CONFORMAL MAPPING AND PDE'S

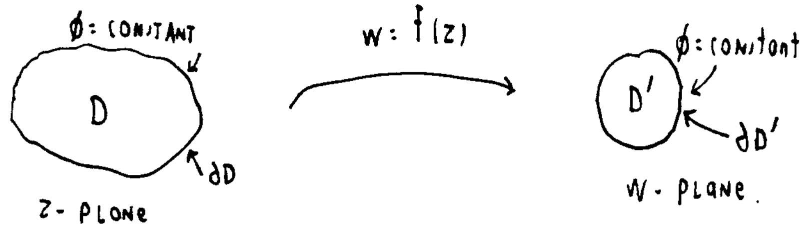
LET $w = f(z)$ BE A CONFORMAL MAP. THEN IF $\phi = \phi(x,y)$ THE TRANSFORMATION PRODUCES

$$(*) \qquad \phi_{xx} + \phi_{yy} = |f'(z)|^2 (\phi_{uu} + \phi_{vv}) \qquad \text{WHERE } w = u + iv.$$

ON THE RIGHT SIDE OF $(*)$ WE HAVE $\phi = \phi(u,v)$.

THUS IF $\phi_{xx} + \phi_{yy} = 0$ IN $D \longrightarrow \phi_{uu} + \phi_{vv} = 0$ IN IMAGE OF D UNDER $f(z)$.

NOTE: AS FOR BOUNDARY CONDITIONS WE HAVE THAT IF $\phi =$ CONSTANT ON ∂D THEN $\phi =$ CONSTANT ON IMAGE OF ∂D UNDER $f(z)$.



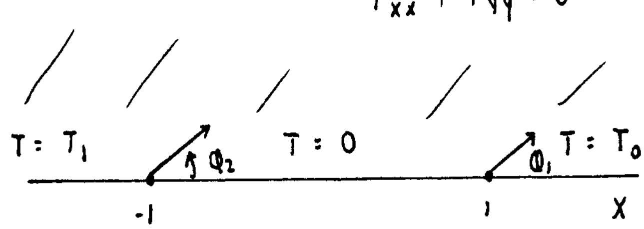
ALSO IF $\nabla\phi \cdot \hat{n} = 0$ ON ∂D THEN IN THE W-PLANE WE ALSO GET $\nabla\phi \cdot \hat{n} = 0$ ON $\partial D'$

I WILL LIST BELOW MANY PROBLEMS INVOLVING CONFORMAL MAPS AND SOLUTIONS TO LAPLACE'S EQUATION. THE IDEA WITH MAPPING IS TO MAP THE PROBLEM TO ONE WHERE WE CAN SPOT THE FORM OF THE SOLUTION ESSENTIALLY BY INSPECTION.

THE FOLLOWING PROBLEMS ARE ONES WHERE WE OBTAIN "EASY" SOLUTIONS.

EASY # 1

FIND $T(x,y)$ SUCH THAT $T_{xx} + T_{yy} = 0$ IN $y \geq 0$ WITH BOUNDARY DATA GIVEN BELOW



LET'S TRY

$$T = A + B Q_1 + C Q_2$$

CLEARLY THIS IS HARMONIC

WE GET 3 EQUATIONS FOR A, B AND C.

$Q_1 = 0, \ Q_2 = 0 \longrightarrow T = A = T_0$

$Q_1 = \pi, \ Q_2 = 0 \longrightarrow T = A + B\pi = 0 \qquad B = -T_0/\pi$

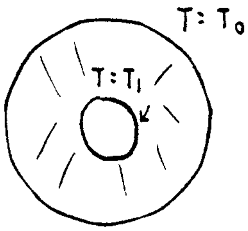
$Q_1 = Q_2 = \pi \longrightarrow T = A + \pi(B + C) = T_1 \qquad C = T_1/\pi$

$$\Longrightarrow \quad T = T_0 - \frac{T_0 Q_1}{\pi} + \frac{T_1 Q_2}{\pi}$$

WITH $Q_1 = \tan^{-1}\left(y/_{x-1}\right)$

$Q_2 = \tan^{-1}\left(y/_{x+1}\right)$

EASY # 2 (CONCENTRIC CIRCLES)



$$T_{xx} + T_{yy} = 0 \quad \text{IN} \quad r_\phi < r < r_0$$

$$T = T_1 \quad \text{ON} \quad r = r_1 \qquad r_\phi = (x^2+y^2)^{1/2}$$

$$T = T_0 \quad \text{ON} \quad r = r_0$$

LOOK FOR $T = T(r)$

$$\longrightarrow \quad T = A + B \log r$$

$$A = T_1 - B \log r_1$$

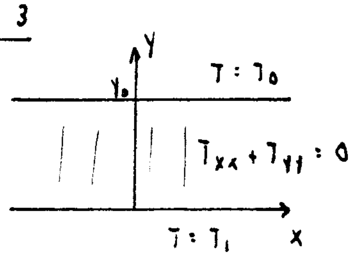
SATISFY THE BOUNDARY CONDITIONS

$$A + B \log r_1 = T_1 \qquad \longrightarrow \qquad B = \frac{T_1 - T_0}{\log(r_1) - \log r_0}$$

