ IS AN ESSENTIAL SINGULARITY. WE MUST COMPUTE THE RESIDUE DIRECTLY.

$$f(z) = e^{1/z}\sin(1/z) = \left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3}\right)\left(\frac{1}{z} - \frac{1}{3!\,z^3} + \dots\right) = \frac{1}{z} + \left(\text{higher powers of } 1/z\right)$$

HENCE $a_{-1} = \text{RES}[f;0] = 1$

$\longrightarrow \quad I = 2\pi i\, a_{-1} = 2\pi i$ BY RESIDUE THEOREM.

**EXAMPLE** FIND ISOLATED SINGULARITIES OF $f(z) = \dfrac{z-1}{\sin z}$ AND COMPUTE RESIDUE AT EACH SINGULARITY. NOTICE

$\sin z = 0$ AT $z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$ etc. $z = n\pi$ $n = 0, \pm 1, \pm 2, \dots$

these are simple poles of $\dfrac{1}{\sin z}$ since $\cos z = \dfrac{d}{dz}\sin z \neq 0$ at $z = n\pi$.

$$\text{RES}[f; n\pi] = \frac{P(n\pi)}{Q'(n\pi)} = \frac{n\pi - 1}{\cos[n\pi]} = (-1)^n[n\pi - 1]$$

$$\text{RES}[f; n\pi] = (-1)^n[n\pi - 1] \qquad n = 0, \pm 1, \pm 2, \dots$$

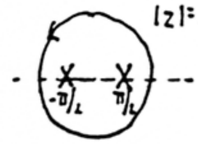
**EXAMPLE** EVALUATE $I = \oint_{|z|=2} \tan z\, dz$

$|z|=2$

NOTICE $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ $\cos z = 0$ AT $z = \pi/2, -\pi/2, 3\pi/2, -3\pi/2, \dots$

ONLY THE SINGULARITIES AT $z = \pi/2$ AND $z = -\pi/2$ LIE INSIDE THE CONTOUR.

$|z|=2$




NOW $z = \pm\pi/2$ CORRESPOND TO SIMPLE POLES.

LET $P(z) = \sin z$, $Q(z) = \cos z$.

$$\left[\ \text{RES}[f; z_i] = \frac{P(z_i)}{Q'(z_i)}\ \right]$$

FOR SIMPLE POLES.

$$\text{RES}[f; \pi/2] = \frac{\sin \pi/2}{-\sin \pi/2} = -1$$

$$\text{RES}[f; -\pi/2] = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

HENCE $I = 2\pi i\left[\text{RES}[f;\pi/2] + \text{RES}[f;-\pi/2]\right] = -4\pi i.$

<u>EXAMPLE</u> EVALUATE $I = \oint_{|z|=2} \frac{dz}{z^3(z+4)}$

LET $f(z) = \frac{1}{z^3(z+4)}$ HAS A SIMPLE POLE AT $z=-4$ ⟵ OUTSIDE CONTOUR $|z|=2$
AND A POLE OF ORDER 3 AT $z=0$. ⟵ INSIDE THE CONTOUR

NOW WE CAN USE THE FORMULA

$$a_{-1} = \frac{1}{2!} \lim_{z\to 0} \left[ \frac{d^2}{dz^2}\left( z^3 f(z)\right)\right] \quad \text{ON P. 3}$$

to compute the residue. Proceeding this way we get

$$a_{-1} = \frac{1}{2!} \lim_{z\to 0} \left[ \frac{d^2}{dz^2}\left(\frac{1}{z+4}\right)\right] = \frac{1}{2!}\, 2\,(z+4)^{-3}\Big|_{z=0} = \frac{1}{64}.$$

$$\to \quad \text{RES}\,[f;0] = \frac{1}{64}.$$

ANOTHER WAY TO COMPUTE $a_{-1}$ IS TO USE THE L. SERIES AND IDENTIFY THE $a_{-1}$ TERM.

i.e. $$f(z) = \frac{1}{z^3(z+4)} = \frac{1}{4z^3(1+z/4)} = \frac{1}{4z^3}\left(1 - z/4 + z^2/16 + \cdots\right)$$

$$f(z) = \frac{1}{4z^3} - \frac{1}{16z^2} + \frac{1}{64z} + \cdots \quad \text{converges FOR } 0<|z|<4.$$

$a_{-1} = \frac{1}{64}$.

THUS BY Residue theorem

$$I = 2\pi i\, a_{-1} \quad \to \quad \underline{I = \pi i/32}$$

<u>SECTION 6.2 TRIGONOMETRIC INTEGRALS</u>

USE RESIDUE THEOREM TO EVALUATE

$$(1) \qquad I = \int_0^{2\pi} U(\cos\varphi, \sin\varphi)\, d\varphi \quad \longleftarrow \text{REAL INTEGRAL}$$

<u>NOTICE</u>: INTEGRATION GOES FROM 0 TO $2\pi$ (i.e. ONCE AROUND the circle).

