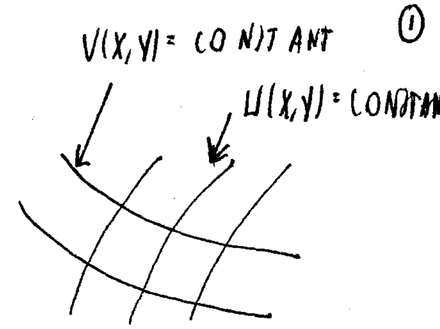
CONFORMAL MAPPING



LET $W = f(z)$ WITH $Z = X + iy$ AND $W = U + iV$

where $f(z)$ IS ANALYTIC.

NOW $U = U(X,Y)$, $V = V(X,Y)$.

THE CAUCHY-RIEMANN (CR) EQUATIONS ARE

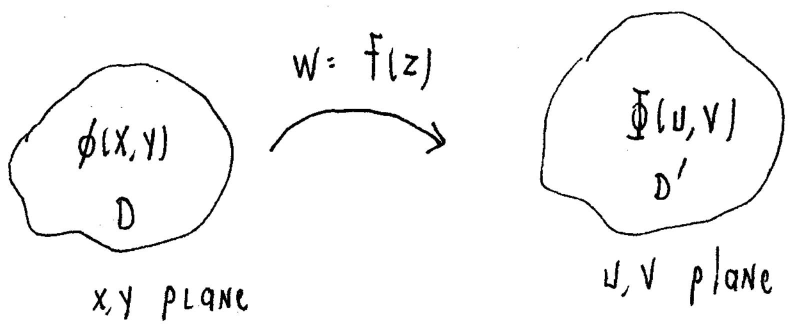
$$\text{CR}\ \left\}\begin{array}{l} U_X = V_y \\ U_y = -V_X \end{array}\right. \Longrightarrow \begin{array}{l} U_{XX} + U_{yy} = 0 \\ V_{XX} + V_{yy} = 0 \end{array} \qquad \text{ALSO}\quad \nabla U \cdot \nabla V = 0$$

SO THAT THE LEVEL CURVES $U(X,Y) = \text{CONSTANT}$ AND $V(X,Y) = \text{CONSTANT}$ ARE ORTHOGONAL

NOW WE CALCULATE $f'(z) = U_X + iV_X = U_X - iU_y$

SO THAT $|f'(z)|^2 = U_X^2 + U_y^2 = V_X^2 + V_y^2$.

MAIN RESULT 1



$Z = X + iy$

$W = U + iV$

SUPPOSE THAT $W = f(z)$ MAPS D INTO D'. LET $\Phi(U,V) = \Phi(U(X,Y), V(X,Y))$

$= \phi(X,Y)$

WE CLAIM THAT $\phi_{XX} + \phi_{yy} = |f'(z)|^2 (\Phi_{UU} + \Phi_{VV})$.

WE CALCULATE

$$\phi_X = \Phi_U U_X + \Phi_V V_X$$

$$\phi_{XX} = U_{XX}\Phi_U + U_X(\Phi_{UU} U_X + \Phi_{UV} V_X) + V_{XX}\Phi_V + V_X(\Phi_{VV} V_X + \Phi_{UV} U_X)$$

THEREFORE,

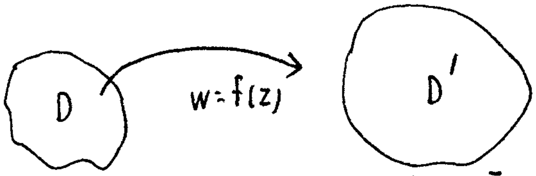
$$\phi_{XX} = U_X^2 \Phi_{UU} + V_X^2 \Phi_{VV} + 2\Phi_{UV} U_X V_X + U_{XX}\Phi_U + V_{XX}\Phi_V$$

$$\phi_{yy} = U_y^2 \Phi_{UU} + V_y^2 \Phi_{VV} + 2\Phi_{UV} U_y V_y + U_{yy}\Phi_U + V_{yy}\Phi_V$$

THUS $\phi_{XX} + \phi_{yy} = (U_X^2 + U_y^2)\Phi_{UU} + (V_X^2 + V_y^2)\Phi_{VV} + 2\Phi_{UV}\nabla U \cdot \nabla V + \Phi_U(U_{XX} + U_{yy}) + \Phi_V(V_{XX} + V_{yy})$

THEN SINCE $\nabla U \cdot \nabla V = 0$, $\Delta U = \Delta V = 0$ WE OBTAIN $\phi_{XX} + \phi_{yy} = |f'(z)|^2 (\Phi_{UU} + \Phi_{VV})$.

THEREFORE IF $f'(z) \neq 0$ INSIDE D WE HAVE $\phi_{xx} + \phi_{yy} = 0 \Longleftrightarrow \Phi_{uu} + \Phi_{vv} = 0$

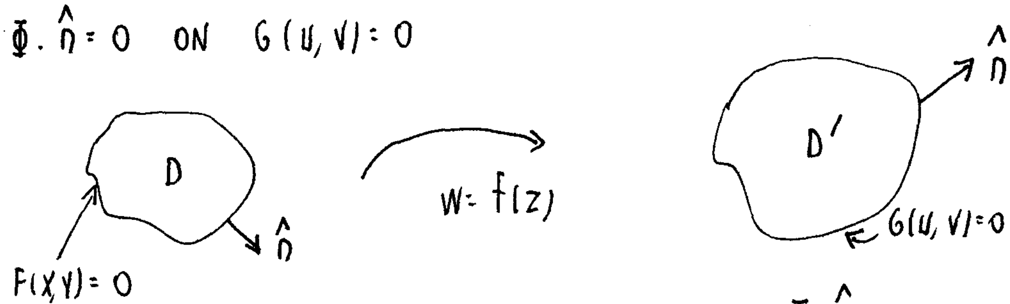


$\phi_{xx} + \phi_{yy} = 0$ IN D $\qquad \Phi_{uu} + \Phi_{vv} = 0$ IN D'.

IT MAY BE THAT $f'(z) = 0$ ON THE BOUNDARY OF D.

