CONFORMAL MAPPING AND FLUID FLOW

CONSIDER A STEADY, TWO-DIMENSIONAL FLOW OF AN INCOMPRESSIBLE, INVISCID FLUID. LET $\underline{V} = (V_1, V_2)$ BE THE VELOCITY VECTOR FOR THE FLUID. WE WILL ASSUME

(i) $\nabla \cdot \underline{V} = 0 \longrightarrow$ INCOMPRESSIBLE FLUID

(ii) $\nabla \times \underline{V} = 0 \longrightarrow$ IRROTATIONAL FLUID (NO VORTICITY)

SINCE $\nabla \times \underline{V} = 0$ THERE IS A FUNCTION $\Phi(x,y)$ CALLED THE <u>VELOCITY POTENTIAL</u> FOR WHICH

$$\underline{V} = \nabla \Phi \longrightarrow V_1 = \partial\Phi/_{\partial x} \quad V_2 = \partial\Phi/_{\partial y}.$$

FROM THE MASS-CONSERVATION CONDITION (i) WE HAVE

$$(1) \qquad \nabla^2 \Phi = 0$$

NOW INTRODUCE A <u>COMPLEX VELOCITY POTENTIAL</u> $\Omega(z)$ GIVEN BY

$$(2) \qquad \Omega(z) = \Phi + i\Psi$$

IF Ψ IS THE HARMONIC CONJUGATE TO Φ, THEN $\Omega(z)$ IS ANALYTIC.

NOW $\Omega'(z) = \Phi_x + i\Psi_x$. BUT $\Psi_x = -\Phi_y$ BY CAUCHY RIEMANN EQUATION

HENCE $\Omega'(z) = \Phi_x - i\Phi_y$.

THEREFORE THE X AND Y COMPONENTS OF $\underline{V}$, LABELLED BY V_1 AND V_2 SATISFY

$$\Omega'(z) = V_1 - iV_2.$$

THUS $V_1 + iV_2 = \overline{\Omega'(z)}$

THE <u>SPEED</u> OF THE FLOW IS $|\underline{V}| = (V_1^2 + V_2^2)^{1/2} = |\overline{\Omega'(z)}|$.

A <u>STAGNATION POINT</u> OF THE FLOW IS A VALUE OF (x,y) FOR WHICH $\underline{V} = 0$. IN OTHER WORDS, SUCH A POINT SATISFIES $|\overline{\Omega'(z_0)}| = 0$

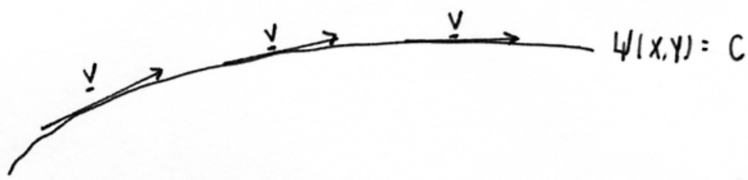
NOW WE INTERPRET Ψ:

FIRST NOTICE THAT $\nabla\Phi \cdot \nabla\Psi = 0$ AS FOLLOWS FROM THE CAUCHY RIEMANN EQUATIONS. BUT $\underline{V} = \nabla\Phi$ AND HENCE

$$\underline{V} \cdot \nabla\Psi = 0$$

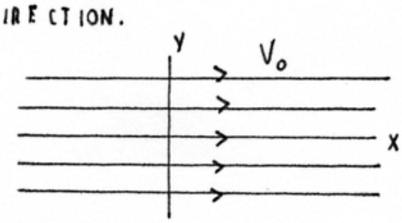
CONSIDER A LEVEL LINE $\Psi(x,y) = C$. THEN $\nabla\Psi$ IS $\perp$ TO THIS LEVEL LINE AND $\underline{V} \cdot \nabla\Psi = 0$. THUS $\underline{V}$ IS PARALLEL TO A TANGENT VECTOR ON THE LEVEL LINE $\Psi(x,y) = C$.

THEREFORE, WE CALL LEVEL LINES Ψ = CONSTANT STREAMLINES SINCE THEY REPRESENT THE DIRECTION OF THE PATHS OF THE FLUID PARTICLES TRANSPORTED BY THE FLOW. Ψ is called the <u>streamfunction</u>.



<u>EXAMPLE</u> FIND THE COMPLEX VELOCITY POTENTIAL CORRESPONDING TO A UNIFORM FLUID FLOW IN THE POSITIVE X-DIRECTION.

WE WANT $\underline{V} = (V_0, 0)$.