CAST $I$ AS A COMPLEX ~~INT~~ CONTOUR INTEGRAL OVER THE UNIT CIRCLE

<u>RECALL</u> IF $z = e^{i\varphi}$, $dz = ie^{i\varphi}\,d\varphi$ $\quad \cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}$ $\quad \sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$

THIS GIVES

$$\cos\varphi = \frac{z + 1/z}{2} \qquad \sin\varphi = \frac{z - 1/z}{2i} \qquad d\varphi = \frac{dz}{iz}.$$

INTEGRATE FROM $\varphi = 0$ TO $\varphi = 2\pi$ $\longrightarrow$ going around unit circle in $z$-plane in counterclockwise direction.

THUS

$$I = \oint_{|z|=1} U\left[\tfrac{1}{2}(z + 1/z), \tfrac{1}{2i}(z - 1/z)\right] \frac{dz}{iz}.$$

THIS IS AN INTEGRAL OF THE FORM $I = \oint_{|z|=1} f(z)\, dz$

AND CAN BE EVALUATED USING RESIDUE CALCULUS. $I = 2\pi i \sum \text{RES}[f; z_i]$

where $z_i$ are singular points of $f(z)$ inside contour $|z| = 1$.

## SECTION 6.3 IMPROPER INTEGRALS OVER REAL LINE $(-\infty, \infty)$.

A FUNCTION $f(x)$ IS INTEGRABLE OVER THE REAL LINE IF THE LIMITS IN

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{c \to -\infty} \int_{c}^{0} f(x)\, dx + \lim_{c \to +\infty} \int_{0}^{c} f(x)\, dx$$

EXIST AND ARE FINITE.

IN SUCH A CASE WE HAVE $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = \lim_{c \to \infty} \int_{-c}^{c} f(x)\, dx.$

HOWEVER EVEN IF $\displaystyle\lim_{c \to \infty} \int_{-c}^{c} f(x)\, dx$ EXISTS IT DOES NOT MEAN THAT THE INDIVIDUAL INTEGRALS $\displaystyle\lim_{c \to -\infty} \int_{c}^{0} f\, dx$ AND $\displaystyle\lim_{c \to \infty} \int_{0}^{c} f\, dx$ EXIST.

EG: LET $f(x) = x$. THEN $\displaystyle\lim_{c \to \infty} \int_{-c}^{c} f\, dx = 0$ BUT $\displaystyle\int_{0}^{c} f\, dx \to \infty$ as $c \to \infty$.

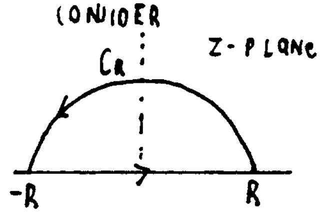
HENCE FOR ANY FUNCTION $f(x)$ WE DEFINE THE CAUCHY PRINCIPAL VALUE OF $f(x)$ OVER $-\infty$ TO $\infty$ (abbreviated by p.v. $\int_{-\infty}^{\infty} f(x)\, dx$) BY

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx \equiv \lim_{c \to \infty} \int_{-c}^{c} f(x)\, dx.$$

WE NOW SHOW HOW TO CALCULATE P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ USING RESIDUE CALCULUS.

<u>IDEA</u> LET $I = \int_{-\infty}^{\infty} f(x)\,dx$. THEN CONSIDER

$$\int_C f(z)\,dz$$




$C$ is contour composed of $C_R$ AND LINE segment $-R$ to $R$.

Let $R \to \infty$ : $\lim_{R\to\infty}\left[\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx\right] = 2\pi i \sum_j \mathrm{RES}\,[f; z_j]$

WHERE $z_j$ is a singularity of $f(z)$ in upper ½ plane.

HENCE P.V. $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_j \mathrm{RES}\,[f; z_j] - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$

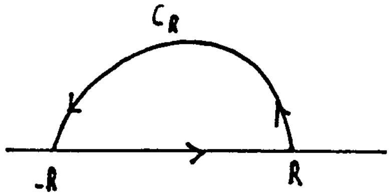
UNDER WHAT CIRCUMSTANCES does the integral over the big circle $\to 0$ as $R \to \infty$?

<u>LEMMA</u> let $f(z) = P(z)/Q(z)$ $\quad$ P, Q POLYNOMIALS

deggree $Q \geq 2 +$ degree $P$.

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{K}{|z|^2}\cdot 2\pi |z| = \frac{2\pi K}{R} \to 0 \text{ as } R\to\infty$$

$\left( \text{SINCE} \quad f(z) = \dfrac{A z^m + \dots}{B z^{m+2} + \dots} \right.$




<u>EXAMPLES</u> EVALUATE

$$I = \int_0^\infty \frac{x^2}{(1+x^2)(4+x^2)}\,dx$$

FIRST WE NOTE $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2I$, by symmetry.

NOW WE INTEGRATE OVER CIRCLE IN UPPER 1/2 PLANE,

singularities in 1/2 plane $Im(z)>0$ ONLY.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(1+x^2)(4+x^2)}\,dx + \int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)}\,dz = 2\pi i \sum_j \text{RES}\,[\,f, z_j\,] \qquad f = \frac{z^2}{(1+z^2)(4+z^2)}$$

NOW $z = 2i$, $z = i$ ARE SING. IN UPPER 1/2 PLANE $\to$ simple poles.