$$A + B \log r_0 = T_0$$

$$T = T_1 + \frac{(T_1 - T_0)}{\log(r_1/r_0)} \log(r/r_1)$$

EASY # 3



$$T = A + By \qquad\qquad T = T_1 + (T_0 - T_1)\, y/a$$

$$A + B y_0 = T_0$$

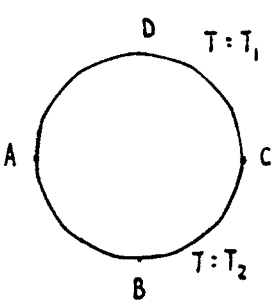
$$A = T_1$$

MAPPING PROB 1

SOLVE $T_{xx} + T_{yy} = 0$ IN CIRCLE $x^2 + y^2 \leq 1$

WITH $T = T_1$ ON upper part of circle

$T = T_2$ ON LOWER part of CIRCLE



WANT TO get upper 1/2 plane

$$w = e^{i\phi}\left(\frac{z-1}{1+z}\right)$$

TAKE $z = i$ INTO $w = 1$ THEN

$$1 = e^{i\phi}\left(\frac{i-1}{i+1}\right) = e^{i\phi}[\,i\,] \qquad \text{THUS} \quad e^{i\phi} = -i$$

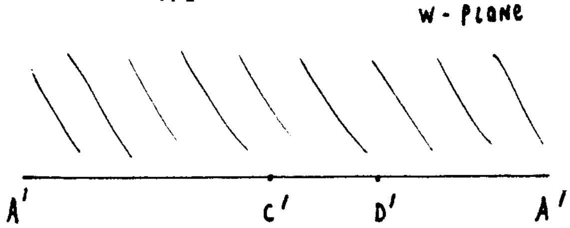
$$\longrightarrow \quad \phi = 3\pi/2.$$

CHOOSE $z = -1$ TO BE POLE

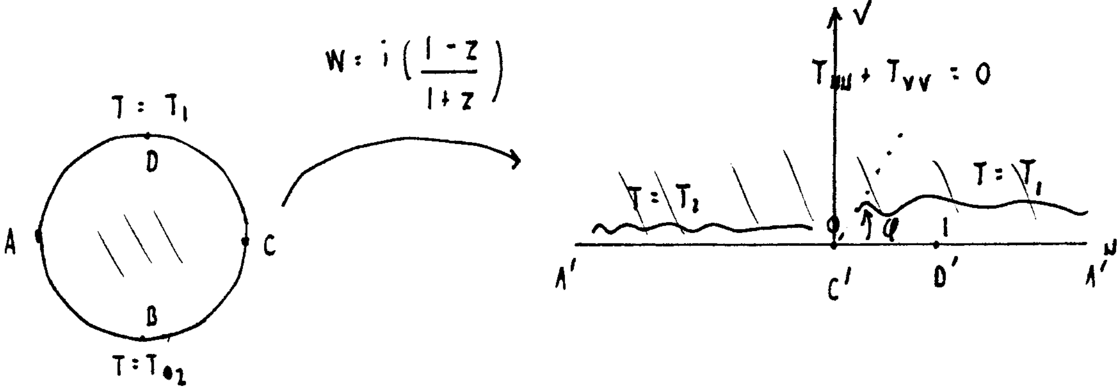
CHOOSE $z = 1$ TO BE ZERO

HENCE $w = -i\left(\frac{z-1}{1+z}\right)$ gives

W - PLANE



NOTICE AS we traverse curve IN z-plane the inside of circle is ON our left. This is preserved by the map and hence we get upper 1/2 plane



NOW LET $Q = \tan^{-1}(v/u) = \arg(w)$.

THEN
$$T = \frac{(T_2 - T_1)}{\pi} Q + T_1$$

$$T = \frac{(T_2 - T_1)}{\pi} \tan^{-1}\left(v/u\right) + T_1$$

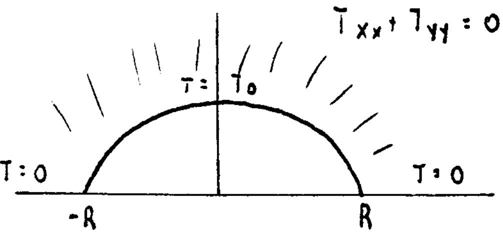
NOW
$$u + iv = i\left(\frac{1-z}{1+z}\right) = i\left[\frac{(1-x) - iy}{(1+x) + iy}\right] = \frac{i\left[(1-x) - iy\right]\left[(1+x) - iy\right]}{(1+x)^2 + y^2}$$

$$u = 2y\Big/(1+x)^2 + y^2 \qquad v = (1 - x^2 - y^2)\Big/(1+x)^2 + y^2$$

$$\tan^{-1}(v/u) = \tan^{-1}\left(\frac{1 - x^2 - y^2}{2y}\right)$$

THUS
$$T = \frac{(T_2 - T_1)}{\pi} \tan^{-1}\left(\frac{1 - x^2 - y^2}{2y}\right) + T_1$$