V_0 constant

THUS $\partial\Phi/\partial x = V_0 \qquad \partial\Phi/\partial y = 0$

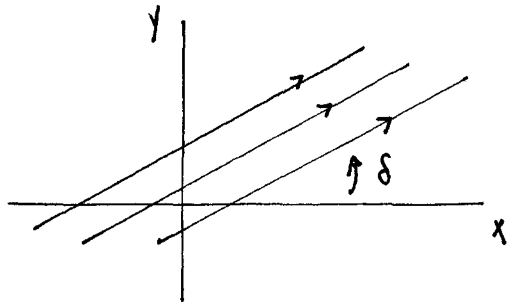
$\longrightarrow \Phi = V_0 x$ (we can neglect the additive constant).

NOW $\Psi_y = \Phi_x \qquad \Psi_x = -\Phi_y \longrightarrow \Psi = V_0 y$. streamfunction

HENCE $\Omega(z) = \Phi + i\Psi = V_0(x + iy)$

$\Omega(z) = V_0 z$ is complex velocity potential FOR UNIFORM FLOW IN positive-x-direction.

EXAMPLE FIND THE VELOCITY POTENTIAL AND COMPLEX VELOCITY POTENTIAL FOR FLOW AT AN ANGLE δ ABOVE THE HORIZONTAL.



WANT $V_1 = V_0 \cos\delta$

$V_2 = V_0 \sin\delta$

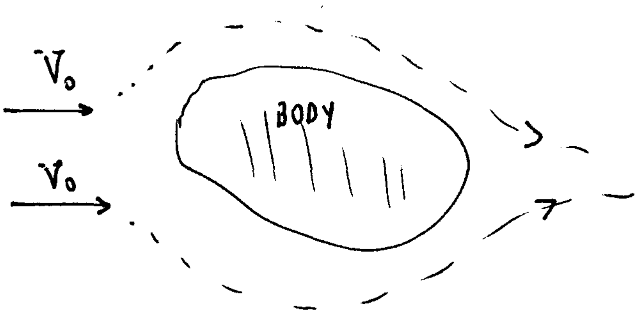
SO $\partial\Phi/\partial x = V_0 \cos\delta \qquad \partial\Phi/\partial y = V_0 \sin\delta$

WE HAVE $\Omega'(z) = \partial\Phi/\partial x - i\,\partial\Phi/\partial y$

THEREFORE $\Omega'(z) = V_0(\cos\delta - i\sin\delta) = V_0 e^{-i\delta} \longrightarrow \Omega(z) = V_0 z e^{-i\delta}$ IS COMPLEX VELOCITY POTENTIAL

FLOW OVER AN OBJECT

WANT TO DETERMINE STREAMLINES AND VELOCITY FOR FLOW OVER A 2-D OBJECT. WE ASSUME THAT THERE IS A UNIFORM STREAM IN X-DIRECTION FAR FROM THE BODY



THE COMPLEX VELOCITY POTENTIAL MUST SATISFY

$$\Omega(z) = V_0 z + G(z)$$

WHERE $G(z)$ IS BOUNDED AS $|z| \to \infty$, OR EQUIVALENTLY

$$\lim_{|z|\to\infty} G'(z) = 0 .$$

WE ALSO REQUIRE THAT THE BOUNDARY OF THE BODY IS A STREAMLINE OF THE FLOW. HENCE, NO FLOW PENETRATES INTO THE BODY. WRITING $\Omega = \Phi + i\psi$, THIS IMPLIES THAT (ψ: STREAMFUNCTION).

$$\mathrm{IM}(\Omega) = \text{CONSTANT ON THE BODY}$$

EXAMPLE CONSIDER THE JOUKOWSKI MAP

$$\Omega(z) = V_0\left(z + a^2/z\right)$$

WHAT TYPE OF FLOW DOES THIS CORRESPOND TO? AS $|z| \to \infty$ WE HAVE $\Omega(z) \to V_0 z$ WHICH IS A UNIFORM STREAM IN X-DIRECTION.