ALSO $\displaystyle\left|\int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)}\,dz\right| \longrightarrow 0$ AS $R\to\infty$ SINCE $P = z^2$

$Q = (4+z^2)(1+z^2)$

$\deg Q = \deg P + 2.$

THEREFORE,

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2\pi i\left(\text{RES}\,[\,f; i\,] + \text{RES}\,[\,f; 2i\,]\right).$$

$Q' = 2z(4+z^2) + 2z(1+z^2).$

$$\text{RES}\,[\,f; i\,] = \frac{P(i)}{Q'(i)} = \frac{-1}{2i\cdot 3} = \frac{i}{6}$$

$$\text{RES}\,[\,f; 2i\,] = \frac{P(2i)}{Q'(2i)} = \frac{-4}{4i(-3)} = -\frac{i}{3}.$$

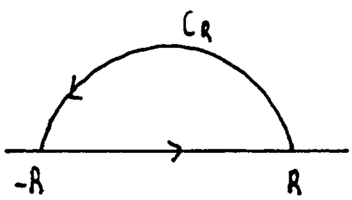
THUS

$$2I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) = \frac{\pi}{3}$$

$$\longrightarrow \quad I = \pi/6.$$

<u>EXAMPLE</u> EVALUATE $\displaystyle I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$. $x = i$ IS A POLE OF ORDER 2.

NOW INTEGRATE OVER THE REGION AS SHOWN BELOW.




$$\lim_{R\to\infty}\left(\int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{C_R} \frac{dz}{(1+z^2)^2}\right) = 2\pi i\ \text{RES}\,[\,f; i\,].$$

NOW $\displaystyle\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{(1+z^2)^2}\right) = \lim_{R\to\infty} \frac{K\cdot R}{R^4} = 0.$

THUS, (*) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \,\text{RES}\,[f, i].$ $\qquad \dfrac{1}{(1+h)^2} = 1-2h+\ldots \quad h \to 0.$

$$f = \frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2(z-i)^2} = \frac{1}{(z-i)^2}\left[(z-i)+2i\right]^{-2} = \frac{1}{(z-i)^2}\frac{1}{(2i)^2\left[1+\frac{(z-i)}{2i}\right]^2}$$

$$f = \frac{1}{-4(z-i)^2}\left[1 - 2\frac{(z-i)}{2i} + \ldots\right] = -\frac{1}{4(z-i)^2} - \frac{i}{4(z-i)} + \ldots \qquad a_{-1} = -\frac{i}{4} \text{ IS RESIDUE.}$$

WE CAN ALSO CALCULATE RESIDUE USING FORMULA OBTAINED BY

$$f(z) = \frac{a_{-2}}{(z-i)^2} + \frac{a_{-1}}{(z-i)} + \ldots$$

$$f(z)(z-i)^2 = a_{-2} + a_{-1}(z-i) + \ldots$$

$$\lim_{z\to i} \frac{d}{dz}\left[f(z-i)^2\right] = a_{-1} \quad \rightarrow \quad a_{-1} = \lim_{z\to i}\frac{d}{dz}\left[(z+i)^{-2}\right] = -\frac{2}{(2i)^3} = \frac{1}{4i} = -\frac{i}{4}.$$

THEREFORE FROM (*) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = 2\pi i\left(-i/4\right) = \pi/2.$

<u>EXAMPLE</u> EVALUATE $\displaystyle I = \int_0^{2\pi}\frac{d\varphi}{2+\cos^2\varphi}$ $\qquad dz = ie^{i\varphi}d\varphi \qquad d\varphi = \dfrac{dz}{iz}$

CONVERT TO AN INTEGRAL OVER THE UNIT CIRCLE.

LET $z = e^{i\varphi}$ $\qquad \cos\varphi = \frac{1}{2}\left(z + 1/z\right) \qquad \cos^2\varphi = \frac{1}{4}\left(z^2+2z+1/z^2\right)$

$$I = \oint_{|z|=1}\frac{dz}{iz\left(2+\frac{1}{4}(z^2+2+1/z^2)\right)} = -i\oint_{|z|=1}\frac{4z\,dz}{z^4+10z^2+1}$$

$$I = -i\oint_{|z|=1}\frac{4z}{z^4+10z^2+1}\,dz$$

SING. POINTS at

$$z^2 = \frac{-10 \pm\sqrt{96}}{2} = -5 \pm 2\sqrt{6}$$

NOTICE THAT THE $-$ ROOT LIES OUTSIDE $|z|=1$ AND

$z^2 = -5+2\sqrt{6} < 0$ LIES INSIDE $|z|=1$

$z = \pm i\sqrt{-2\sqrt{6}+5}$ ARE SIMPLE ROOTS OF DENOMINATOR

(i.e. ARE SIMPLE POLES).

LET $z_+ = i\sqrt{-2\sqrt{6}+5}$ $\qquad z_- = -i\sqrt{-2\sqrt{6}+5}$ ARE SIMPLE POLES IN $|z|=1$

THEN LET $P = -4iz$, $Q = z^4+10z^2+1$. $\quad Q'(z_\pm) = 4z_\pm^3 + 20z_\pm$.

