BILINEAR, OR MOBIUS MAPPINGS

A BILINEAR, OR MOBIUS, TRANSFORMATION HAS THE FORM

$$w = f(z) = \frac{az+b}{cz+d} \quad \text{FOR SOME COMPLEX } a, b, c, d. \quad \frac{a}{b} \neq \frac{c}{d}$$

NOW THIS MAP HAS SOME KEY PROPERTIES

(i) IT IS THE COMPOSITION OF SEVERAL SIMPLE TRANSFORMATIONS: TRANSLATION, LINEAR, INVERSION.

WE CAN WRITE $f(z)$ AS

$$w = f(z) = \frac{a}{c} + \frac{(b - ad/c)}{cz+d}$$

HENCE THE COMPOSITION IS AS FOLLOWS:

$$* \left\{ \begin{array}{ll} z_1 = cz + d & \text{LINEAR} \\ z_2 = 1/z_1 & \text{INVERSION} \\ w = \dfrac{a}{c} + \dfrac{(bc - ad)}{c} z_2 & \text{LINEAR} \end{array} \right.$$

(ii) THE <u>POLE</u> OF THE MAP IS AT $z = -d/c$. WE CALCULATE

$$f'(z) = \frac{ad - bc}{(cz+d)^2} \neq 0 \quad \text{SINCE } ad \neq bc.$$

THUS THE MAP IS CONFORMAL EVERYWHERE EXCEPT AT THE POLE

(iii) THE INVERSE MAP IS FOUND BY SOLVING $z = f^{-1}(w)$.

WE OBTAIN $w(cz+d) = az+b \qquad z(cw-a) = b - dw$

OR $z = f^{-1}(w) = \dfrac{-dw + b}{cw - a}$

$f(\infty) = a/c$

$f(-d/c) = \infty$

THUS A BILINEAR MAP IS A 1-1 MAPPING OF EXTENDED PLANE TO ITSEL[F]

(iv) KEY POINT A BILINEAR TRANSFORMATION MAPS THE CLASS OF CIRCLES AND LINES TO CIRCLES AND LINES

z-plane		w-plane
line through pole of map	$\rightarrow$	line through pole of map
line not through pole	$\rightarrow$	circle +
circle through pole of map	$\rightarrow$	line

PROOF A BILINEAR MAP IS COMPOSED OF THE BASIC STEPS (*)

EACH LINEAR MAP $z_1 = cz + d$, $w = \frac{a}{c} + \frac{(bc - ad)}{c} z_2$ MAPS CLASS OF CIRCLES AND LINES TO ITSELF, THUS WE NEED ONLY LOOK AT PROPERTIES OF INVERSION MAP.

CONSIDER $W = \frac{1}{Z}$ (WITH pole at $z = 0$).

WE LET $W = u + iv$, $z = x + iy$

$$u + iv = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2} \Rightarrow u = \frac{x}{x^2+y^2} \quad v = \frac{-y}{x^2+y^2}$$

SO IN ADDITION WE CAN WRITE $z = 1/W$

$$x + iy = \frac{1}{u+iv} = \frac{u - iv}{u^2 + v^2} \Rightarrow x = \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2}$$

(i) CONSIDER THE LINE $Ax + By = C$

$$\rightarrow \frac{Au}{u^2+v^2} - \frac{Bv}{u^2+v^2} = C$$

IF $C \neq 0$ (NOT THROUGH pole of MAP) $\Rightarrow u^2 + v^2 = \frac{Au}{C} - \frac{Bv}{C} \Rightarrow$ CIRCLE THROUGH ORIGIN

IF $C = 0$ (LINE THROUGH POLE) $\Rightarrow Au = Bv \Rightarrow$ LINE THROUGH ORIGIN

(ii) NOW CONSIDER THE CIRCLE

$$x^2 + y^2 + Ax + By = C$$

THEN

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{Au}{u^2+v^2} - \frac{Bv}{u^2+v^2} = C$$

SO THAT

$$1 + Au - Bv = C(u^2+v^2)$$

IF $C \neq 0$ (CIRCLE NOT THROUGH POLE) $\Rightarrow$ CIRCLE NOT THROUGH POLE

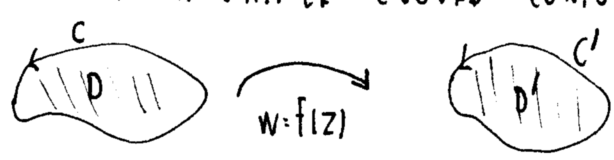
IF $C = 0$ (CIRCLE THROUGH POLE) $\Rightarrow$ LINE THROUGH POLE.