<u>MAIN RESULT 2</u>

CONSIDER A LEVEL CURVE $F(x,y) = 0$ UPON WHICH $\nabla\phi \cdot \hat{n} = 0$. THEN UNDER $w = f(z)$ THE LEVEL CURVE MAPS TO $G(u,v) = 0$. WE WILL SHOW THAT $\nabla\Phi \cdot \hat{n} = 0$ ON $G(u,v) = 0$



$\Phi(u,v) = \phi(x(u,v), y(u,v))$

IF $\nabla\phi \cdot \hat{n} = 0$ ON $F = 0 \longleftrightarrow \nabla\Phi \cdot \hat{n} = 0$ ON $G = 0$

CONSIDER THE MAP $w = f(z) \longrightarrow w = u + iv$ SO $u = u(x,y)$, $v = v(x,y)$.

SUPPOSE $f(z)$ IS ANALYTIC SO $u_x = v_y$, $u_y = -v_x$.

THEN
$$\phi_x = \Phi_u u_x + \Phi_v v_x$$
$$\phi_y = \Phi_u u_y + \Phi_v v_y$$
SO $\begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} = M \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix}$ $\qquad M = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$

LET $\nabla\Phi \equiv (\Phi_u, \Phi_v)^T$, $\nabla G = (G_u, G_v)^T$. THEN
$$\nabla\phi = M\nabla\Phi, \qquad \nabla F = M\nabla G.$$

RECALL $\underline{a} \cdot \underline{b} = a^T b$
$(\underline{a} \cdot \underline{a})^{1/2} = (a^T a)^{1/2} = |a|$

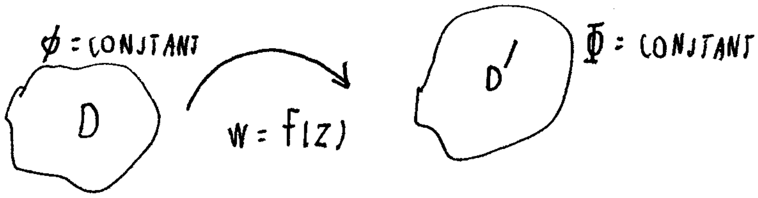
NOW (*)
$$\frac{\partial\phi}{\partial n} = \nabla\phi \cdot \frac{\nabla F}{|\nabla F|} = \frac{M\nabla\Phi \cdot (M\nabla G)}{|M\nabla G|} = \frac{(\nabla\Phi)^T M^T M \nabla G}{[(M\nabla G)^T (M\nabla G)]^{1/2}} = \frac{(\nabla\Phi)^T M^T M \nabla G}{((\nabla G)^T M^T M \nabla G)^{1/2}}$$

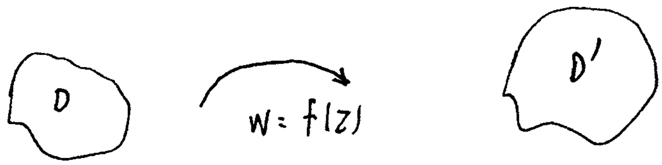
NOW $M = \begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix}$ SO $M^T M = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix} = \begin{pmatrix} u_x^2 + u_y^2 & 0 \\ 0 & u_x^2 + u_y^2 \end{pmatrix} I$.

SO $M^T M = |f'(z)|^2 I$. SUBSTITUTING INTO (*) WE OBTAIN
$$\frac{\partial\phi}{\partial n} = \nabla\phi \cdot \frac{\nabla F}{|\nabla F|} = |f'(z)| \nabla\Phi \cdot \frac{\nabla G}{|\nabla G|} = \frac{\partial\Phi}{\partial n}.$$

THIS RESULT PROVES THAT IF $\dfrac{\partial \phi}{\partial n} = 0$ ON THE BOUNDARY OF D THEN $\partial \Phi / \partial n = 0$ ON THE BOUNDARY OF D', PROVIDED THAT $|f'(z)| \neq 0$ ON THE BOUNDARY OF D.

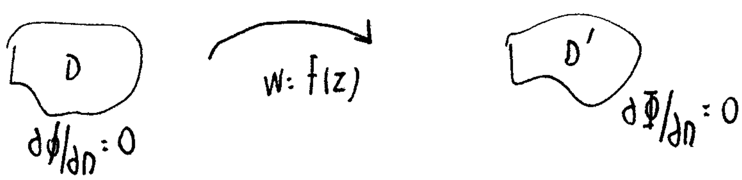
THEREFORE, WE HAVE





$$\phi_{xx} + \phi_{yy} = |f'(z)|^2 (\Phi_{uu} + \Phi_{vv})$$

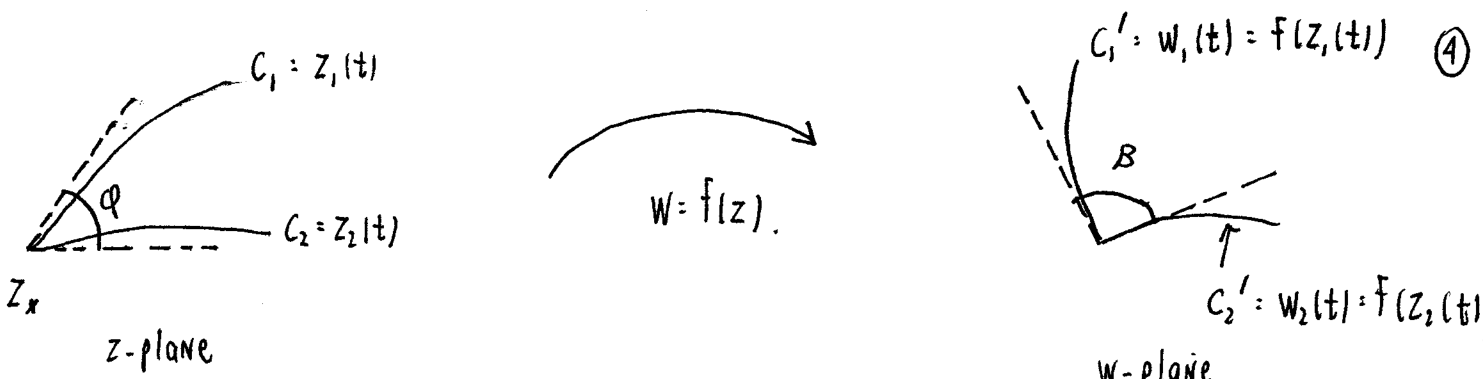
IF $\phi_{xx} + \phi_{yy} = 0$ IN D AND $|f'(z)| \neq 0$ IN D THEN $\Phi_{uu} + \Phi_{vv} = 0$ IN D'



IF $f'(z) \neq 0$ ON THE BOUNDARY OF D THEN $\partial \phi/\partial n = 0$ ON $\partial D \Leftrightarrow \dfrac{\partial \Phi}{\partial n} = 0$ ON $\partial D'$

DEFINITION A MAPPING IS CALLED CONFORMAL IF FOR EACH $z \in D$, $f(z)$ IS ANALYTIC, IS NOT A GLOBAL CONSTANT, AND $f'(z) \neq 0$ AT EACH z IN D.