THEN BY RESIDUE THEOREM,

$$I = 2\pi i\left[\operatorname{RES}\left(P/Q\,;\, z_+\right) + \operatorname{RES}\left(P/Q\,;\, z_-\right)\right].$$

NOW CALCULATE, $$I = 2\pi i\left[\frac{-4i\, z_+}{4z_+^3+20z_+} - \frac{4i\, z_-}{4z_-^3+20z_-}\right] = 2\pi\left[\frac{1}{z_+^2+5}+\frac{1}{z_-^2+5}\right].$$

NOW $z_+^2+5 = -5+2\sqrt{6}+5 = 2\sqrt{6}$

$z_-^2+5 = 2\sqrt{6}$ $\longrightarrow$ $I = 2\pi\left(1/2\sqrt{6} + 1/2\sqrt{6}\right)$

$I = 2\pi/\sqrt{6}$.

<u>EXAMPLE</u> $$I = \int_0^{2\pi}\frac{d\varphi}{a+\cos\varphi}. \quad a>1 \qquad z = e^{i\varphi}, \quad dz = ie^{i\varphi}\,d\varphi = iz\,d\varphi$$

NOW $\cos\varphi = \dfrac{(z+1/z)}{2}$ $$I = \oint_{|z|=1}\frac{1}{iz}\,\frac{dz}{a+\frac{(z+1/z)}{2}} = \oint_{|z|=1}\frac{-2i}{z^2+2az+1}\,dz$$

NOW $f(z) = \dfrac{-2i}{z^2+2az+1}$ HAS SIMPLE POLES AT $$z_\pm = \frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$$

NOTICE THAT FOR $a>1$;

$z_+ = -a+\sqrt{a^2-1}$ SATISFIES $z_+'(a)>0$, $z_+(1)=-1$, $z_+(a)\to 0$ AS $a\to\infty$

HENCE $z_+(a)$ LIES INSIDE CIRCLE $|z|=1$.

$z_- = -a-\sqrt{a^2-1}$ LIES OUTSIDE CIRCLE $|z|=1$

NOW $$I = 2\pi i\operatorname{RES}[f\,;z_+] = 2\pi i\left[\frac{-2i}{2z_++2a}\right] = \frac{\pi}{z_++a} = \frac{\pi}{(a^2-1)^{1/2}}.$$

HENCE $$I = \int_0^{2\pi}\frac{d\varphi}{a+\cos\varphi} = \frac{\pi}{(a^2-1)^{1/2}}.$$

<u>EXAMPLE</u> $$I = \int_0^{2\pi}\frac{\cos 2\varphi}{1-2a\cos\varphi+a^2}\,d\varphi \quad |a|<1.$$

$\cos w = \dfrac{e^{iw}+e^{-iw}}{2}$ $\qquad dz = ie^{i\varphi}\,d\varphi$

$\cos 2\varphi = \dfrac{e^{2i\varphi}+e^{-2i\varphi}}{2}$ $\qquad z = e^{i\varphi}$

$\cos 2\varphi = \dfrac{z^2+z^{-2}}{2}$. $\qquad d\varphi = \dfrac{1}{iz}\,dz$

$az^2+a-(a^2+1)z =$

$\longrightarrow z = a$

$z = 1/a$

$$I = \oint_{|z|=1}\frac{z^{-2}\left[z^4+1\right]}{2\left(1-a(z+\frac{1}{z})+a^2\right)}\,\frac{1}{iz\,dz} = \frac{i}{2}\oint_{|z|=1}\frac{(z^4+1)}{z^2\left[az^2-(a^2+1)z+a\right]}\,dz.$$

NOTICE WE HAVE POLE OF ORDER 2 AT $z=0$ AND SIMPLE POLES AT $z=a$, $z=1/a$

NOW SINCE $|a|<1$ WE HAVE THAT ONLY $z=0$ AND $z=a$ LIE INSIDE $|z|=1$.

THUS $\quad I = 2\pi i \left[ \text{RES}(f;0) + \text{RES}(f;a) \right]$.

FIND $\text{RES}[f;0]$ USING LAURENT SERIES.

$$f = \frac{i}{2} \frac{(1+z^4)}{az^2\left[1+\left(z^2 - \frac{(a^2+1)}{a}z\right)\right]} \sim \frac{i}{2az^2}(1+z^4)\left[1+\left(\frac{a^2+1}{a}\right)z - z^2 + \dots\right] \quad \text{as } z\to 0.$$

$$f = \frac{(\ )}{z^2} + \frac{i}{2a^2}\frac{(a^2+1)}{z} + \dots \quad \text{AS } z\to 0. \qquad \text{RES}[f;0] = \frac{i(a^2+1)}{2a^2}.$$