MAPPING PROBLEM 2



NOW RECALL THE JOUKOWSKI MAP

$$W = \frac{1}{2}\left(Z + R^2/Z\right)$$

let $z = re^{iQ}$. THEN
$$W = \frac{1}{2}\left(re^{iQ} + \frac{R^2}{r}e^{-iQ}\right)$$

$$W = \frac{1}{2}\left(r + R^2/r\right)\cos Q + \frac{i}{2}\left(r - R^2/r\right)\sin Q = u + iv$$

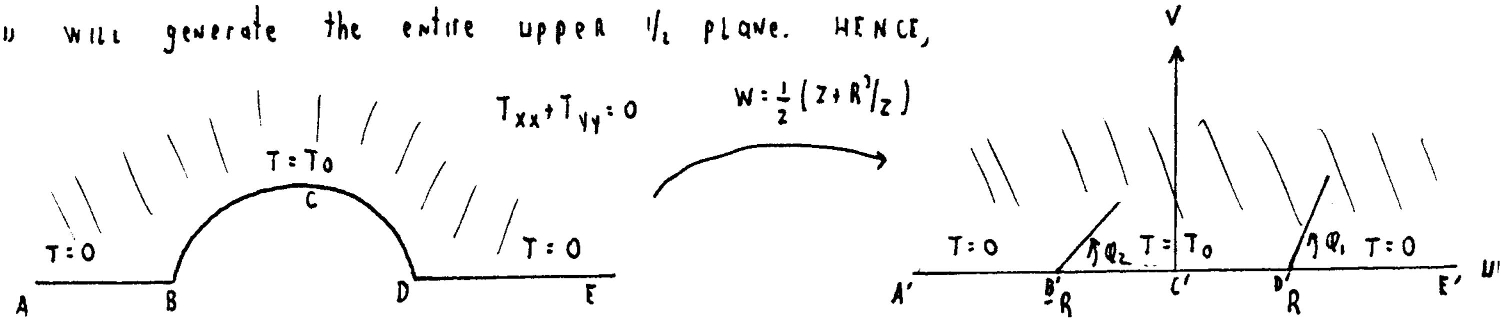
let $r = R \longrightarrow$ $u = R\cos Q$ $\quad v = 0$ $\quad$ AS Q RANGES FROM $-\pi$ TO π we get the LINE $|u| \le R$ $\quad v = 0$.

let $Q = 0$ OR $Q = \pi \longrightarrow$ $W = \pm\frac{1}{2}\left(r + R^2/r\right)$ $\quad$ For $r > R$ we get $u < -R$ OR $u > R$ WITH $v = 0$.

NOTICE FOR $r > R$ FIXED WE HAVE

$$\frac{u^2}{\frac{1}{4}\left(r + R^2/r\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - R/r\right)^2} = 1 \quad \text{ellipses}$$

THIS WILL generate the entire upper 1/2 plane. HENCE,



NOW $\quad T = C_0 + C_1 \varphi_1 + C_2 \varphi_2$

$$\varphi_1 = \varphi_2 = 0 \longrightarrow T = 0 \longrightarrow C_0 = 0$$

$$\varphi_1 = \pi,\ \varphi_2 = 0 \longrightarrow T = T_0 \longrightarrow C_1 = T_0/\pi$$

$$\varphi_1 = \varphi_2 = \pi \longrightarrow T = 0 \longrightarrow C_2 = -T_0/\pi$$

THEREFORE $\quad T = \frac{T_0}{\pi}\left(\varphi_1 - \varphi_2\right) \qquad \varphi_1 = \tan^{-1}\left(\frac{v}{u-R}\right) \qquad \varphi_2 = \tan^{-1}\left(\frac{v}{u+R}\right)$

RECALL $\quad \tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB}\right) \qquad A = v/_{u-R} \qquad B = v/_{u+R}$

THIS GIVES $\quad T = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2vR}{u^2+v^2-R^2}\right)$

NOW $\quad u = \frac{1}{2}\left(r + R^2/r\right)\cos\varphi \qquad v = \frac{1}{2}\left(r - R^2/r\right)\sin\varphi$

$$u^2 + v^2 = \frac{1}{4}r^2 + \frac{1}{4}\frac{R^4}{r^2} + \frac{R^2}{2}\left(\cos^2\varphi - \sin^2\varphi\right) = \frac{1}{4}r^2 + \frac{1}{4}\frac{R^4}{r^2} + \frac{R^2}{2}\cos 2\varphi$$

$$-R^2 + u^2 + v^2 = \frac{1}{4}\left(r - R^2/r\right)^2 + \frac{R^2}{2}\left(\cos 2\varphi - 1\right)$$