PROPERTY 1 A KEY PROPERTY OF CONFORMAL MAPS. SUPPOSE THAT TWO CURVES $z_1(t)$, $z_2(t)$, WHERE t IS A REAL PARAMETRIZATION, INTERSECT AT SOME $t = t_*$ WHERE $z_1 = z_2 = z_*$. THEN THE "ANGLE" BETWEEN THE TWO CURVES IS PRESERVED BY THE MAP $w = f(z)$ PROVIDED THAT $f'(z_*) \neq 0$.



slope of tangent lines are
$z_1'(t_0)$, $z_2'(t_0)$ at $t = t_0$

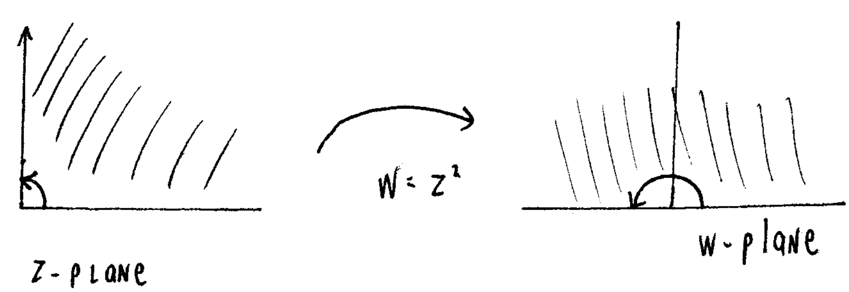
tangent lines are
$w_1'(t_0) = f'(z_1(t_0))\, z_1'(t_0)$
$w_2'(t_0) = f'(z_2(t_0))\, z_2'(t_0)$

$\arg(f'(z_*))$

THEREFORE
$$\arg(w_1'(t_0)) = \arg(f'(z_1(t_0))) + \arg(z_1'(t_0))$$
$$\arg(w_2'(t_0)) = \arg(f'(z_2(t_0))) + \arg(z_2'(t_0))$$

SUBTRACTING WE OBTAIN $\beta = \varphi$ SINCE
$$\beta = \arg(w_1'(t_0)) - \arg(w_2'(t_0)) = \arg(z_1'(t_0)) - \arg(z_2'(t_0)) = \varphi$$
∎

<u>REMARKS</u> (i) IF $f'(z_*) = 0$ THE ANGLE MAY NOT BE PRESERVED.
CONSIDER $w = f(z) = z^2$ THEN WE HAVE $f'(0) = 0$ AND



SO THAT THE ANGLE AT $z = 0$ IS NOT PRESERVED BUT IS DOUBLED.

(ii) BY TAYLOR SERIES SUPPOSE $f'(z_*) = 0$. THEN
$$w_* = f(z_*) + \frac{f''(z_*)}{2}(z - z_*)^2 + \ldots$$
SO FOR $|z - z_*|$ SMALL $\longrightarrow$ $\arg(w_* - f(z_*)) = \arg(f(z_*))$
$+ 2\arg(z - z_*) + \ldots$

EXAMPLES OF MAPPINGS

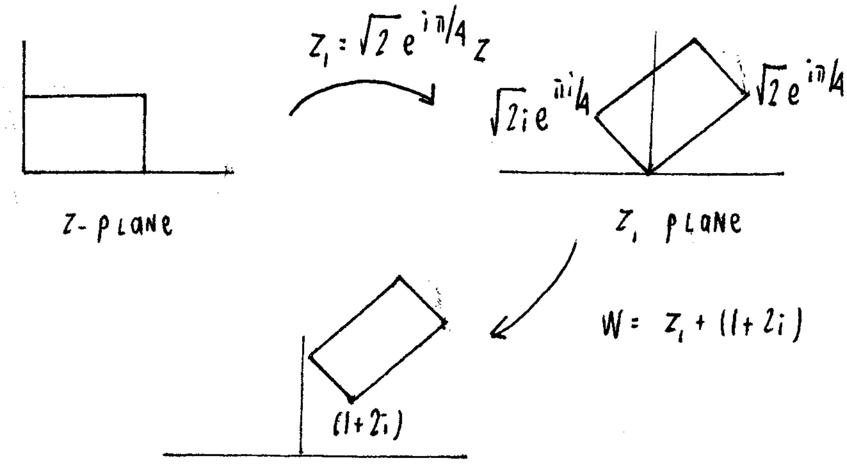
EX 1 LET $D: \{0 < x < 2,\ 0 < y < 1\}$

AND CONSIDER THE MAP $w = (1+i)z + (1+2i)$.

WE LET $1+i = \sqrt{2}\, e^{i\pi/4}$

AND WRITE $z_1 = \sqrt{2}\, e^{i\pi/4} z$ (ROTATION + DILATION)

$w = z_1 + (1+2i)$ (TRANSLATION)



EX 2 ANY MAP OF THE FORM $w = \alpha z + \beta$ WITH $\alpha = |\alpha| e^{i\varphi}$

CAN BE DECOMPOSED AS

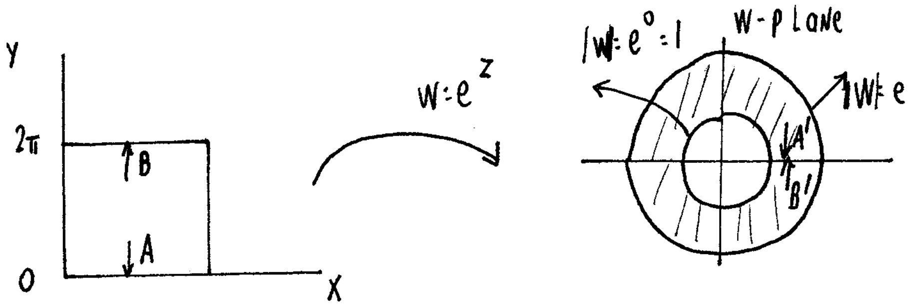
$z_1 = \alpha e^{i\varphi} z$ rotation + stretching

$w = z_1 + \beta$ translation.