NOW $\quad \text{RES}[f;a] = \dfrac{\frac{i}{2}(a^4+1)}{a^2\left(2a^2-(a^2+1)\right)} = \dfrac{i}{2}\dfrac{(a^4+1)}{a^2(a^2-1)} = \dfrac{i}{2}\dfrac{(a^4+1)}{a^2(a^2-1)}$

$$I = 2\pi i\left[\frac{i(a^2+1)}{2a^2} + \frac{i}{2}\frac{(a^4+1)}{a^2(a^2-1)}\right] = -\pi\left[\frac{(a^2+1)(a^2-1)+(a^4+1)}{a^2(a^2-1)}\right]$$

$$I = -\pi\left[\frac{2a^4}{a^2(a^2-1)}\right] = \frac{-2\pi a^2}{a^2-1} = \frac{2\pi a^2}{1-a^2}.$$

$$\longrightarrow \int_0^{2\pi} \frac{\cos 2\varphi}{1-2a\cos\varphi + a^2}\, d\varphi = \frac{2\pi a^2}{1-a^2}.$$

EXAMPLE $\quad I = \displaystyle\int_0^{\pi} \sin^{2n}\varphi\, d\varphi. \longrightarrow I = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\varphi\, d\varphi.$

BY SYMMETRY, $\quad I = + \displaystyle\int_{\pi}^{2\pi} \sin^{2n}\varphi\, d\varphi. \qquad \sin\varphi = \frac{z-z^{-1}}{2i} \qquad z = e^{i\varphi},\ dz = ie^{i\varphi}d\varphi \qquad d\varphi = \frac{dz}{iz}$

HENCE $\quad I = \dfrac{1}{2}\displaystyle\oint_{|z|=1} \frac{1}{2^{2n}}\left(z-\frac{1}{z}\right)^{2n}\frac{1}{i^{2n-1}}\frac{1}{z}\,dz = 2\pi i\, \text{RES}[f;0]$

NOTICE $z=0$ IS A POLE OF ORDER $2^{n+1}$. TO FIND RESIDUE GET $a_{-1}$ TERM IN L.SERIES.

RECALL THAT, $\quad \left(z-\frac{1}{z}\right)^{2n} = z^{2n} + \binom{2n}{1}z^{2n-1}\left(-\frac{1}{z}\right) + \dots \binom{2n}{n}z^n\left(-\frac{1}{z}\right)^n + \dots + \binom{2n}{n-1}z\left(-\frac{1}{z}\right)^{2n-1} + \left(-\frac{1}{z}\right)^{2n}$

$\left(z-\frac{1}{z}\right)^{2n} = (-1)^n\binom{2n}{n}$ + other powers $z^k \quad k\neq 0.$

HENCE $\quad \text{RES}[f;0] = \dfrac{1}{2^{2n+1}}\dfrac{1}{i^{2n+1}}(-1)^n\dbinom{2n}{n} = \dfrac{1}{2^{2n+1}}\dfrac{1}{i^{2n+1}}(-1)^n\dfrac{(2n)!}{(n!)^2}.$

$$I = 2\pi i \; \frac{1}{2^{2n+1}} \; \frac{1}{i^{2n+1}} (-1)^n \frac{(2n)!}{(n!)^2}. \qquad \begin{array}{l} i^3 = -i \\ i^5 = i \end{array} \longrightarrow i^{2n+1} = (-1)^n i.$$

$$I = \frac{1}{2^{2n}} \frac{(2n!)}{(n!)^2} \pi. \qquad \longrightarrow \int_0^\pi \sin^{2n}\varphi \, d\varphi = \frac{(2n)!\,\pi}{2^{2n}(n!)^2} \qquad n = 1, 2, 3, \ldots$$

<u>APPLICATION</u> NOW CALCULATE INTEGRALS OF THE FORM

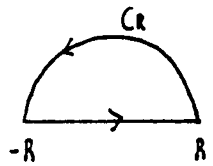
$$I = \int_{-\infty}^{\infty} \frac{P(x)\cos Bx}{Q(x)} \, dx \quad \text{OR} \quad I = \int_{-\infty}^{\infty} \frac{P(x)\sin Bx}{Q(x)} \, dx. \longrightarrow I = \mathrm{IM}\left(\int_{-\infty}^{\infty} e^{iBx} \frac{P(x)}{Q(x)} dx\right)$$

SUPPOSE $P, Q$ ARE POLYNOMIALS IN $X$.

<u>TECHNIQUE</u>: NOTICE $I = \int_{-\infty}^{\infty} \frac{P}{Q} \cos Bx \, dx = \mathrm{RE}\left[ \int_{-\infty}^{\infty} \frac{P e^{iBx}}{Q} dx \right]$

NOW CONSIDER

$$\int_C \frac{P(z) e^{iBz}}{Q(z)} dz \quad \text{WHERE}$$




$$\int_C = \int_{C_R} + \int_{-R}^{R}.$$

IF $B > 0$ $\quad |e^{iBz}| = |e^{iB(x+iy)}| = |e^{+iBx} e^{-By}| = e^{-By} \longrightarrow 0$ AS $y \to \infty$ IF $B > 0$.