THEREFORE, SINCE INVERSION MAP $W = 1/z$ MAPS CIRCLES AND LINES TO CIRCLES AND LINES, AND SINCE A BILINEAR MAP IS A COMPOSITION OF ELEMENTARY MAPS OF THIS SAME PROPERTY IN (X), THEN WE ARE DONE: BILINEAR MAPS LEAVE CLASS OF CIRCLES AND LINES <u>INVARIANT</u>.

<u>REMARK</u> (i) IF A CIRCLE OR LINE PASSES THROUGH THE POLE OF THE MAP $w = \frac{az+b}{cz+d}$, THEN THE IMAGE MUST BE UNBOUNDED, AND HENCE A LINE.

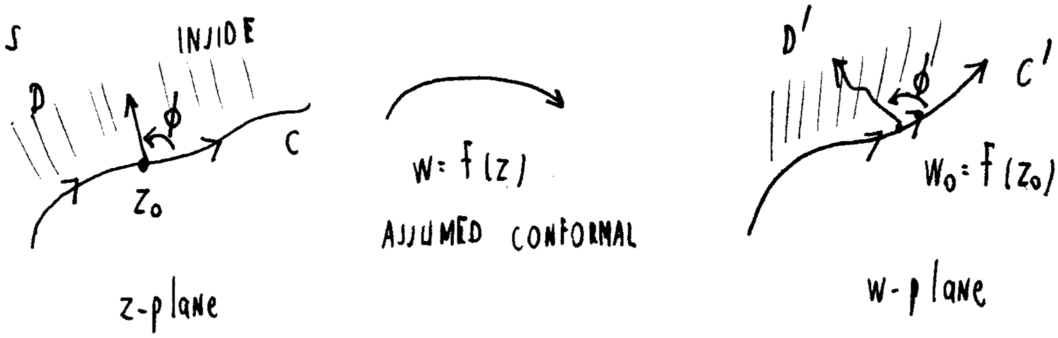
(ii) CAN THINK OF A LINE AS BEING CIRCLE OF INFINITE RADIUS.

<u>THEOREM 1</u> LET C BE A SIMPLE CLOSED CONTOUR ENCLOSING A DOMAIN D AND LET $f(z)$ BE ANALYTIC ON C AND IN D. THEN $w = f(z)$ TAKES C ENCLOSING D TO A SIMPLE CLOSED CONTOUR C' ENCLOSING A REGION D'



AN IMPORTANT PROPERTY IS THE PRESERVATION OF ORIENTATION

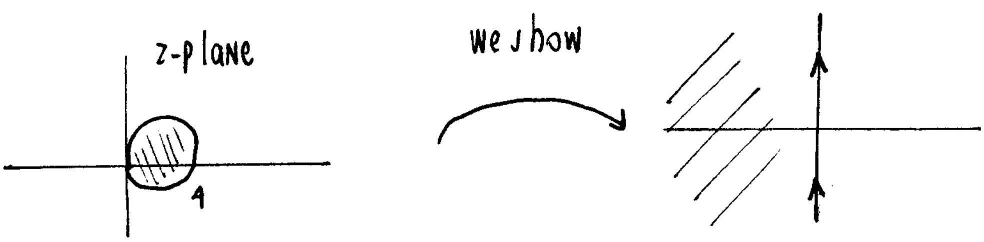
EXPRESSED AS



THUS IF IN z-plane, D IS TO MY LEFT AS WE TRAVERSE A CURVE C ORIENTED AS SHOWN, THE IMAGE DOMAIN IS ALSO ON MY LEFT ALONG THE IMAGE C' UNDER THE CONFORMAL MAP $w = f(z)$.

EXAMPLE 1 FIND THE IMAGE OF $\cancel{w = \frac{z}{2z-8}}$

$C: |z-2| \leq 2$ UNDER THE MAP $w = \dfrac{z}{2z-8}$.



SINCE $z = 4$ IS A pole OF the map AND IT IS ON C THEN THE IMAGE OF C MUST BE A LINE. THAT LINE MUST GO THROUGH THE ORIGIN SINCE $z = 0$ MAPS TO $w = 0$. HENCE IMAGE MUST BE LINE THROUGH ORIGIN.

NOW TAKE ANY POINT ON C SUCH AS $z = 2 + 2i$.