NOW CALCULATE Φ AND Ψ.

$$\Omega = \Phi + i\Psi = V_0\left(re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\right)$$

$$\Omega = V_0\left(r + \frac{a^2}{r}\right)\cos\varphi + i\,V_0\left(r - a^2/r\right)\sin\varphi$$

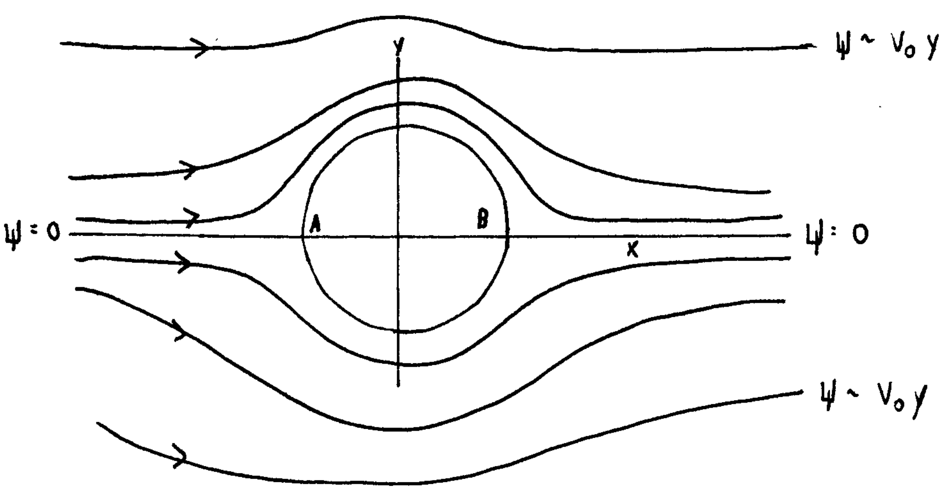
THUS $\quad \Phi = V_0\left(r + a^2/r\right)\cos\varphi \qquad \Psi = V_0\left(r - a^2/r\right)\sin\varphi$

NOTICE THAT $r = a$ IS A STREAMLINE $\Psi = 0$. THUS WE CAN INTERPRET THIS COMPLEX POTENTIAL AS REPRESENTING A UNIFORM FLOW OVER A CIRCULAR CYLINDER OF RADIUS a. THE STREAMLINES $\Psi = C$ SATISFY

$$r^2 - a^2 - \frac{Cr}{V_0\sin\varphi} = 0 \qquad r = \frac{C}{2V_0\sin\varphi} \pm \frac{1}{2}\sqrt{\frac{C^2}{V_0^2\sin^2\varphi} + 4a^2} \qquad C > 0.$$

+ (root is needed)

PLOTTING, WE GET THE FOLLOWING PICTURE FOR STREAMLINES



NOW CALCULATE THE VELOCITY. RECALL $\underline{V} = (V_1, V_2)$ AND $\overline{\Omega'(z)} = V_1 + iV_2$.

NOW $\quad \Omega'(z) = V_0\left(1 - a^2/z^2\right) = V_0\left(1 - \frac{a^2}{r^2}e^{-2i\varphi}\right)$

$$\overline{\Omega'(z)} = \left(V_0 - \frac{a^2}{r^2}V_0 e^{2i\varphi}\right) = \left(V_0 - V_0\frac{a^2}{r^2}\cos 2\varphi\right) - iV_0\frac{a^2}{r^2}\sin 2\varphi$$

HENCE $\quad V_1 = V_0\left(1 - a^2/r^2\cos 2\varphi\right) \qquad V_2 = -V_0\frac{a^2}{r^2}\sin 2\varphi.$

NOW $\quad |\underline{V}| = \left[1 - \frac{2a^2}{r^2}\cos 2\varphi + \frac{a^4}{r^4}\right]^{1/2} V_0 \quad$ magnitude of velocity.

THE STAGNATION POINTS ARE WHERE $\Omega'(z) = 0$.

SINCE $\Omega'(z) = V_0 \left(1 - a^2/z^2\right) \longrightarrow \Omega'(z) = 0$ AT $z = \pm a$.

AT THESE POINTS LABELLED A AND B WE HAVE $\underline{v} = \underline{0}$.

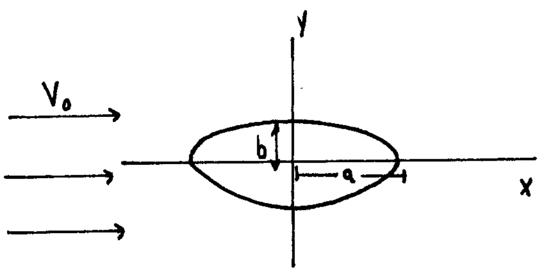
NOW THE SPEED OF THE FLOW IS MAXIMIZED AT $\varphi = \pi/2$ AND $\varphi = 3\pi/2$, WITH $\Gamma =$

AT THESE POINTS $|\underline{v}| = V_0 \left[\frac{a^4}{a^4} + \frac{2a^2}{a^2} + 1 \right]^{1/2}$

$\longrightarrow |\underline{v}| = 2V_0$ AT THESE POINTS.