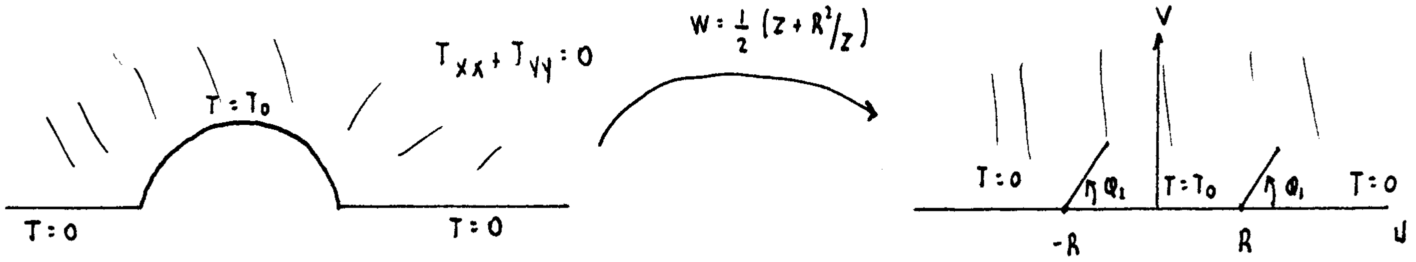
$$u^2 + v^2 - R^2 = \frac{1}{4}\left(r - R^2/r\right)^2 + R^2\sin^2\varphi$$

$$T = \frac{T_0}{\pi}\tan^{-1}\left[\frac{R\left(r - R^2/r\right)\sin\varphi}{\frac{1}{4}\left(r - R^2/r\right)^2 + R^2\sin^2\varphi}\right]$$

NOTICE FOR $r > R$ FIXED WE HAVE

$$\frac{u^2}{\frac{1}{4}\left(r + R^2/r\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - R^2/r\right)^2} = 1$$

ellipses. This gives upper 1/2 plane.



let $T = c_0 + c_1\theta_1 + c_2\theta_2$

Linear system for c_i's give

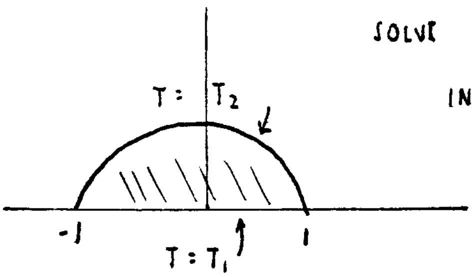
$$T = \frac{T_0}{\pi}\left(\theta_1 - \theta_2\right)$$

$$T = \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{v}{u-R}\right) - \tan^{-1}\left(\frac{v}{u+R}\right)\right)$$

WITH $u = \frac{1}{2}\left(r + R^2/r\right)\cos\varphi \qquad v = \frac{1}{2}\left(r - R^2/r\right)\sin\varphi$

USING $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$ WE CAN WRITE $T = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2vR}{u^2+v^2-R^2}\right)$

MAPPING PROBLEM 3



SOLVE $T_{xx} + T_{yy} = 0$ IN semi-CIRCLE AS SHOWN

NOW MAP -1 TO 0 AND 0 TO ∞

TRY $w = \dfrac{1+z}{1-z}$

THEN IMAGE OF THE DIAMETER OF SEMI-CIRCLE $|x| < 1$ GIVES $0 \le u \le \infty$ $v = 0$ (i.e. positive real axis). Since mapping is conformal at $z = -1$ image of semi-circle will give a line perpendicular to real axis in w-plane and this line will go through the origin.

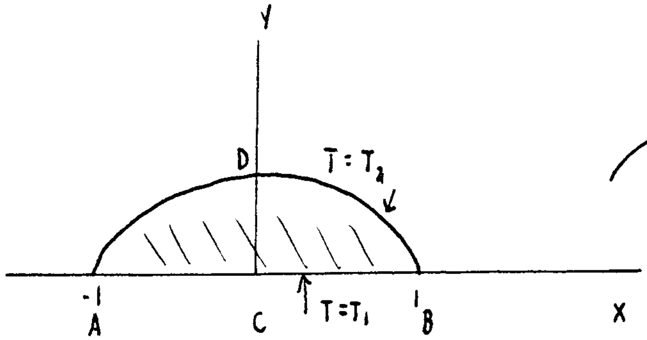
TO SHOW THAT semi-circle yields the entire imaginary axis let $z = e^{i\varphi}$ $\varphi \in (0, \pi)$.

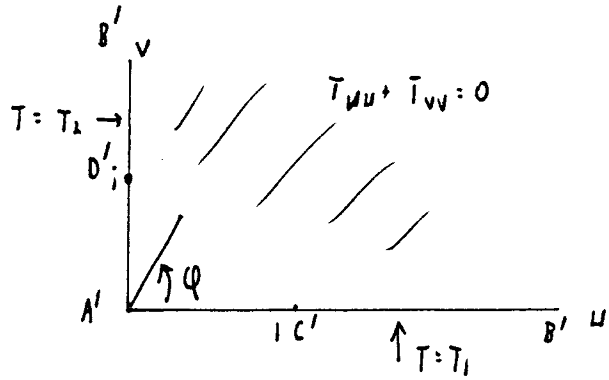
THEN $w = \dfrac{1+e^{-i\varphi}}{1-e^{-i\varphi}} = i\cot\left(\varphi/2\right)$ THIS ranges FROM $0i$ TO ∞i AS φ ranges FROM π TO 0.