EX 3 SHOW THAT $w = e^z$ TAKES $0 < x < 1,\ 0 < y < 2\pi$ ONTO INTERIOR OF THE ANNULUS (BUT WITH THE NEED FOR A BRANCH CUT)

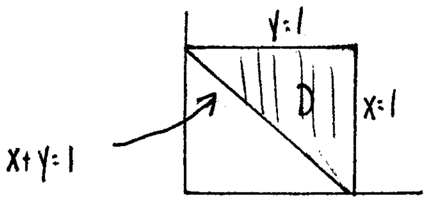
$u + iv = e^x \cos y + i e^x \sin y$ SO $u = e^x \cos y$ $v = e^x \sin y$.

$\rightarrow u^2 + v^2 = e^{2x}$

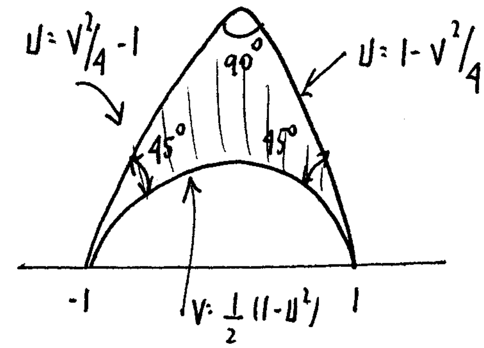


EX 4 LET D BE THE TRIANGLE AS SHOWN.

FIND ITS IMAGE UNDER THE MAP $W = Z^2$.



$W = Z^2 = f(Z)$



LET $Z = X + iY$ $W = U + iV$. SO THAT

$$U + iV = (X+iY)^2 = X^2 - Y^2 + 2iXY$$

$$\rightarrow \quad U = X^2 - Y^2, \quad V = 2XY.$$

MAPPING OF $Y = 1$

$$U = X^2 - 1, \quad V = 2X$$

SO $U = V^2/4 - 1$

MAPPING OF $X = 1$ $\rightarrow$ $U = 1 - Y^2$, $V = 2Y$

SO $U = 1 - V^2/4$

MAPPING OF $X + Y = 1$

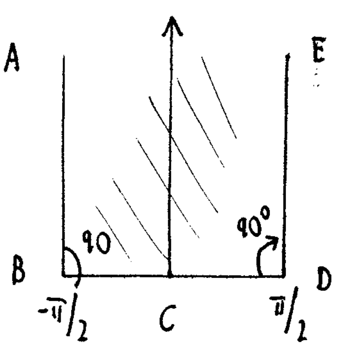
$$U = (1-Y)^2 - Y^2 = -2Y + 1 \rightarrow Y = \frac{1-U}{2}$$

$$V = 2Y(1-Y) = 2\left(\frac{1-U}{2}\right) - 2\left(\frac{1-U}{2}\right)^2$$

$$\rightarrow V = (1-U) - \frac{1}{2}(1-U)^2 = \frac{1}{2}(1-U^2)$$

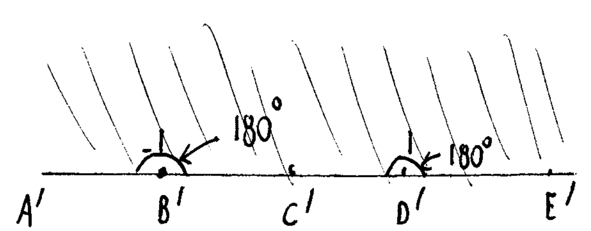
SINCE $f'(Z) \neq 0$ AT THE VERTICES THE MAP IS CONFORMAL THERE AND THE ANGLES ARE PRESERVED.

EX 5 CONSIDER THE INFINITE STRIP AS SHOWN AND THE MAP $W = \sin Z$



$W = \sin Z$

$$Z = \sin^{-1}(W) = -i\log\left(iW + \sqrt{1-W^2}\right)$$



NOW $W = \sin Z \rightarrow U + iV = \sin(X+iY) = \sin X \cosh Y + i \cos X \sinh Y$

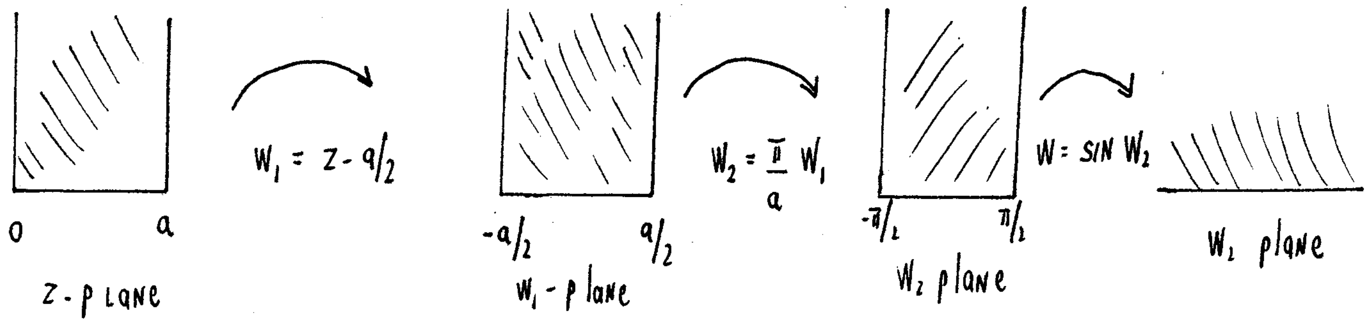
SO $U = \sin X \cosh Y$ SINCE $-\pi/2 < X < \pi/2 \rightarrow$ TOP ½ OF ELLIPSE

$V = \cos X \sinh Y$ SINCE $V \geq 0$.