WE CALCULATE $$W = \frac{2+2i}{-4+4i} = \frac{2\sqrt{2}\, e^{\pi i/4}}{4\sqrt{2}\, e^{3\pi i/4}} = \frac{1}{2} e^{-i\pi/2} = \frac{-i}{2}.$$

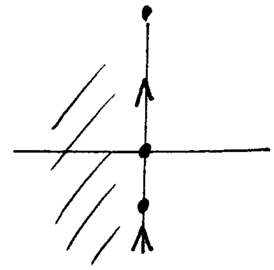
HENCE WE MUST HAVE THAT IMAGE OF C IS IMAGINARY AXIS

ORIENTATION THE POINTS GENERATE AN ORIENTATION

$Z = 4 \longrightarrow W = \infty$ (pole)

$Z = 2+2i \longrightarrow W = \frac{-i}{2}$

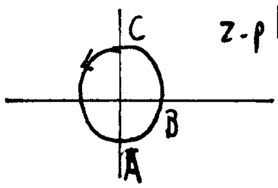
$Z = 0 \longrightarrow W = 0$



PRESERVING ORIENTATION IMPLIES $|Z-2| \leq 2 \Longrightarrow \mathrm{RE}\, W \leq 0$

CHECK: $Z = 2 \longrightarrow W = \frac{2}{-4} = -\frac{1}{2}$ IN $\mathrm{RE}\, W < 0$:

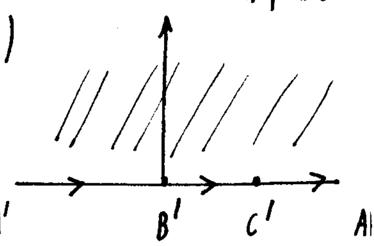
EXAMPLE 2 MAP THE UNIT CIRCLE $|Z| \leq 1$ TO UPPER HALF PLANE WITH A BILINEAR MAP



TRY $Z = -1 \longrightarrow$ POLE OF MAP (this gives a line)

$Z = 1 \rightarrow W = 0$

$Z = i \rightarrow W = 1$



PUTTING IN THE ZERO AND POLE WE GET

$$W = B\left(\frac{Z-1}{Z+1}\right) \quad \text{FOR SOME } B.$$

WE THEN SET $Z = i$ TO $W = 1$ $\qquad 1 = B\left(\frac{-1+i}{1+i}\right) = \frac{B\, e^{3\pi i/4}}{e^{\pi i/4}} = Bi$

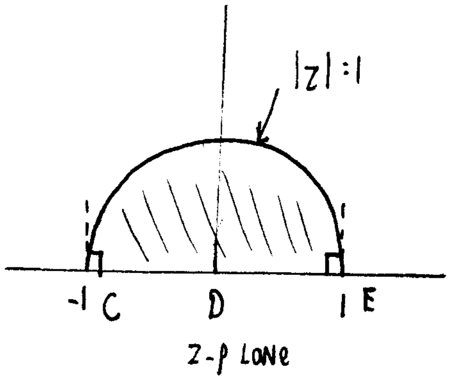
$\rightarrow B = -i$

OR EQUIVALENTLY $W = -i\left(\frac{Z-1}{Z+1}\right)$ IS SUCH A MAP

IT PRESERVES ORIENTATION SO THAT $|Z| \leq 1$ MAPPED TO $\mathrm{IM}\, W \geq 0$.

WE CHECK THAT $Z = 0 \longrightarrow W = i$ IN $\mathrm{IM}\, W > 0$.

EXAMPLE MAP THE SEMI-CIRCLE TO SOME PORTION OF THE UPPER 1/2 PLANE USING A BILINEAR MAP



WE CHOOSE

$z=-1 \rightarrow w=0$

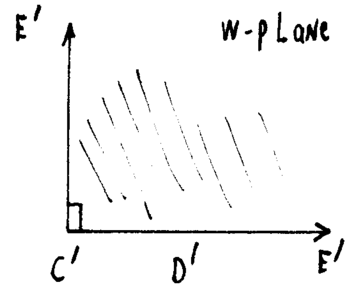
$z=1 \rightarrow w=\infty$

$\Rightarrow$ BOTH THE SEMI-CIRCLE AND ITS DIAMETER ARE MAPPED TO LINES PASSING THROUGH ORIGIN

THE MAP THEN HAS THE FORM

$$w = B\frac{(z+1)}{z-1} \quad \text{FOR SOME } B.$$

IF WE TAKE $B<0$ REAL THEN DIAMETER MAPS TO SOME PORTION OF REAL AXIS $\operatorname{Im} w = 0$, $\operatorname{Re} w \geq 0$.



MAP $z=0$ TO $w=1 \rightarrow B=-1$. $\quad w = \dfrac{1+z}{1-z}$.

IT PRESERVES ORIENTATION AND SINCE MAP IS CONFORMAL AT $z=-1$ WE MUST HAVE ANGLE PRESERVATION

$\Rightarrow$ SEMI-CIRCLE $|z|=1$, $\operatorname{Im} z \geq 0 \Rightarrow$ positive imaginary axis

THE MAP IS $\quad w = \dfrac{1+z}{1-z}$.