NOW TAKE A POINT INSIDE AT $Z = i/2$

THEN $W = \frac{2+i}{2-i} = \frac{3}{4} + i$ (first quadrant).

THEREFORE





$T_{uu} + T_{vv} = 0$

LET $T = C_0 + C_1 \varphi$

THEN $T = T_1 + 2\left(\frac{T_2 - T_1}{\pi}\right) \varphi$

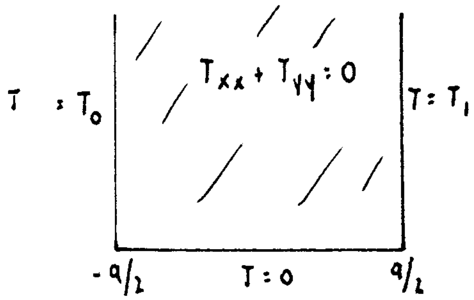
$$T = \frac{2(T_2 - T_1)}{\pi} \tan^{-1}\left(V/U\right) + T_1$$

$$U + iV = \frac{(1 + X + iy)(1 - X + iy)}{(1 - X - iy)(1 - X + iy)} = \frac{(1 - X^2 - y^2) + 2iy}{(1-X)^2 + y^2}$$

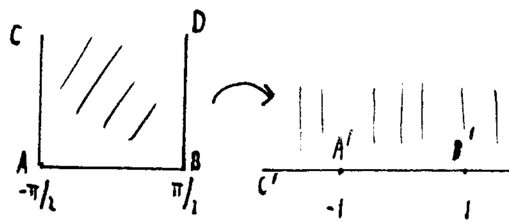
$$\frac{V}{U} = \frac{2y}{1 - X^2 - y^2}$$

$$T = \frac{2(T_2 - T_1)}{\pi} \tan^{-1}\left(\frac{2y}{1 - X^2 - y^2}\right) + T_1$$

MAPPING PROBLEM 4



RECALL $W = \sin Z$ MAPS

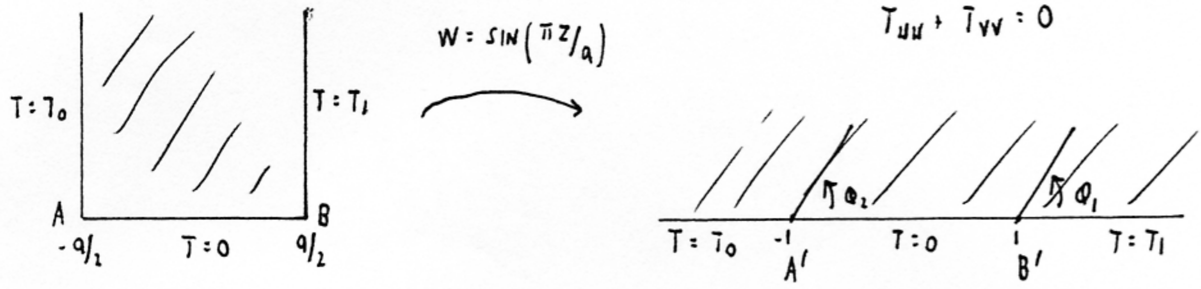


THEREFORE TAKE

$$W = \sin\left(\pi Z/a\right)$$

MAPPING NOT CONFORMAL AT $Z = \pm \pi/2$.

THEN



NOW $\quad T = C_0 + C_1 Q_1 + C_2 Q_2$

SATISFYING THE CONDITIONS WE GET

$$T = T_1 - \frac{T_1}{\pi} Q_1 + \frac{T_0}{\pi} Q_2 \qquad Q_1 = \tan^{-1}\left(\frac{V}{U-1}\right)$$

$$Q_2 = \tan^{-1}\left(\frac{V}{U+1}\right)$$

NOW $\quad W = \sin\left(\pi z / a\right)$ GIVES

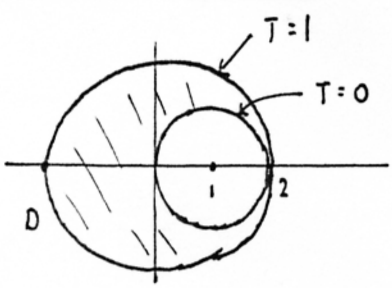
$$U + iV = \sin\left(\pi x/a\right)\cosh\left(\pi y/a\right) + i\cos\left(\pi x/a\right)\sinh\left(\pi y/a\right)$$

$$U = \sin\left(\pi x/a\right)\cosh\left(\pi y/a\right)$$

$$V = \cos\left(\pi x/a\right)\sinh\left(\pi y/a\right)$$

$$T = T_1 - \frac{T_1}{\pi}\tan^{-1}\left(\frac{V}{U-1}\right) + \frac{T_0}{\pi}\tan^{-1}\left(\frac{V}{U+1}\right)$$

MAPPING PROBLEM 5



$T_{xx} + T_{yy} = 0$ between circles

$T = 1$ on big circle $x^2 + y^2 = 4$

$T = 0$ on small circle $(x-1)^2 + y^2 = 1$

WANT TO CHOOSE $z = 2$ TO BE POLE OF MAP. THEN images of EACH circle is a line. let $z = 0$ get mapped to origin. let diameter of circle (x-axis) get mapped to a real line segment.