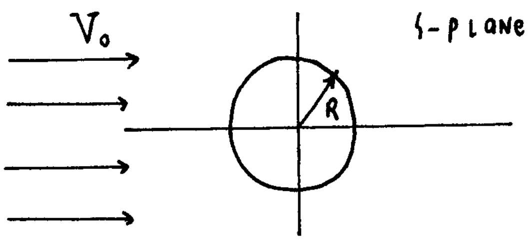
FLOW OVER AN ELLIPSE

FIND VELOCITY POTENTIAL AND COMPLEX VELOCITY POTENTIAL FOR FLOW OVER AN ELLIPSE AS SHOWN

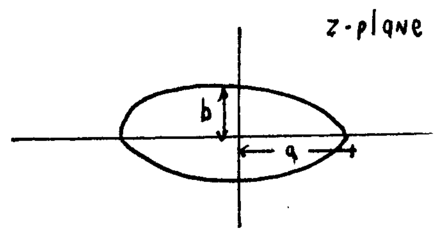


ASSUME $a > b$.

FIRST RECALL THAT $\Omega(s) = V_0\left(s + R^2/s\right)$ CORRESPONDS TO A UNIFORM FLOW IN X-DIRECTION OVER A CIRCULAR CYLINDER OF RADIUS R.



$z = s + \tau^2/s$



NOW MAP s-PLANE TO z-PLANE to get an ellipse with semi-axes a and b.

NOW $s = Re^{i\varphi}$ gives $z = Re^{i\varphi} + \frac{\tau^2}{R}e^{-i\varphi}$

$\rightarrow x = (R + \tau^2/R)\cos\varphi \qquad y = (R - \tau^2/R)\sin\varphi$

WE WANT
$a = R + \tau^2/R$
$b = R - \tau^2/R$
$\Longrightarrow R = \frac{a+b}{2} \qquad \tau = \frac{1}{2}(a^2 - b^2)^{1/2}$

NOW $\varsigma + \tau^2/\varsigma = z$ OR $\varsigma^2 - \varsigma z + \tau^2 = 0$

$$\varsigma = \frac{z \pm (z^2 - 4\tau^2)^{1/2}}{2}$$

WE NEED + ROOT SINCE WE WANT $\varsigma \sim z$ AS $|z| \to \infty$ TO GET UNIFORM FLOW.

HENCE THE COMPLEX VELOCITY POTENTIAL IS

$$\Omega(z) = V_0 \left[\frac{z + (z^2 - (a^2 - b^2))^{1/2}}{2} + \frac{(a+b)^2}{2\left[z + (z^2 - (a^2 - b^2))^{1/2}\right]} \right]$$

<u>REMARK</u>: NOW $\dfrac{d\Omega}{dz} = \dfrac{d\Omega}{d\varsigma}\dfrac{d\varsigma}{dz}$ BUT

$$\frac{d\Omega}{d\varsigma} = V_0 \left(1 - R^2/\varsigma^2\right)$$

$$\frac{d\varsigma}{dz}\left(1 - \tau^2/\varsigma^2\right) = 1$$

THIS GIVES,

$$\frac{d\Omega}{dz} = V_0\left(1 - R^2/\varsigma^2\right)\frac{1}{1 - \tau^2/\varsigma^2} = V_0\left[\frac{\varsigma^2 - R^2}{\varsigma^2 - \tau^2}\right]$$

NOW AT THE POINT $\varsigma = iR$, WE GET THE POINT $z = ib$ ON ELLIPSE

" $\varsigma = -iR$, " " " $z = -ib$ ON ELLIPSE

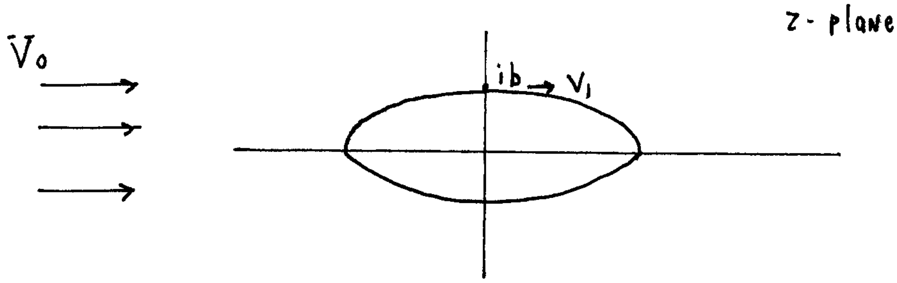
THEREFORE AT $\varsigma = iR$ (i.e. $z = ib$) WE HAVE

$$\left.\frac{d\Omega}{dz}\right|_{z=ib} = \frac{-V_0\, 2R^2}{-(R^2+\tau^2)} = 2V_0\left[\frac{1}{1+\tau^2/R^2}\right] \qquad \frac{\tau^2}{R^2} = \frac{(a^2-b^2)}{(a+b)^2} = \frac{a-b}{a+b}$$