$$U^2/\cosh^2 Y_0 + V^2/\sinh^2 Y_0 = 1 \rightarrow \text{ELLIPSE}$$

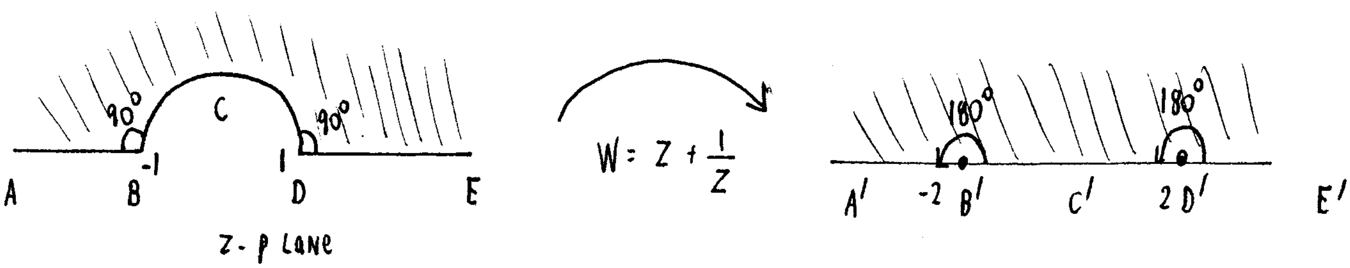
NOTICE $f'(Z) = \cos Z = 0$ AT $Z = \pm\pi/2 \rightarrow$ MAP IS NOT CONFORM THERE. $\rightarrow$ ANGLE IS DOUBLED.

EX 6 FIND A MAP TO TAKE THE S-INFINITE STRIP $0 \leq \mathrm{Re}\, z \leq a$, $\mathrm{Im}\, z \geq 0$ TO THE UPPER 1/2 OF W-PLANE



THEREFORE, $W_2 = \frac{\pi}{a}\left(z - a/2\right)$ SO THAT $W = \sin\left(\frac{\pi}{a}\left(z - a/2\right)\right)$.

EX 7 CONSIDER THE MAP $W = z + \frac{1}{z}$ OF THE REGION AS SHOWN



WE LET $W = u + iv$, $z = re^{i\varphi}$ SO THAT $u + iv = re^{i\varphi} + \frac{1}{r}e^{-i\varphi}$.

HENCE
$$u = \left(r + \frac{1}{r}\right)\cos\varphi$$
$$v = \left(r - \frac{1}{r}\right)\sin\varphi$$
FOR $r \geq 1$.

ON $r = 1$, $0 \leq \varphi \leq \pi \longrightarrow u = 2\cos\varphi$ IN $(-2, 2)$ AND $v = 0$.

THE CIRCLE $r = r_0 > 1$ MAPS TO $\frac{u^2}{[a(r)]^2} + \frac{v^2}{[b(r)]^2} = 1$ ELLIPSE

WITH
$$\left.\begin{array}{lll} a(r) = r + \frac{1}{r} & a(1) = 2, & a'(r) > 0 \text{ FOR } r > 1 \\ b(r) = r - \frac{1}{r} & b(1) = 0, & b'(r) > 0 \text{ FOR } r > 1 \end{array}\right\} \text{ ELLIPSE.}$$

NOW SINCE $0 < \varphi < \pi \longrightarrow v > 0$

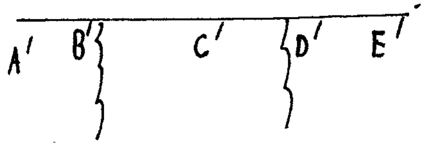
THE DERIVATIVE IS $f'(z) = 1 - \frac{1}{z^2}$ WITH $f'(\pm 1) = 0 \rightarrow$ NOT CONFORMAL AT $z = \pm 1$ (POINTS B AND D). THE ANGLE IS DOUBLED BY THE MAP.

NOW CALCULATE THE INVERSE MAP

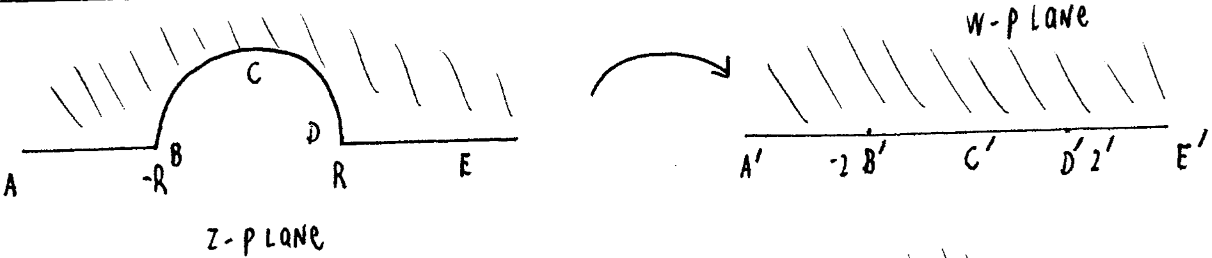
$$z^2 - zw + 1 = 0 \quad \text{so} \quad z = \frac{w + \sqrt{w^2 - 4}}{2}.$$