HENCE $|e^{iBz}| \to 0$ AS $|z| \to \infty$ IN UPPER $1/2$ PLANE. This will ensure that

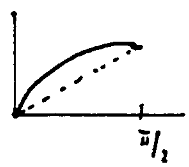
$$\left| \int_{C_R} e^{iBz} \frac{P(z)}{Q(z)} dz \right| \longrightarrow 0 \quad \text{AS} \quad R \to \infty \quad \text{IN UPPER } \tfrac{1}{2} \text{ PLANE.}$$

<u>JORDAN'S LEMMA</u> SHOW $\left| \int_{\substack{|z| = R \\ \mathrm{IM}(z) > 0}} e^{iz} dz \right| \leq \pi (1 - e^{-R}) < \pi$ FOR ALL $R$.

<u>PROOF</u>: $\left| \int_0^\pi i e^{iz} R e^{i\varphi} d\varphi \right| \leq R \int_0^\pi |e^{iz}| \, d\varphi = R \int_0^\pi |e^{i(R\cos\varphi + iR\sin\varphi)}| \, d\varphi = R \int_0^\pi e^{-R\sin\varphi} d\varphi$

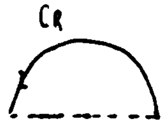
NOW $|I| \leq R \int_0^\pi e^{-R\sin\varphi} d\varphi = 2R \int_0^{\pi/2} e^{-R\sin\varphi} d\varphi$ ON $[0, \pi/2]$; $\sin\varphi > 2\varphi/\pi$.

HENCE $|I| \leq 2R \int_0^{\pi/2} e^{-2R\varphi/\pi} d\varphi = 2R \frac{\pi}{2R} \left[ -e^{-2R\varphi/\pi} \right] \Big|_{\varphi = 0}^{\pi/2}$




$$|I| \leq \pi \left( 1 - e^{-R} \right) < \pi \quad \text{FOR ALL} \quad R > 0.$$

**REMARK** CONSIDER $I = \int_{C_R} \frac{P(z)}{Q(z)} e^{iz}\, dz$




$C_R: |z| = R.$

THEN WE HAVE THAT IF $\left|\frac{P(z)}{Q(z)}\right| \to 0$ AS $|z| \to \infty$ IN UPPER 1/2 PLANE then $I \to 0$ AS $R \to \infty$.

**PROOF** $|I| \le \max_{z \in C_R} \left|\frac{P}{Q}\right| \left|\int_{C_R} e^{iz}\, dz\right| \le \pi \max_{z \in C_R} |P/Q|$ BY J. LEMMA.

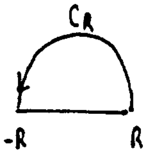
THUS $|I| \to 0$ AS $R \to \infty$ IF $\max_{z \in C_R} |P/Q| \to 0$, AS $R \to \infty$.

**REMARK** FOR $I = \int_{-\infty}^{\infty} e^{iBz} \frac{P(z)}{Q(z)}\, dz$ ENCLOSE IN UPPER 1/2 PLANE IF $B > 0$

ENCLOSE IN LOWER 1/2 PLANE IF $B < 0$.

**EXAMPLE** $I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + \alpha^2}\, dx.$ $(\alpha > 0)$

$I = \mathrm{RE}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2}\, dx\right).$ HAS POLES AT $z = \pm i\alpha$
$\alpha > 0$ $\qquad z = i\alpha$ IN upper 1/2 plane. ← simple pole.




$$\lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{ix}}{x^2+\alpha^2}\, dx + \int_{C_R} \frac{e^{iz}}{z^2+\alpha^2}\, dz\right) = 2\pi i \,\mathrm{RES}[f, i\alpha].$$

NOW $\left|\int_{C_R} \frac{e^{iz}}{z^2+\alpha^2}\, dz\right| \le \frac{K}{R^2} \to 0$ AS $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2}\, dx = 2\pi i \left[\frac{e^{iz}}{2z}\right]\Bigg|_{z = i\alpha} = \frac{\pi}{\alpha} e^{-\alpha} \qquad \alpha > 0$$

NOW TAKE REAL PART. $\rightarrow$ $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2} = \frac{\pi}{\alpha} e^{-\alpha}.$

**EXAMPLE** WE JUST SHOWED $I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2}\, dx = \frac{\pi}{\alpha} e^{-\alpha}.$

let $x = By$. THEN $dx = B\,dy$.

$$\int_{-\infty}^{\infty} \frac{B \cos By}{B^2 y^2 + \alpha^2} = \frac{\pi}{\alpha} e^{-\alpha}$$

$$\int_{-\infty}^{\infty} \frac{\cos By}{y^2+\delta^2} = \frac{\pi}{\delta} e^{-\alpha} = \frac{\pi}{\delta} e^{-B\delta} ; \ \delta = \alpha/B .$$

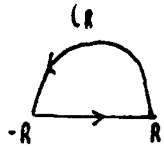
NOW DIFFERENTIATE W.R.T. $B$.

$$\int_{-\infty}^{\infty} \frac{y \sin By}{y^2+\delta^2} \, dy = \pi e^{-B\delta} .$$