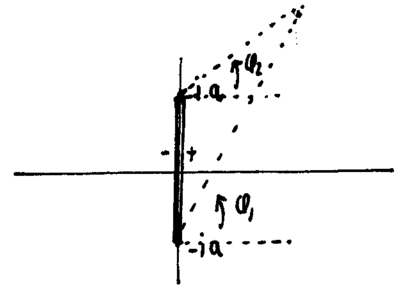
$$\longrightarrow \quad \left.\frac{d\Omega}{dz}\right|_{z=ib} = V_0\left[\frac{a+b}{a}\right]$$

THEREFORE THE FLOW VELOCITY IN X-DIRECTION AT TIP OF THE ELLIPSE IS

$$V_1 = V_0\left[1 + b/a\right] \quad \text{ALSO} \quad V_2 = 0$$



IF $b < a$ then $V_1 < 2V_0$.



EXAMPLE (FLOW PAST A PLATE)

CONSIDER THE COMPLEX VELOCITY POTENTIAL

$$\Omega(z) = (z^2 + a^2)^{1/2}$$

WE CHOOSE THE BRANCH CUT BETWEEN $z = ia$, $z = -ia$.

THE ANGLES ARE DEFINED BY $\frac{-3\pi}{2} < \theta_1 < \frac{\pi}{2}$, $\frac{-3\pi}{2} < \theta_2 < \frac{\pi}{2}$, WHICH GIVES THE BRANCH CUTS AS SHOWN.

NOW ON THE + SIDE OF THE PLATE $\theta_1 = \pi/2$, $\theta_2 = -\pi/2$, WHICH GIVES

$$\Omega = (z^2 + a^2)^{1/2} = |z+ia|^{1/2}|z-ia|^{1/2}.$$

ON THE - SIDE OF THE PLATE $\theta_1 = -3\pi/2$, $\theta_2 = -\pi/2$

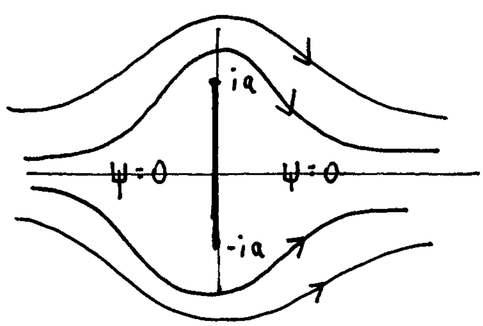
$$\Omega = (z^2 + a^2)^{1/2} = |z+ia|^{1/2}|z-ia|^{1/2} e^{-2\pi i/2} = -|z+ia|^{1/2}|z-ia|^{1/2}$$

HENCE, ON EITHER SIDE OF THE PLATE WE HAVE $IM(\Omega) = 0$, SO THAT THE PLATE IS THE STREAMLINE $\psi = 0$. NOW

$$\Omega(z) = |z^2 + a^2|^{1/2} e^{i(\theta_1 + \theta_2)/2}$$

SO $\psi = IM(\Omega) = |z^2 + a^2|^{1/2} \sin\left((\theta_1 + \theta_2)/2\right)$ IS THE STREAMFUNCTION

WHERE $-\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$, $-\frac{3\pi}{2} < \theta_2 < \frac{\pi}{2}$. NOTICE THAT $\theta_1 = \theta_2 = 0 \longrightarrow \psi = 0$, AND THAT $\theta_1 = \theta_2 = -\pi \longrightarrow \psi = 0$. IF WE PLOT LEVEL CURVES OF ψ WE GET



NOTE

(i) $\Omega(z) = (z^2)^{1/2}\left[1 + \frac{a^2}{z^2}\right]^{1/2} \sim z\left(1 - \frac{a^2}{2z^2} + \ldots\right)$ FOR $|z| >> 1$

$\longrightarrow \Omega(z) \approx z$ FOR $|z| \to \infty$ (free stream)

(ii) $\Omega'(z) = z(z^2 + a^2)^{-1/2}$. VELOCITY $\underline{V}$ IS $\underline{V} = \overline{\Omega'(z)}$. NOTE $\underline{V} = 0$ WHEN $z = 0$, AND $\underline{V}$ IS infinite when $z = \pm ia$.