THUS $\quad W = \gamma z / (z-2) \implies$ INNER circle is a line through the origin. which line? let $z = 1+i$. we get

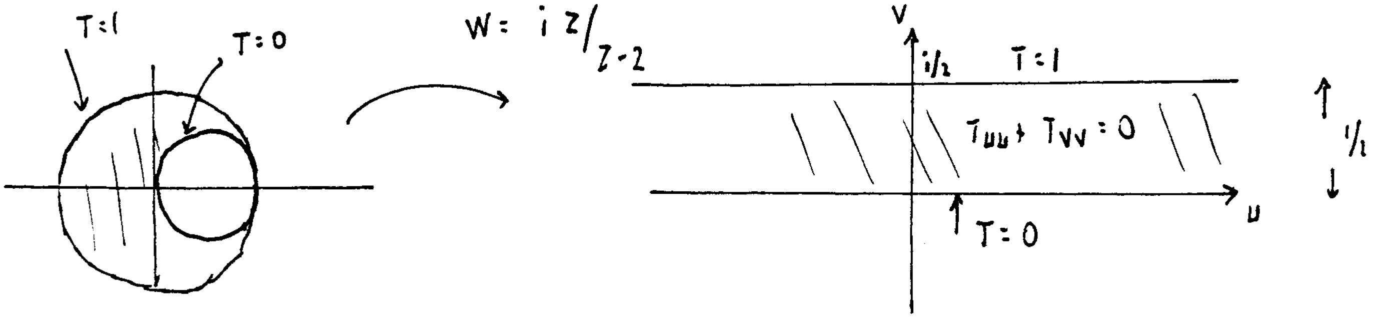
$|\gamma| = 1 \longrightarrow \gamma = \pm i$

$$W = \gamma\left(\frac{1+i}{-1+i}\right) = -i\gamma \qquad \text{CHOOSE } \gamma = \pm i \text{ TO}$$

get real axis.

THUS IF $\gamma = \pm i$ the image of INNER circle is the real axis. The diameter $|x| \le 2$ WITH $y = 0$ gets mapped to a segment OF imaginary axis. Thus the image of big CIRCLE must be a LINE $\perp$ TO imaginary axis SINCE map is conformal at point D. TO get this LINE we let $z = -2$.

$\longrightarrow \quad W = \gamma/2$. CHOOSE $\gamma = i \longrightarrow W = i/2$. THE LINE runs parallel TO X-axis and has $y = 1/2$.



THUS TRY $T = C_0 + C_1 V$

$V = 0 \rightarrow T = 0 \rightarrow C_0 = 0$

$V = 1/2 \rightarrow T = 1 \rightarrow C_1 = 2$

THE SOLUTION IN V-PLANE IS

$$T = 2V$$

NOW
$$u + iv = \frac{i(x+iy)}{(x-2)+iy} = \frac{i(x+iy)(x-2-iy)}{(x-2)^2+y^2}$$

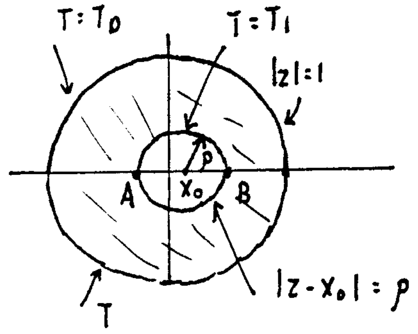
$$V = \frac{x(x-2)+y^2}{(x-2)^2+y^2}$$

HENCE
$$T = 2\left[\frac{(x-1)^2+y^2-1}{(x-2)^2+y^2}\right]$$

CAN CHECK THAT $T=0$ ON $(x-1)^2+y^2=1$

AND $T=1$ ON $(x^2+y^2) = 4$.

MAPPING PROBLEM 6 — SOLVE LAPLACE'S EQUATION BETWEEN NONCONCENTRIC CIRCLES.



ASSUME $\left\{\begin{array}{l} \rho + X_0 < 1 \\ X_0 > 0 \end{array}\right.$

IT WAS PROVED IN CLASS THAT THE MAP, WITH $|\alpha| < 1$,

$$W = \frac{z - \alpha}{\alpha z - 1}$$

TAKES UNIT CIRCLE $|z| \le 1$ INTO ITSELF (i.e. $|w| \le 1$).

WE THEN TRY α REAL TO GET CONCENTRIC CIRCLES. TAKE $-1 < \alpha < 1$. THEN REAL AXIS GETS MAPPED TO REAL AXIS AND MAP IS CONFORMAL AT A AND B. THUS CENTER OF IMAGE OF SMALL CIRCLE LIES ON REAL AXIS IN W-PLANE.