WE WANT $z = f^{-1}(w)$ BE ANALYTIC IN RE $w > 0$

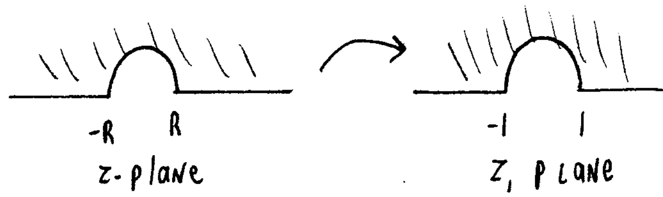
SO TAKE BRANCH CUTS AS SHOWN



EXAMPLE 8 FIND A MAP AS SHOWN



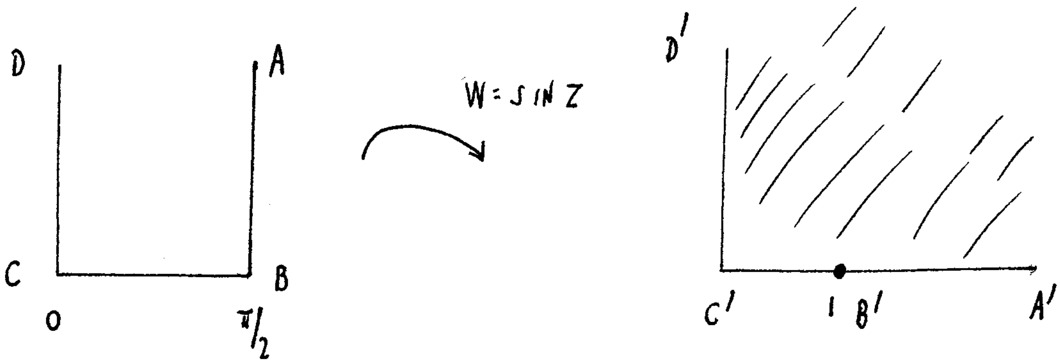
WE FIRST LET $z_1 = z/R$ TO GET



THEN $w = z_1 + \dfrac{1}{z_1}$

THIS GIVES $w = R^{-1}z + \dfrac{R}{z} = \dfrac{1}{R}\left(z + R^2/z\right).$

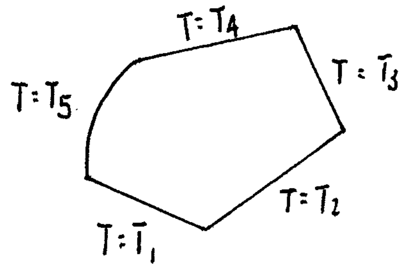
EXAMPLE 9



$f'(z) = -\cos z$ NOT CONFORMAL AT $z = \pi/2$.

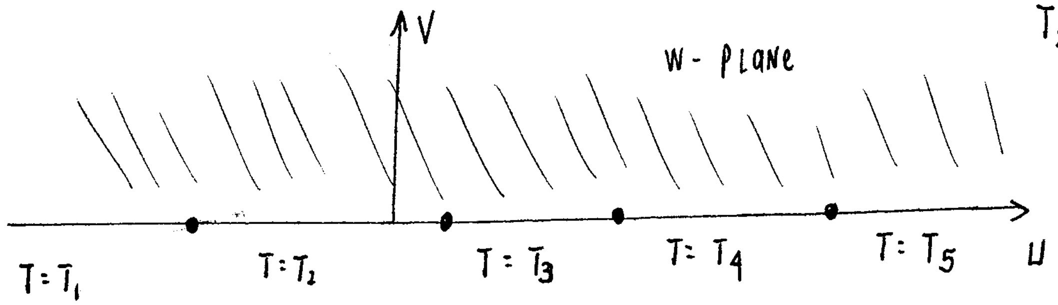
CONFORMAL MAPPING AND LAPLACE'S EQUATION

SUPPOSE WE WANT TO SOLVE $T_{xx}+T_{yy}=0$ IN GENERAL REGION SHOWN WITH $T=T_i$ ON EACH SEGMENT WITH T_i CONSTANT

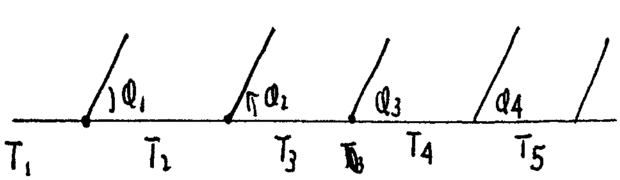


WE'D LIKE TO TRY TO MAP TO UPPER 1/2 PLANE WITH SOME MAP $W=f(z)$ TO GET

$$T_{xx}+T_{yy}=|f'(z)|^2(T_{uu}+T_{vv})$$



IF THE MAP IS CONFORMAL <u>INSIDE</u> DOMAIN THEN WE MUST SOLVE $T_{uu}+T_{vv}=0$ IN UPPER 1/2 PLANE WITH PIECEWISE CONSTANT DATA AS SHOWN. THIS IS EASY TO SOLVE BY LOOKING FOR A LINEAR COMBINATION OF SPECIAL SOLUTIONS



WE WRITE

$$T=A+B\theta_1+C\theta_2+D\theta_3+E\theta_4$$

AND WRITE A LINEAR SYSTEM FOR A, B, C, D, E.

CLEARLY $T_{uu}+T_{vv}=0$.

$\theta_1=\theta_2=\theta_3=\theta_4=0 \longrightarrow T=T_5 \longrightarrow A=T_5$

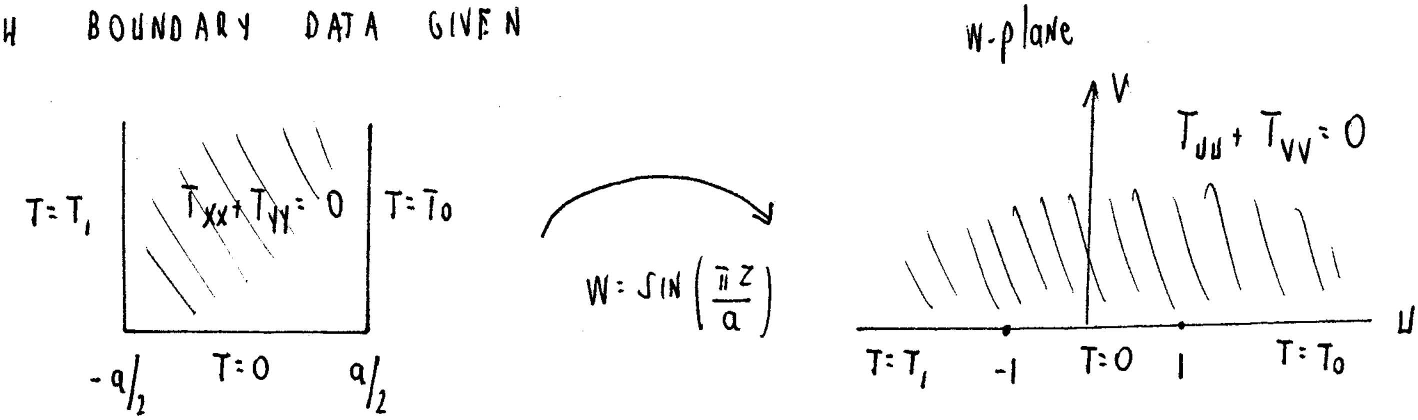
$\theta_1=\theta_2=\theta_3=0,\ \theta_4=\pi \longrightarrow T=T_4 \qquad T_4=A+E\pi \longrightarrow E=\dfrac{T_4-T_5}{\pi}$

$\theta_1=\theta_2=0,\ \theta_3=\theta_4=\pi \longrightarrow T=T_3 \qquad T_3=A+(D+E)\pi \longrightarrow D=\dots$

ETC..

