LAPLACE TRANSFORM AND PERIODIC FUNCTIONS

SUPPOSE $f(t)$ IS T-periodic so that $f(T+t)=f(t)$. THEN, EARLIER WE CALCULATED THAT

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \qquad (*)$$

WE CAN ALSO REPRESENT A PERIODIC FUNCTION WITH PERIOD T IN A DIFFERENT WAY. LET $f_0(t)$ SATISFY

$$f_0(t) = \begin{cases} 0 & t<0 \\ f(t) & 0<t<T \\ 0 & t>T \end{cases}$$

THEN, WE CALCULATE $f(t) = \sum_{n=0}^{\infty} f_0(t-nT) = f_0(t) + f_0(t-T) + \dots + f_0(t-NT) + \dots$

WE CLAIM $f(t+T) = f(t)$. TO SHOW THIS WE CALCULATE

$$f(t+T) = f_0(t+T) + f_0(t) + \dots + f_0(t-(N-1)T) + \dots$$

HOWEVER, $f_0(t+T) = 0$, SO THAT $f(t+T) = f(t)$.

WE THEN CAN ALSO WRITE $F_0(s) \equiv \int_0^\infty e^{-st} f_0(t) = \int_0^T e^{-st} f(t)\,dt$.

AND SO $\mathcal{L}(f(t)) = \dfrac{F_0(s)}{1-e^{-sT}}$.

<u>REMARK</u> SINCE WE ARE INTEGRATING OVER A FINITE RANGE IN $(*)$ THEN WE KNOW THAT $F_0(s)$ IS ANALYTIC FOR ALL s (WE CAN DIFFERENTIATE REPEATEDLY WRT s UNDER THE INTEGRAL SIGN).

HENCE, WE HAVE $F_0(s)$ IS ANALYTIC FOR ALL s AND $F_0(s) \to 0$ FOR $\mathrm{RE}(s) > 0$ WITH $\mathrm{RE}(s) \to +\infty$.

<u>THEOREM</u> SUPPOSE THAT $F(s) = \dfrac{F_0(s)}{1-e^{-sT}}$

AND THAT $F_0(s)$ IS ANALYTIC FOR ALL s WITH $F_0(s) \to 0$ FOR $\mathrm{RE}(s) > 0$ WITH $\mathrm{RE}(s) \to \infty$. SUPPOSE ALSO THAT $F_0(s) \to 0$ FOR $\mathrm{RE}(s) < 0$ AND $|\mathrm{RE}(s)| \to \infty$.

THEN, WE HAVE

(i) $f_0(t) = \mathcal{L}^{-1}[F_0(s)]$ WITH $f_0(t) \equiv 0$ FOR $t<0$ AND $t>T$.

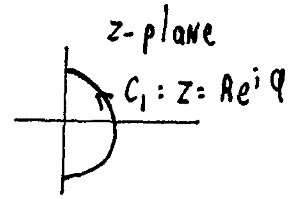
(ii) $f(t) = \mathcal{L}^{-1}[F(s)] = f_0(t) + \sum_{n=1}^{\infty} f_0(t-nT)$

WHERE $f(t+T) = f(t)$.

PROOF WE RECALL JORDAN'S LEMMAS, WHICH CAN BE STATED AS

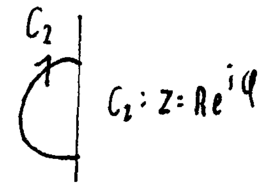
(I) SUPPOSE $\lim_{R\to\infty} F(Re^{i\varphi}) = 0$ FOR $-\pi/2 \le \varphi \le \pi/2$

THEN WHEN $b<0$, $\lim_{R\to\infty} \int_{C_1} F(z)e^{bz}dz = 0$



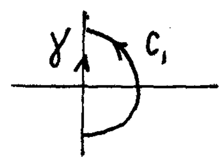
(II) SUPPOSE $\lim_{R\to\infty} F(Re^{i\varphi}) = 0$ FOR $\pi/2 \le \varphi \le 3\pi/2$

THEN when $b>0$ $\lim_{R\to\infty} \int_{C_2} F(z)e^{bz}dz = 0$



WE NOW SHOW FOR (i) THAT $f_0(t) \equiv 0$ FOR $t<0$. SINCE $F_0(s)$ IS ANALYTIC THE MELLIN-INVERSION FORMULA GIVES

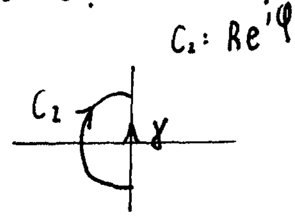
$$f_0(t) = \frac{1}{2\pi i}\int_\gamma F_0(s)e^{st}ds$$



γ = imaginary AXIS

WE DEFORM γ TO $C_1 : z = Re^{i\varphi}$ $|\varphi| < \pi/2$ AND LET $R\to\infty$. SINCE $t<0$ WE CAN USE (I) ABOVE TOGETHER WITH ANALYTICITY OF $F_0(s)$ TO OBTAIN

$$f_0(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_1} F_0(s)e^{st}ds = 0 \quad \text{FOR } t<0.$$



NOW WE SHOW $f_0(t) \equiv 0$ FOR $t>T$.