TO GET THE CENTER AT THE ORIGIN WE NEED THAT

$$W(X_0 + \rho) = -W(X_0 - \rho)$$

i.e.

$$\frac{X_0 + \rho - \alpha}{\alpha(X_0 + \rho) - 1} = \frac{\alpha - (X_0 - \rho)}{\alpha(X_0 - \rho) - 1}$$

get QUADRATIC EQUATION $\quad \alpha^2 X_0 - \alpha(1 + X_0^2 - \rho^2) + X_0 = 0$

$$\alpha = -\frac{1}{2X_0}(\rho^2 - X_0^2 - 1) \pm \frac{1}{2}\left[\frac{1}{X_0^2}(\rho^2 - X_0^2 - 1)^2 - 4\right]^{1/2}$$

ROOTS ARE REAL IF $\quad (\rho^2 - X_0^2 - 1)^2 > 4X_0^2 \qquad |\rho^2 - X_0^2 - 1| > 2X_0$

this is satisfied when $\rho + X_0 < 1$ (NOT TOUCHING)

SINCE THE PRODUCT OF THE TWO ROOTS α_1 AND α_2 ARE 1

i.e. $\alpha_1 \alpha_2 = 1$

$\Big[\ (\alpha - \alpha_1)(\alpha - \alpha_2) = 0$

$\longrightarrow \alpha^2 - (\alpha_1 + \alpha_2)\alpha + \alpha_1\alpha_2 = 0$

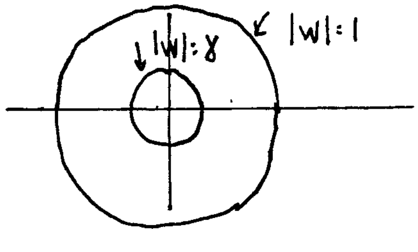
$\alpha_1 + \alpha_2 = \dfrac{(1 + X_0^2 - \rho^2)}{X_0}$

$\alpha_1 \alpha_2 = 1\ \Big]$

WE MUST HAVE ONE ROOT WITH $-1 < \alpha < 1$.

THE OTHER WILL LIE OUTSIDE THIS INTERVAL.

HENCE CHOOSE α TO BE THE SMALLER ROOT.



$$\gamma = \left|\frac{X_0 + \rho - \alpha}{\alpha(X_0 + \rho) - 1}\right|$$

AS AN EXAMPLE LET $p = .30$ AND $X_0 = .30$.

THEN WE GET $\alpha = 1/3$.

$$\alpha^2 - \frac{10\alpha}{3} + 1 = 0$$

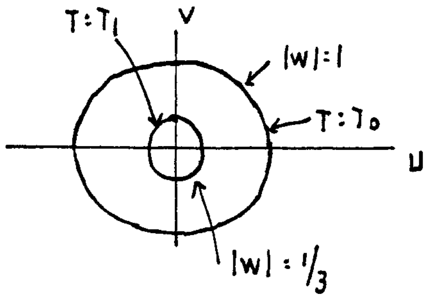
$$\alpha = \frac{\frac{10}{3} \pm \sqrt{100/9 - 36/9}}{2}$$

$$\alpha = \frac{10/3 \pm 8/3}{2} = 1/3$$

$$W = \frac{z - 1/3}{1/3\, z - 1} = \frac{3z - 1}{z - 3} = f(z)$$

SINCE $z = 0$ IS ON SMALL CIRCLE WE GET THAT RADIUS OF SMALL CIRCLE IN IMAGE PLANE IS $|f(0)| = 1/3$.

THUS



$T_{uu} + T_{vv} = 0$ between $|w| = 1/3$ AND $|w| = 1$

THUS $\left\{ \begin{array}{l} T = T_0 \text{ ON } |w| = 1 \\ T = T_1 \text{ ON } |w| = 1/3 \end{array} \right.$

$r = |w|$

NOW LET $r^2 = u^2 + v^2$.

THEN $T = A + B \log r$.

$$T = \left\{ \begin{array}{ll} T_0 & \text{ON } r = 1 \\ T_1 & \text{ON } r = 1/3 \end{array} \right.$$

NOW USE RESULT ON page 2 WITH $r_0 = 1$, $r_1 = 1/3$. THIS GIVES

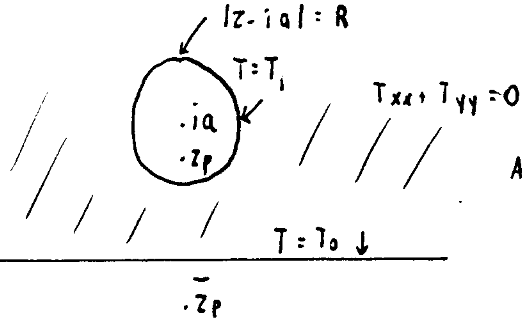
$$T = T_1 + \frac{(T_1 - T_0)}{\log(1/3)} \log(3r)$$

BUT $r = |f(z)|$

$$r = \left| \frac{3z-1}{z-3} \right| = \frac{|3z-1|}{|z-3|} = \left[\frac{(3x-1)^2 + 9y^2}{(x-3)^2 + y^2} \right]^{1/2}$$

$$T = T_1 + \frac{(T_1 - T_0)}{\log(1/3)} \log\left[3 \left[\frac{(3x-1)^2 + 9y^2}{(x-3)^2 + y^2} \right]^{1/2} \right]$$

MAPPING PROBLEM 7



ASSUME $R < a$

WE TRY TO MAP TO CONCENTRIC CIRCLES.