EXAMPLE 1

FIND THE SOLUTION TO LAPLACE'S EQUATION IN THE REGION SHOWN WITH BOUNDARY DATA GIVEN

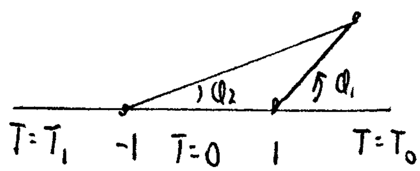


STEP 1 MAP REGION TO UPPER 1/2 PLANE.

THE REQUIRED MAP IS $W = \sin\left(\frac{\pi z}{a}\right)$ FROM EARLIER IN THE NOTES

STEP 2 SOLVE FOR T IN W-plane BY WRITING

$$T = C_0 + C_1 \theta_1 + C_2 \theta_2$$



WHEN $\theta_1 = \theta_2 = 0 \rightarrow T = T_0 \longrightarrow C_0 = T_0$

$\theta_2 = 0,\ \theta_1 = \pi \rightarrow T = 0 \longrightarrow 0 = T_0 + C_1 \pi \qquad C_1 = -T_0/\pi.$

$\theta_1 = \theta_2 = \pi \rightarrow T = T_1 \longrightarrow T_1 = T_0 + C_1\pi + C_2\pi \rightarrow C_2 = +T_1/\pi$

THEREFORE, $$T = T_0 + \frac{1}{\pi}\left(-T_0 \theta_1 + T_1 \theta_2\right)$$

$$\tan^{-1}\varsigma = \begin{cases} ATAN(\varsigma) & \varsigma > \\ \pi + ATAN\varsigma & \varsigma < \end{cases}$$

$-\frac{\pi}{2} \le ATAN\varsigma \le \frac{\pi}{2}$

STEP 3 MAP BACK TO X AND Y:

THEN $\theta_1 = \tan^{-1}\left(V/_{U-1}\right)$, $\theta_2 = \tan^{-1}\left(V/_{U+1}\right)$

WE HAVE $$T = T_0 + \frac{1}{\pi}\left(T_1 \tan^{-1}\left(\frac{V}{U+1}\right) - T_0 \tan^{-1}\left(\frac{V}{U-1}\right)\right).$$

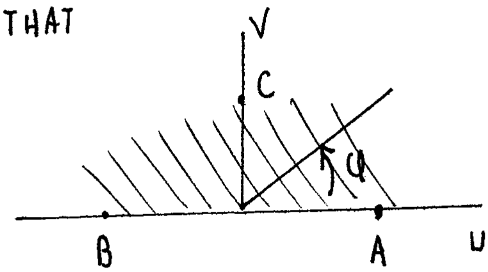
AND $$U + iV = \sin\left(\frac{\pi z}{a}\right) = \sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right) + i\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)$$

SO $U = \sin(\pi x/a)\cosh(\pi y/a)$, $V = \cos(\pi x/a)\sinh(\pi y/a)$

IMPORTANT REMARK

CONSIDER THE PLOT BELOW WHERE WE WANT $0 \le \varphi \le \pi$

SO THAT



WE CAN'T TAKE

$$\varphi = \text{ATAN}\left(\frac{V}{U}\right)$$

SINCE ATAN ζ IS IN $-\pi/2 \le \text{ATAN}\,\zeta \le \pi/2$.

THEREFORE WE LABEL $\varphi = \tan^{-1}\left(\frac{V}{U}\right)$

WHERE (*) $\tan^{-1}(\zeta) = \begin{cases} \text{ATAN}\,\zeta & \zeta > 0 \\ \pi + \text{ATAN}\,\zeta & \zeta < 0 \end{cases}$

WITH $-\pi/2 \le \text{ATAN}\,\zeta \le \pi/2$

EX AT POINT A : $U = U_0 > 0 \quad V \to 0^+$

SO $\tan^{-1}\left(\frac{V}{U}\right) = \text{ATAN}\,0 = 0$

AT POINT B : $U = U_0 < 0 \quad V \to 0^+$

SO $\tan^{-1}\left(\frac{V}{U}\right) = \tan^{-1}(0^-) = \pi + \text{ATAN}\,0 = \pi$.

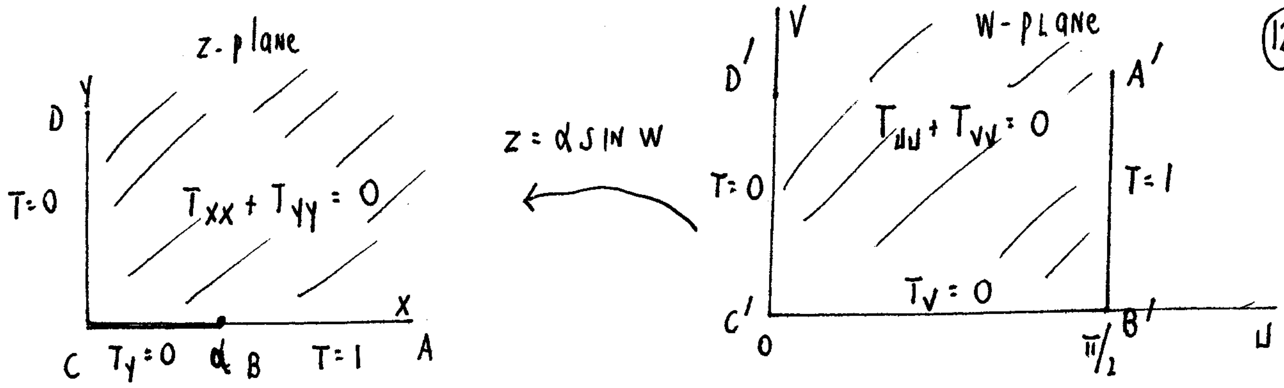
AT POINT C : $U = 0, \quad V \to 0^{\pm}$

SO $\tan^{-1}\left(\frac{V}{U}\right) = \tan^{-1}(\pm\infty) = \begin{cases} \text{ATAN}\,\infty = \pi/2 \\ \pi + \text{ATAN}(-\infty) = \pi/2 \end{cases}$

SO THE DEFINITION (*) FOR $\tan^{-1}$ IS NEEDED.