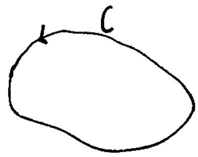
FORCE DUE TO FLUID PRESSURE

THE BERNOULLI'S LAW IS

$$P + \frac{1}{2}\rho |\underline{v}|^2 = \alpha \qquad \alpha = \text{constant along each streamline.}$$

MAIN RESULT LET C BE A STREAMLINE REPRESENTING A BODY. LET $\Omega(z)$ BE THE CORRESPONDING VELOCITY POTENTIAL. LET $\underline{F} = F_1 + iF_2$ BE THE FORCE ON THE BODY. THEN

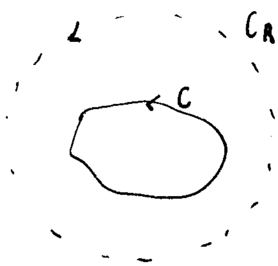
$$\bar{\underline{F}} = \frac{i\rho}{2}\oint_C [\Omega'(z)]^2\, dz.$$



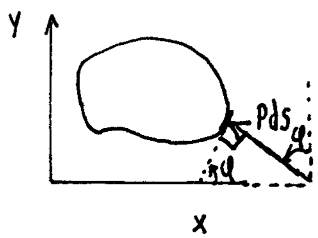
EQUIVALENTLY SINCE $\Omega'(z)$ IS ANALYTIC OUTSIDE C THEN WE CAN USE THE RESIDUE THEOREM TO CALCULATE $\underline{F}$ AS

$$\underline{F} = \frac{i\rho}{2}\lim_{R\to\infty}\oint_{C_R} [\Omega'(z)]^2\, dz$$

WHERE C_R IS A CIRCLE OF RADIUS R ($|z| = R$) THAT ENCLOSES C.



DERIVATION CONSIDER THE FORCE $d\underline{F}$ induced ON A SMALL segment ds OF C AS SHOWN



$$d\underline{F} = dF_1 + i\,dF_2 = -P\sin\varphi\, ds + iP\cos\varphi\, ds$$

THUS $\quad d\underline{F} = iP e^{i\varphi}\, ds.$

NOW $\quad dz = dx + i\,dy = \cos\varphi\, ds + i\sin\varphi\, ds = e^{i\varphi} ds. \rightarrow dz = e^{i\varphi} ds$

NOW $\quad \Omega'(z) = V_1 - iV_2 \quad$ WITH $\quad (V_1, V_2) = |v| e^{i\varphi} \longrightarrow \Omega'(z) = |v| e^{-i\varphi}.$

NOW USE BERNOULLI'S LAW $\quad d\underline{F} = i\left(\alpha - \frac{1}{2}\rho|v|^2\right) e^{i\varphi}\, ds.$

NOW $\int_C \alpha e^{i\varphi}\, ds = \alpha\int_C dz = 0.$ THUS,

$$\underline{F} = -\frac{i\rho}{2}\oint_C |v|^2 e^{i\varphi}\, ds \longrightarrow \bar{\underline{F}} = \frac{i\rho}{2}\oint_C |v|^2 e^{-i\varphi}\, ds = \frac{i\rho}{2}\oint_C |v|^2 e^{-2i\varphi}\, dz.$$

HOWEVER $\Omega'(z) = |v| e^{-i\varphi}$, SO THAT

$$\bar{\underline{F}} = \frac{i\rho}{2}\oint_C [\Omega'(z)]^2\, dz.$$

SINKS, SOURCES, AND VORTICES

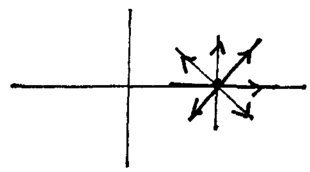
A **POINT SOURCE** HAS THE FORM

$$\Omega(z) = K \log(z-a), \quad \text{WHERE } K>0 \text{ AND } K \text{ REAL.}$$

LET $\Omega'(z) = \dfrac{K}{z-a}$. WE CALCULATE, WITH $z-a = re^{i\varphi}$

$$\Omega'(z) = \frac{K}{z-a} = \frac{K}{r} e^{-i\varphi} = \frac{K}{r}(\cos\varphi - i\sin\varphi). = V_1 - iV_2.$$

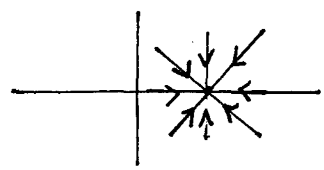
THUS $V_1 = \dfrac{K\cos\varphi}{r}$, $V_2 = \dfrac{K\sin\varphi}{r}$ $\quad |V| = K/r$



THIS REPRESENTS OUTWARD FLOW FROM A SINGULAR POINT.