LET z_p BE A POINT INSIDE CIRCLE

$\bar{z}_p$ IS THE SYMMETRIC POINT WITH RESPECT TO X-AXIS.

NOW $z = z_p \to w = 0$

$z = \bar{z}_p \to w = \infty$

$w = 0$ AND $w = \infty$ ARE SYMMETRIC POINTS W.R.T. IMAGE OF THE X-AXIS UNDER THE MAP.

HENCE IMAGE OF X-AXIS MUST BE A CIRCLE CENTRED AT ORIGIN (IT IS ONLY OBJECT WITH $w=0$, $w=\infty$ AS SYM. POINTS).

TAKE $w = \dfrac{z - z_p}{z - \bar{z}_p}$ THEN IF z IS REAL WE HAVE $|w| = 1 \to$ CIRCLE OF RADIUS 1 IS IMAGE OF X-AXIS.

NOW WE WANT z_p AND $\bar{z}_p$ TO BE SYMMETRIC POINTS W.R.T. THE CIRCLE. THIS WILL YIELD THAT IMAGE OF CIRCLE IS A CIRCLE CENTRED AT THE ORIGIN.

NOW z_p AND $\bar{z}_p$ ARE SYMMETRIC W.R.T. SMALL CIRCLE IF

$$\bar{z}_p = ai + \frac{R^2}{\bar{z}_p + ai}$$

$$\bar{z}_p(\bar{z}_p + ai) = (\bar{z}_p + ai)(ai) + R^2$$

$$\to (\bar{z}_p)^2 = R^2 - a^2$$

$$\to z_p = i(a^2 - R^2)^{1/2}$$

CLAIM THAT z_p LIES INSIDE $|z - ia| = R$ SINCE

$$z_p = i(a-R)^{1/2}(a+R)^{1/2}\frac{(a-R)^{1/2}}{(a-R)^{1/2}}$$

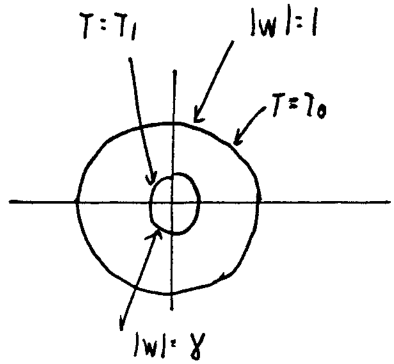
$$z_p = i(a-R)\left(\frac{a+R}{a-R}\right)^{1/2} < i(a-R) \quad \text{SINCE } R < a$$

$$\to z_p - ia < -iR$$

$$|z_p - ia| < R. \checkmark$$

THEREFORE TAKE THE MAP

$$W = \frac{Z - Z_p}{Z - \bar{Z}_p} \qquad \text{WITH} \qquad Z_p = i\,(a^2 - R^2)^{1/2}$$



THEN WE WILL GET CONCENTRIC CIRCLES IN W-PLANE

TO FIND γ TAKE A POINT $Z = i\,(a - R)$ AND THEN CALCULATE $|W|$.

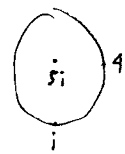
<u>EXAMPLE</u> LET $a = 5$ $R = 4$ THEN $Z_p = 3i$

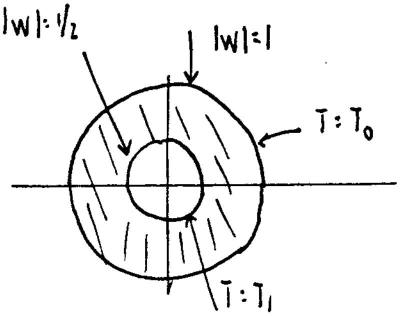
$$W = \frac{Z - 3i}{Z + 3i}$$

X-AXIS $\longrightarrow$ CIRCLE OF RADIUS 1

$|Z - 5i| = 4 \longrightarrow \quad |W| = \left| \dfrac{i - 3i}{i + 4i} \right| = \dfrac{1}{2}$

LET $Z = i$





$T_{uu} + T_{vv} = 0 \qquad 1/2 \le r \le 1$

$T = T_0$ ON $r = 1$

$T = T_1$ ON $r = 1/2$

NOW USE p. 2 WITH $r_0 = 1$, $r_1 = 1/2$

$$T = T_1 + \frac{(T_1 - T_0)}{\log(1/2)} \log(2r)$$

BUT $|W| = r$

$$r = \frac{|z - 3i|}{|z + 3i|} \qquad r = \left[\frac{x^2 + (y-3)^2}{x^2 + (y+3)^2} \right]^{1/2}$$

$$\Longrightarrow \qquad T = T_1 + \frac{(T_1 - T_0)}{\log(1/2)} \log\left[2 \left(\frac{x^2 + (y-3)^2}{x^2 + (y+3)^2} \right)^{1/2} \right]$$