EXAMPLE 2

SOLVE



WE FIRST MAP TO A STRIP USING $z = \alpha \sin w$

NOTICE THIS IS SIMILAR TO MAP ON PAGE 8 EXAMPLE 9.

SINCE MAP IS CONFORMAL EXCEPT AT $w = \pi/2 \implies$ WE MUST SOLVE LAPLACE'S EQUATION AS SHOWN.

WE LOOK FOR $T = T(u)$. THEN $T_{uu} + T_{vv} = 0 \longrightarrow T_{uu} = 0$.

THIS GIVES $T = Au + B$ AT $u = 0 \rightarrow T = 0 \rightarrow B = 0$

$u = \frac{\pi}{2} \rightarrow T = 1 \rightarrow A = \frac{2}{\pi}$.

THEREFORE, $T = \frac{2}{\pi} u$. NOTICE THAT $T_v = 0$ ON $v = 0$ IS SATISFIED!

NOW SOLVE FOR $u = u(x,y)$ FROM $z = \alpha \sin w$

SO $x + iy = \alpha \sin(u + iv) = \alpha \sin u \cosh v + i \cos u \sinh v$.

HENCE $x = \alpha \sin u \cosh v$

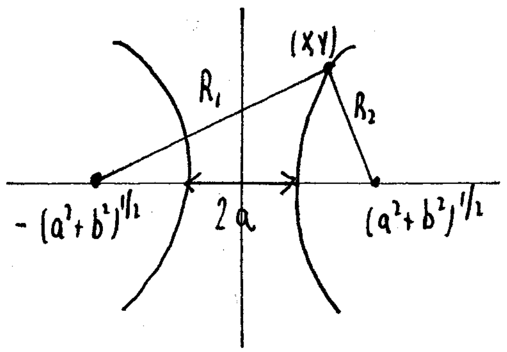
$y = \alpha \cos u \sinh v$

BUT $\cosh^2 v - \sinh^2 v = 1 \longrightarrow \dfrac{x^2}{\alpha^2 \sin^2 u} - \dfrac{y^2}{\alpha^2 \cos^2 u} = 1$

WE COULD WRITE $\cos^2 u = 1 - \sin^2 u$ AND SOLVE $u = u(x,y)$

BUT THERE IS AN EASIER WAY.

RECALL PROPERTIES OF HYPERBOLA



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$R_1 - R_2 = 2a$$

SO IF $a = \alpha \sin U$, $b = \alpha \cos U \rightarrow (a^2+b^2)^{1/2} = \alpha$.

THUS $\quad R_1 - R_2 = \alpha\, 2 \sin U$

OR $\quad \alpha\, 2 \sin U = \left[(x+\alpha)^2 + y^2 \right]^{1/2} - \left[(x-\alpha)^2 + y^2 \right]^{1/2}$

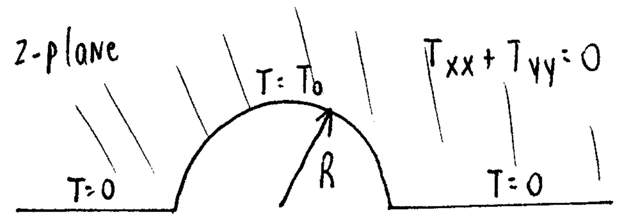
THUS $\quad T = \frac{2}{\pi} U = \frac{2}{\pi} \sin^{-1}\left[\frac{1}{2\alpha} \left(\left[(x+\alpha)^2 + y^2 \right]^{1/2} - \left[(x-\alpha)^2 + y^2 \right]^{1/2} \right) \right]$

NOW WITH $y = 0$ AND $0 < x < \alpha$

WE HAVE $\quad T(x, 0) = \frac{2}{\pi} \sin^{-1}\left[x/\alpha \right]$

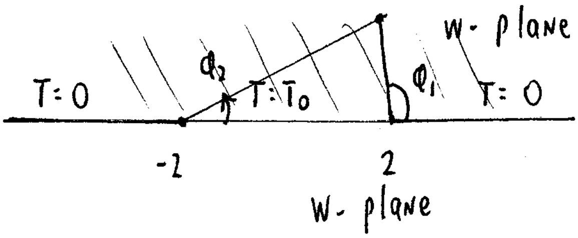
THIS VARIES FROM 0 TO 1 AS x GOES FROM 0 TO α.

<u>EXAMPLE</u> SOLVE FOR THE TEMPERATURE FIELD



WE FIRST USE MAP $\quad W = \frac{1}{R}\left(z + R^2/z \right)$

TO GET



$$T = A + B\,\Phi_1 + C\,\Phi_2$$

$T = 0 \quad \Phi_1 = \Phi_2 = 0 \rightarrow \quad A = 0$

$T = T_0 \quad \Phi_1 = \pi,\ \Phi_2 = 0 \qquad B = T_0/\pi$

$T = 0, \quad \Phi_1 = \Phi_2 = \pi \qquad C = -T_0/\pi$

HENCE $\quad T = \frac{T_0}{\pi}\left(\Phi_1 - \Phi_2 \right)$

NOW IF $Z = Re^{i\varphi}$

$$W = U + iV = \frac{1}{R}\left(Re^{i\varphi} + Re^{-i\varphi}\right) \longrightarrow W = 2\cos\varphi \in (-2, 2)$$

NOW IF $Z = \rho e^{i\varphi}$ $\rho \geq R$

WE GET

$$(*) \quad \left\{ \begin{aligned} U &= \frac{1}{R}\left(\rho + R^2/\rho\right)\cos\varphi \\ V &= \frac{1}{R}\left(\rho - \frac{R^2}{\rho}\right)\sin\varphi \end{aligned} \right.$$

HOWEVER $$T = \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{V}{U-2}\right) - \tan^{-1}\left(\frac{V}{U+2}\right)\right]$$

WHERE V, U WRITTEN IN TERMS OF $\rho = (x^2 + y^2)^{1/2}$ AND φ

IN $(*)$