WE WRITE $f_0(t) = \frac{1}{2\pi i}\int_\gamma \left(F_0(s)e^{sT}\right)e^{s(t-T)}dt$

DEFINE $\hat{F}_0(s) = F_0(s)e^{sT}$. THEN $|\hat{F}_0(s)| \to 0$ FOR $s = Re^{i\varphi}$ WITH $\pi/2 < \varphi < 3\pi/2$. NOTICE ALSO $t-T>0$ SO $|e^{s(t-T)}| \to 0$ FOR $RE(s)<0$

AND $|\mathrm{RE}(s)| \to \infty$.

HENCE BY DEFORMING CONTOUR FROM γ TO C_2:

$$f_0(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_2} \hat{F}_0(s) e^{s(t-T)} ds = 0 \quad \text{BY JORDAN'S LEMMA (II).}$$

NOW WE PROVE (ii). SUPPOSE $f_0(t) = \mathcal{L}^{-1}[F_0(s)]$. THEN,

$$F(s) = \frac{F_0(s)}{1-e^{-sT}} = F_0(s) + F_0(s)e^{-sT} + F_0(s)e^{-2sT} + F_0(s)e^{-3sT} + \ldots$$

SO $f(t) = \mathcal{L}^{-1}[F(s)] = f_0(t) + \sum_{n=1}^{\infty} f_0(t-nT)\, u_{nT}(t)$

BUT $u_{nT}(t) = \begin{cases} 0 & t-nT<0 \\ 1 & t-nT>0 \end{cases}$

HENCE $f(t) = f_0(t) + \sum_{n=1}^{\infty} f_0(t-nT)$ WHICH IS T-PERIODIC $\square$

<u>APPLICATION</u> SUPPOSE THAT WE WANT TO SOLVE

(*) $L(y) = r$ WITH $r(t+T) = r(t)$ AND ZERO INITIAL CONDITIONS.

HERE $L(y)$ IS A DIFFERENTIAL OPERATOR WITH CONSTANT COEFFICIENTS (SUCH AS $L(y) = y'' + 5y' + 6y$). THEN WE CAN TAKE LAPLACE TRANSFORMS OF (*) TO OBTAIN

$$Y(s) = R(s)\,H(s) \qquad R(s) = \frac{R_0(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st} r(t)\,dt}{1-e^{-sT}}$$

AND $H(s)$ IS THE TRANSFER FUNCTION.

HENCE $$Y(s) = \frac{R_0(s)\,H(s)}{1-e^{-sT}} \qquad (1)$$

WE WANT TO REWRITE THIS IN THE FORM

(2) $$Y(s) = \frac{P_0(s)}{1-e^{-sT}} + A(s)$$ WHERE $P_0(s)$ IS ANALYTIC

WITH $|P_0(s)| \to 0$ AS $t \to \infty$. IF WE CAN WRITE (1) AS (2) WE CAN THEN USE THE THEOREM TO FIND $y(t) = \mathcal{L}^{-1}[Y(s)]$.

TO DO SO, WE DEFINE

$$a(t) = \mathcal{L}^{-1}[A(s)], \qquad p_0(t) = \mathcal{L}^{-1}[P_0(s)] \to p_0(t) = 0 \quad t<0,\ t>T.$$

THEN,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \underbrace{p_0(t) + \sum_{n=1}^{\infty} p_0(t-nT)}_{p(t)} + a(t)$$

NOTICE $p(t+T) = p(t)$ IS THE PERIODIC PART OF THE SOLUTION AND $a(t)$ IS THE APERIODIC OR TRANSIENT RESPONSE. TYPICALLY WE CALL $p(t)$ THE PERIODIC OR "STEADY-STATE" RESPONSE.

THE CRUX OF THE CALCULATION THEN IS TO WRITE (1) AS (2). WE SUPPOSE THAT $H(s)$ HAS SIMPLE POLES AT $s = s_j$ FOR $j = 1, 2, \dots, n$ AND THAT THEY DO NOT COINCIDE WITH THE ZEROES OF $1 - e^{-sT} = 0$. ASSUME THAT $H(s)$ HAS NO OTHER SINGULARITIES.