THE STREAMLINES CORRESPOND TO $\mathrm{IM}[\Omega(z)]$ = CONSTANT. $\longrightarrow$ streamlines CORRESPOND TO φ = CONSTANT.

A **POINT SINK** LIKEWISE HAS THE FORM $\Omega(z) = -K\log(z-a)$ WITH K REAL AND $K>0$.



A **VORTEX** CORRESPONDS TO SWIRLING FLOW AROUND SOME POINT.

CONSIDER $\Omega(z) = iK\log(z-a)$ WITH $K>0$ AND REAL.

WE LET $z = a + re^{i\varphi}$ AND CALCULATE $\Omega'(z) = \dfrac{iK}{z-a} = \dfrac{iK}{r}e^{-i\varphi}$.

WE GET $\Omega'(z) = \dfrac{K}{r}(\sin\varphi + i\cos\varphi) = V_1 - iV_2$.

THUS $V_1 = \dfrac{K}{r}\sin\varphi$, $V_2 = -\dfrac{K}{r}\cos\varphi$.

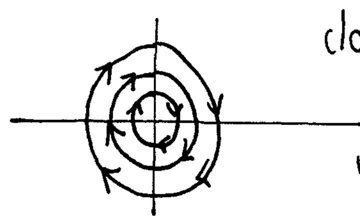
NOTICE (WITH $K>0$)

QUADRANT	V_1	V_2
I	>0	<0
II	>0	>0
III	<0	>0
IV	<0	<0

NOW STREAMLINES ARE $\mathrm{IM}(\Omega)$ = CONSTANT.

THIS IMPLIES WITH $\log(z-a) = \log|z-a| + i\varphi$

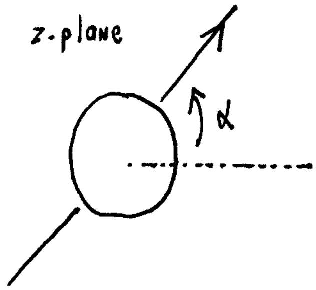
THAT r = CONSTANT. THIS GIVES THE FLOW



clockwise SWIRL when $K>0$.

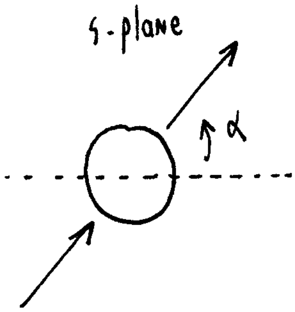
IF $K<0 \longrightarrow$ counterclockwise SWIRL.

PROBLEM CONSIDER FLOW OVER A CIRCLE OF RADIUS a AT AN ANGLE α OF ATTACK



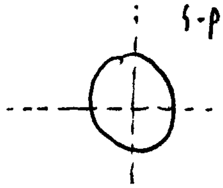
$$\Omega(z) = V_0 \left(z e^{-i\alpha} + \frac{a^2}{z e^{-i\alpha}} \right)$$

PROBLEM FLOW OVER AN ELLIPSE IN z-plane AT angle OF ATTACK α.



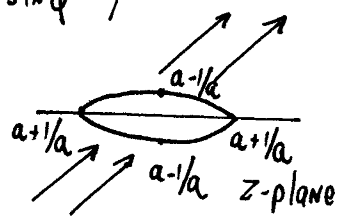
$$\Omega = V_0 \left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}} \right)$$ WITH $a > 1$.

NOW let $z = \zeta + 1/\zeta$. IF $\zeta = a e^{i\varphi}$ THEN



$$z = a e^{i\varphi} + \frac{1}{a} e^{-i\varphi}$$

$$\left. \begin{aligned} x &= (a + 1/a) \cos\varphi \\ y &= (a - 1/a) \sin\varphi \end{aligned} \right\} \text{ ellipse.}$$



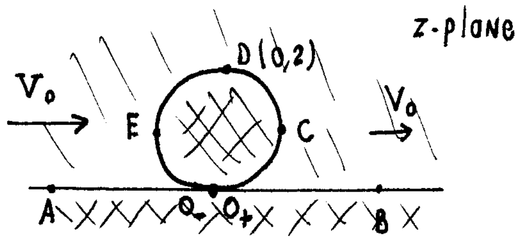
HENCE $$\Omega(z) = V_0 \left[\zeta(z) e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{\zeta(z)} \right].$$