NOW LET $\delta \to 0$ $$\int_{-\infty}^{\infty} \frac{\sin By}{y} \, dy = \pi e^{-B} .$$

let $B = 1 \longrightarrow \displaystyle\int_{-\infty}^{\infty} \frac{\sin y}{y} \, dy = \pi .$

<u>EXAMPLE</u> $$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5} \, dx$$




NOTICE $$I = \mathrm{IM}\left[ \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+2x+5} \, dx \right].$$

NOW SIMPLE POLES AT $z = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ $z = -1+2i$ IN UPPER $1/2$ PLANE

$$\lim_{R\to\infty} \left[ \int_{-R}^{R} \frac{x e^{i\pi x}}{x^2+2x+5} \, dx + \int_{C_R} \frac{z e^{i\pi z}}{z^2+2z+5} \, dz \right] = 2\pi i \, \mathrm{RES}\left[ f ; -1+2i \right] = \frac{2\pi i \, (-1+2i) e^{-i\pi - 2\pi}}{(-1+2i)2+2}$$

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+2x+5} \, dx + \lim_{R\to\infty} \int_{C_R} \frac{z e^{i\pi z}}{z^2+2z+5} \, dz = \frac{\pi}{2}(1-2i) e^{-2\pi}$$

NOW $$\lim_{z\to\infty} \left| \int_{C_R} \frac{z e^{i\pi z}}{z^2+2z+5} \, dz \right| = \frac{K}{R} \longrightarrow 0 \text{ AS } R \to \infty .$$

THUS $$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+2x+5} \, dx = \frac{\pi}{2} (1-2i) e^{-2\pi}$$

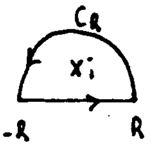
TAKE THE REAL AND IMAGINARY PARTS TO CONCLUDE,

$$\int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2+2x+5} \, dx = \frac{\pi}{2} e^{-2}$$

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5} \, dx = -\pi e^{-2\pi} .$$

**EXAMPLE** EVALUATE $I = \int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\,dx$ FOR $B>0$ AND $B<0$.

**CASE 1** $B>0$ ENCLOSE IN UPPER 1/2 PLANE SINCE $|e^{iBz}| = |e^{iBx-By}| = e^{-By} \to 0$ AS $y \to +\infty$

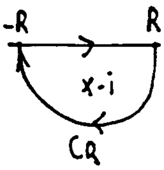



$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iBx}}{x^2+1}\,dx + \int_{C_R} \frac{e^{iBz}}{z^2+1}\,dz = 2\pi i \,\text{RES}[f;i] = 2\pi i \,\frac{e^{iBi}}{2i}$$

simple pole at $z=i$. ALSO $\left|\int_{C_R}\right| \to 0$ AS $R\to\infty$

THUS $\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\,dx = \pi e^{-B}$ FOR $B>0$.

**CASE 2** IF $B<0$ ENCLOSE IN LOWER 1/2 PLANE




notice - sign due to clockwise contour.

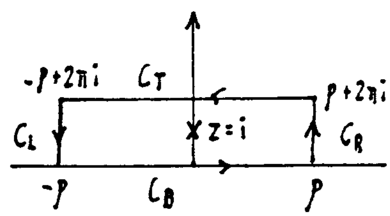
$$\lim_{R\to\infty} \left( \int_{-R}^{R} \frac{e^{iBx}}{x^2+1}\,dx + \int_{C_R} \right) = -2\pi i \,\text{RES}[f,-i]$$

NOW WE GET $|e^{iBz}| = |e^{iB(x+iy)}| = |e^{-By}| \to 0$ AS $y\to-\infty$ FOR $B<0$.

$$\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\,dx = -2\pi i \left[ \frac{e^{iBz}}{2z} \right]\Bigg|_{z=-i} = \frac{-2\pi i}{-2i} e^{+B} = \pi e^{B} \qquad B<0.$$

$$\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\,dx = \begin{cases} \pi e^{-B} & B>0 \\ \pi e^{B} & B<0. \end{cases}$$

**EXAMPLE** EVALUATE $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \qquad 0<a<1.$




INTEGRATE OVER RECTANGLE $-p<x<p$; $|y|<2\pi i$.

simple poles at $1+e^z=0 \quad z=\pm\pi i, \pm 3\pi i, \ldots$ ONLY $z=\pi i$ LIES INSIDE CONTOUR.

$$\int_{C_B} + \int_{C_T} + \int_{C_R} + \int_{C_L} = 2\pi i \,\text{RES}[f;\pi i] \qquad \text{RES}[f;\pi i] = \frac{e^{a\pi i}}{e^{\pi i}} = e^{(a-1)\pi i}.$$

NOW $$\lim_{p\to\infty} \left( \int_{-p}^{p} \frac{e^{ax}}{1+e^x}\,dx + \int_{C_R} + \int_{C_L} + \int_{2\pi i+p}^{2\pi i-p} \frac{e^{az}}{1+e^z}\,dz \right) = 2\pi i e^{(a-1)\pi i}.$$

$z = 2\pi i + x$

THUS GIVES,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx + \int_{C_R} + \int_{C_L} + e^{2\pi i a}\int_{\infty}^{-\infty} \frac{e^{ax}}{1+e^{x}}\,dx = e^{(a-1)\pi i}.$$

$$\left|\int_{C_R} \frac{e^{az}}{1+e^{z}}\,dz\right| = \left|\int_{0}^{1} \frac{e^{ap+2\pi i a y}}{1+e^{p+2\pi i y}}\right| dy = O\left(e^{(a-1)p}\right) \to 0 \text{ AS } p\to\infty \text{ IF } a<1.$$

$z = p + 2\pi i y$

SIMILARLY, WE CAN SHOW THAT $\left|\int_{C_L} \frac{e^{az}}{1+e^{z}}\,dz\right| \to 0$ AS $p\to\infty$ IF $a>0$.