THEN WE CAN WRITE (1) AS

$$(3) \qquad Y(s) = \left(\frac{H(s) R_0(s)}{1 - e^{-sT}} - A(s) \right) + A(s)$$

AND CHOOSE $A(s)$ TO "CANCEL" THE SIMPLE POLES AT $s = s_j$ $j = 1, \dots, n$ OF THE FIRST TERM. THIS MAKES THE FIRST TERM ANALYTIC EXCEPT AT $1 = e^{-sT}$.

SUPPOSE $H(s) = \dfrac{a_j}{s - s_j}$ + ANALYTIC TERMS NEAR $s = s_j$

THEN CHOOSE
$$A(s) = \sum_{j=1}^{n} \frac{R_0(s_j)}{1 - e^{-s_j T}} \frac{a_j}{s - s_j}$$

THEREFORE (3) BECOMES

$$Y(s) = \frac{P_0(s)}{1-e^{-sT}} + A(s) \qquad P_0(s) = H(s)R_0(s) - A(s)(1-e^{-sT})$$

$\rightarrow P_0(s)$ IS ANALYTIC

WHERE $p_0(t) = \mathcal{L}^{-1}[P_0(s)] = 0$ FOR $t<0$ AND $t>T$ (BY THE THEOREM)

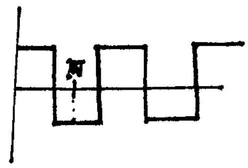
THEN $$y(t) = \sum_{n=0}^{\infty} p_0(t-nT) + a(t)$$

where $a(t) = \mathcal{L}^{-1}[A(s)] = \sum_{j=1}^{n} \frac{R_0(s_j)}{1-e^{-s_jT}}\, a_j e^{s_j t}$ IS THE TRANSIENT RESPONSE AND SATISFIES $a(t) \rightarrow 0$ AS $t \rightarrow \infty$ WHEN $RE(S_j) < 0$ $j=1,\dots,n$.

<u>EXAMPLE 1</u> FIND $f(t)$ SO THAT $F(s) = \mathcal{L}(f(t)) = \frac{1}{s} \tanh(sT/2)$

WE WRITE $$F(s) = \frac{1}{s}\left(\frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}}\right) = \frac{1}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}}\right) = \frac{1}{s}(1-e^{-sT})(1+e^{-sT})^{-1}$$

NOW FOR $|e^{-sT}| < 1 \rightarrow (RE\, s > 0)$ WE EXPAND TO OBTAIN



$$F(s) = \frac{1}{s}(1-e^{-sT})(1-e^{-sT}+e^{-2sT}-e^{-3sT}+\dots)$$

SO $F(s) = \frac{1}{s}(1-2e^{-sT}+2e^{-2sT}-2e^{-3sT}+\dots\dots) \rightarrow f(t) = 1-2u_T(t)+2u_{2T}(t)-2u_{3T}(t)+\dots$

NOTICE THAT THIS FUNCTION IS PERIODIC WITH PERIODIC $2T$.

FOR THE SECOND METHOD WE WRITE

$$F(s) = \frac{1}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}}\right)\left(\frac{1-e^{-sT}}{1-e^{-sT}}\right) = \frac{P_0(s)}{1-e^{-2sT}} \qquad P_0(s) = \frac{1}{s} - \frac{2}{s}e^{-sT} + \frac{1}{s}e^{-2sT}$$

NOTICE THAT $P_0(s)$ IS ANALYTIC (EVEN AT $s=0$). ALSO $|P_0(s)| \rightarrow 0$ AS $RE(s) \rightarrow +\infty$.

HENCE $$f(t) = \sum_{n=0}^{\infty} p_0(t-n2T)$$ where $p_0(t) = \mathcal{L}^{-1}[P_0(s)] = 1 - 2u_T(t) + u_{2T}(t)$ ON $0<t<2T$.

HENCE $p_0(t) = 1-2u_T(t)$ ON $0<t<2T$ SINCE $u_{2T}(t) = 0$ FOR $t<2T$.

EXAMPLE SUPPOSE $y'' + 5y' + 6y = f(t)$ $\quad f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 \le t \le 2 \end{cases}$

WITH $f(t+2) = f(t)$. ASSUME ALSO $y(0) = y'(0) = 0$. CALCULATE $y(t)$ IN THE FORM $y(t) = p(t) + a(t)$ where $p(t+2) = p(t)$ AND $a(t)$ is aperiodic.