SOLVING FOR $\zeta = \zeta(z)$ WE GET $$\zeta = \frac{1}{2} \left[z + \sqrt{z^2 - 4} \right]$$

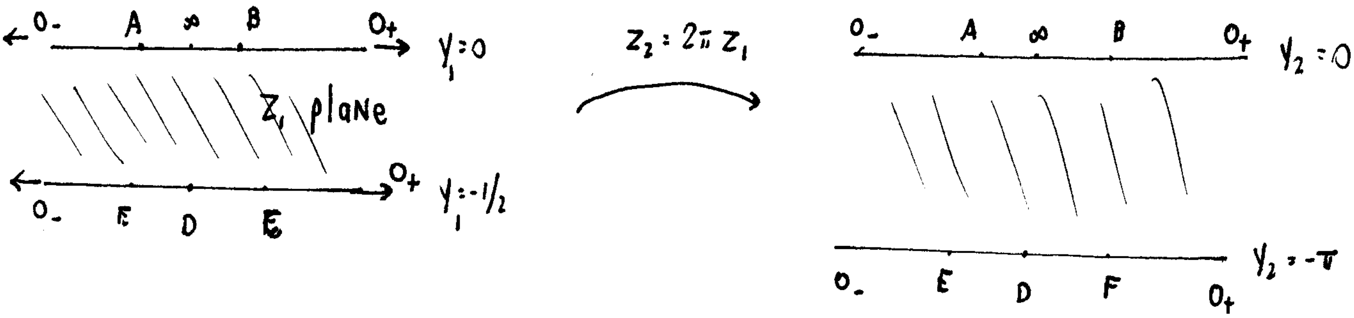
NOW $$\frac{d\Omega}{dz} = \frac{d\Omega}{d\zeta} \frac{d\zeta}{dz} = \frac{d\Omega/d\zeta}{dz/d\zeta} = \frac{V_0 \left[e^{-i\alpha} - a^2 e^{i\alpha}/\zeta^2 \right]}{\left(1 - 1/\zeta^2 \right)} = V_1 - i V_2$$

NOW IF WE SET $\zeta = a e^{i\varphi}$ WE CAN calculate the velocity at each point ON the ellipse.

EXAMPLE FIND THE COMPLEX POTENTIAL $\Omega(z)$ FOR THE FLOW AS SHOWN.



NOW let $z_1 = 1/z$. BOTH the circle and line get mapped to lines. the lines are parallel as shown (NOTE: z REAL $\to z_1$ REAL. $z = 2i \to z_1 = -i/2$

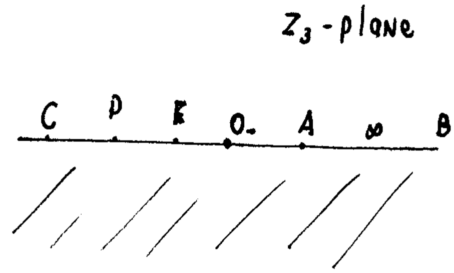


NOW LET $z_3 = e^{z_2} = e^{x_2}(\cos y_2 + i \sin y_2)$.

WE GET $y_2 = 0 \to x_3 = e^{x_2} > 0$

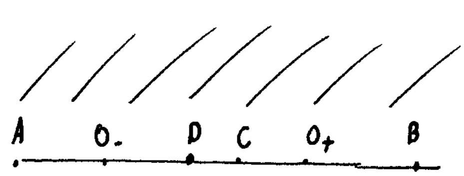
$y_2 = -\pi \to x_3 = -e^{x_2} < 0$

$z_3 = e^{z_2} \implies$



ALSO NOTE $\text{IM}(z_3) < 0$. NOTE $y_2 = -\pi$, $x_2 = 0$ MAPS TO $z_3 = -1$.

NOTICE $z_2 = 0$ IN THE MAP BECOMES $z_3 = 1$. WE DO NOT GET A UNIFORM STREAM YET. NOW LET $z_4 = \dfrac{z_3 + 1}{z_3 - 1}$



NOTE $z_3 \to +\infty \implies z_4 = 1$

$z_3 \to 0 \implies z_4 = -1$

$z_3 \to 1 \implies z_4 = \infty$

NOW $z_3 = e^{2\pi z_1}$ SO THAT

$$z_4 = \frac{e^{2\pi z_1} + 1}{e^{2\pi z_1} - 1} = \frac{e^{\pi z_1} + e^{-\pi z_1}}{e^{\pi z_1} - e^{-\pi z_1}}$$

WITH $z_1 = 1/z$ WE GET $\Omega = A \coth\left(\pi/z\right)$. NEED $A = \pi V_0$.