THEREFORE, WE GET

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx - e^{+2\pi i a}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = 2\pi i e^{(a-1)\pi i}$$

$$(1-e^{+2\pi i a})\, I = 2\pi i\, e^{(a-1)\pi i}$$

$$(e^{-a\pi i} - e^{+a\pi i})\, I = 2i\,\pi\,[\cos(-\pi)]$$

$$I = \frac{-2i\pi}{(e^{-a\pi i} - e^{+a\pi i})} \longrightarrow I = \pi/\sin(a\pi)$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin(a\pi)}$$

<u>EXAMPLE</u> $I = \int_{-\infty}^{\infty} \frac{\sin x}{x+i}\,dx$. WE CAN'NOT TAKE $\sin x = \mathrm{Im}(e^{ix})$ AND WORK WITH $\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\,dx$.

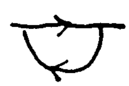
INSTEAD WE WRITE,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

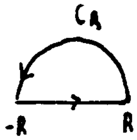
$$I = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\,dx$$


ENCLOSE IN UPPER ½ PLANE

ENCLOSED IN LOWER ½ PLANE.



$$J_1 \equiv \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx .$$




$$\longrightarrow \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{ix}}{x+i}\, dx + \int_{C_R} \right) = 0 \leftarrow$$ NO. SING. POINTS IN UPPER ½ PLANE.

SINGULARITY IS AT $x=-i$ NOT IN UPPER ½ PLANE. BUT $\lim_{R\to\infty} \int_{C_R} = 0$.

THUS $$\int_{-\infty}^{\infty} e^{ix}/_{x+i}\, dx = 0 .$$

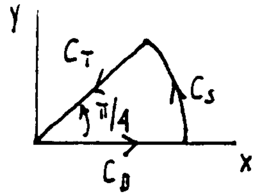
NOW $$J_2 = \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx \longrightarrow J_2 = -2\pi i \operatorname{RES}[f_1, -i] = -2\pi i e^{-1}$$


$$\left( \lim_{R\to\infty} \int_{-R}^{R} + \int_{C_R} = -2\pi i \operatorname{RES}[f, -i] = -2\pi i e^{-1} \right).$$

HENCE $$I = +\frac{1}{2i}\left(0 + 2\pi i e^{-1}\right) = \pi e^{-1}.$$

<u>EXAMPLE</u> SHOW $$\int_0^{\infty} \sin(x^2)\,dx = \int_0^{\infty} \cos(x^2)\,dx = \frac{1}{2}\left(\pi/2\right)^{1/2}.$$

CONSIDER $I = \oint_C e^{iz^2}\,dz$ WHERE




$C_S : \{ z \mid |z| = R,\ 0 < \arg z < \pi/4 \}$.

$$\int_{C_B} e^{iz^2}\,dz + \int_{C_S} e^{iz^2}\,dz + \int_{C_T} e^{iz^2}\,dz = 0$$ SINCE NO SINGULARITIES ARE INSIDE THE CONTOUR.

ON $C_B$ : $z = x$

ON $C_T$ : $z = \rho e^{i\pi/4}$ $\quad 0 < \rho < R.$ $\quad z^2 = i\rho$

$$(*) \quad \int_0^R e^{ix^2}\,dx + \int_{C_S} e^{iz^2}\,dz + \int_R^0 e^{-\rho^2} e^{i\pi/4}\,d\rho = 0$$

NOW $$\left|\int_{C_S} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2[\cos 2\varphi + i\sin 2\varphi]}\, iRe^{i\varphi}\,d\varphi\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\varphi}\,d\varphi = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\varphi}\,d\varphi$$

$z = Re^{i\varphi}$ $$\left|\int_{C_S} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\varphi/\pi}\,d\varphi = \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \to 0 \text{ as } R\to\infty.$$

THUS LET $R\to\infty$ IN $(*)$ TO GET
$$\int_0^{\infty} e^{ix^2}\,dx + \int_{\infty}^{0} e^{i\pi/4} e^{-x^2}\,dx = 0$$

$$\int_0^{\infty} e^{ix^2}\,dx = e^{i\pi/4}\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}e^{i\pi/4}$$

$$\int_0^{\infty} \cos x^2 = \frac{\sqrt{\pi}}{2}\cos(\pi/4)$$
$$\int_0^{\infty} \sin x^2 = \sqrt{\pi}\,\frac{\sin(\pi/4)}{2}$$