SOLUTION TAKE LAPLACE TRANSFORMS:

$$(s^2 + 5s + 6)Y(s) = \frac{F_0(s)}{1-e^{-2s}} \qquad \mathcal{L}(f(t)) = \frac{F_0(s)}{1-e^{-2s}} \qquad F_0(s) = \int_0^2 f(t)e^{-st}\,dt.$$

HENCE
$$Y(s) = \frac{F_0(s)\,H(s)}{1-e^{-2s}} \qquad H(s) = \frac{1}{s^2+5s+6} = \frac{-1}{s+3} + \frac{1}{s+2}$$

NOW WRITE
$$Y(s) = \left(\frac{F_0(s)H(s)}{1-e^{-2s}} - A(s)\right) + A(s) \qquad A(s) = \frac{F_0(-2)}{(1-e^4)}\frac{1}{s+2} - \frac{F_0(-3)}{(1-e^6)(3+s)}$$

NOW
$$Y(s) = \frac{P_0(s)}{1-e^{-2s}} + A(s)$$

where
$$P_0(s) = F_0(s)H(s) - A(s)(1-e^{-2s}) \qquad p_0(t) = \mathcal{L}^{-1}[P_0(s)] = 0 \text{ FOR } t<0 \text{ AND } t>2$$

$$a(t) = \mathcal{L}^{-1}(A(s)) = \frac{F_0(-2)}{(1-e^4)}e^{-2t} - \frac{F_0(-3)}{(1-e^6)}e^{-3t} \longrightarrow \text{transient response}$$

NOW calculate
$$p_0(t) = \mathcal{L}^{-1}[P_0(s)] = \mathcal{L}^{-1}\left[\frac{F_0(s)}{s^2+5s+6} - \frac{F_0(-2)}{(1-e^4)(s+2)} + \frac{F_0(-3)}{(1-e^6)(s+3)} + A(s)e^{-2s}\right]$$

BUT $\mathcal{L}^{-1}[A(s)e^{-2s}] = 0$ FOR $t > 2$.

HENCE
$$p_0(t) = \mathcal{L}^{-1}\left[\frac{F_0(s)}{s^2+5s+6}\right] - \frac{F_0(-2)}{(1-e^4)}e^{-2t} + \frac{F_0(-3)}{(1-e^6)}e^{-3t} \quad \text{ON } 0<t<2.$$

WE CALCULATE
$$F_0(s) = \int_0^1 t e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^1 = \frac{1}{s}(1-e^{-s}).$$

NOW $\quad F_0(-2) = -\frac{1}{2}(1-e^2) \qquad F_0(-3) = -\frac{1}{3}(1-e^3)$

HENCE $\quad a(t) = \mathcal{L}^{-1}(A(s)) = -\frac{1}{2}\frac{(1-e^2)}{(1-e^4)}e^{-2t} + \frac{1}{3}\frac{(1-e^3)}{(1-e^6)}e^{-3t}$

$$a(t) = -\frac{1}{2}\frac{(1-e^2)}{(1-e^2)(1+e^2)}e^{-2t} + \frac{1}{3}\frac{(1-e^3)}{(1-e^3)(1+e^3)}e^{-3t}$$

HENCE, (**) $\quad a(t) = -\frac{1}{2}\frac{1}{1+e^2}e^{-2t} + \frac{1}{3}\frac{1}{1+e^3}e^{-3t}.$

NOW $\quad p_0(t) = \mathcal{L}^{-1}\left[\frac{(1-e^{-s})}{s(s+3)(s+2)}\right] + \frac{1}{2}\frac{1}{1+e^2}e^{-2t} - \frac{1}{3}\frac{1}{1+e^3}e^{-3t}$ ON $0<t<2$

NOW CALCULATING $\quad \frac{1}{s(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}$

HENCE $\quad p_0(t) = \mathcal{L}^{-1}\left[\frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)} - \frac{e^{-s}}{6s} + \frac{e^{-s}}{2(s+2)} - \frac{e^{-s}}{3(s+3)}\right]$

$$+\frac{1}{2}\frac{1}{1+e^2}e^{-2t} - \frac{1}{3}\frac{1}{1+e^3}e^{-3t} \quad \text{ON } 0<t<2$$

THIS GIVES

$$p_0(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} - u_1(t)\left(\frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)}\right)$$

$$+\frac{1}{2}\frac{1}{1+e^2}e^{-2t} - \frac{1}{3}\frac{1}{1+e^3}e^{-3t} \quad \text{ON } 0<t<2$$

HENCE $\quad y(t) = \underbrace{\sum_{n=0}^{\infty} p_0(t-2n)}_{y_p(t)} + a(t)$

WITH $a(t)$ GIVEN ABOVE IN (**)