Math 400-201 2013-2014 Term 2

Applied Partial Differential Equations

Section 1: First-Order Partial Differential Equations

In this section, we will develop the method of characteristics to solve first order partial differential equations of the following type

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \qquad (1.0)$$

Linear: $a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)$ (1.1)

Quasilinear: a, b, c depends on u, e.g.

$$u_x + u\,u_y = 0 \qquad (1.2)$$

Fully nonlinear: $F(x,y,u,u_x,u_y)=0$

The method will also work well for $p(x,t)$ satisfying

(1.3) $p_t + f'(p)\,p_x = 0$ $\longrightarrow$ special case of quasi-linear

(arises in traffice flow problems)

(1.4) $\phi_x^2 + \phi_y^2 = n(x,y)$ $\longrightarrow$ Eikonal equation of optics

- special case of fully nonlinear 1st order PDE

Let us now solve (1.1), starting from the simplest

$\boxed{\text{Key Idea of Solving (1.0): Reduce to System of ODEs, via Method of Characteristics}}$

Example 1: $u_x = 0$

This means that u doesnot depend on x, that is u only depends on y. Hence general solution is

$$u = f(y) \qquad \text{— (1.5)}$$

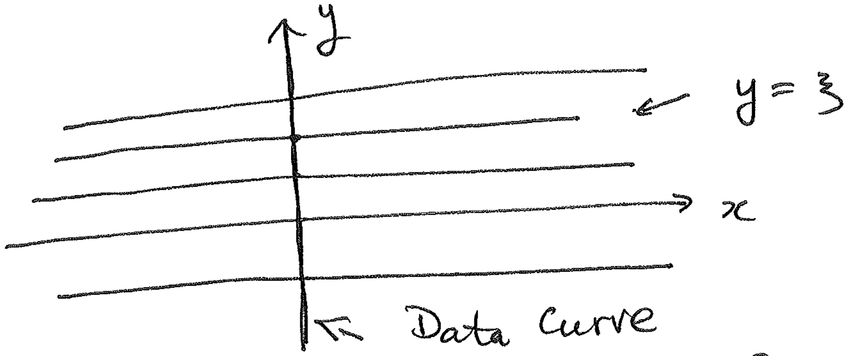
To find out what $f(y)$ is, we have to specify u on a particular curve, e.g.

$$u(0, y) = y^2 \qquad \text{— (1.6)}$$

Then the particular solution is

$$u(x, y) = y^2 = f(y) \qquad \text{— (1.7)}$$

Let us look at (1.5) more closely: $u \equiv$ Constant on all lines parallel to x-axis



On the line (curve) $y = \xi$, $u = f(\xi) \equiv$ Constant

Intial condition: $u(0, y) = y^2$. Data Curve $\{(0, y)\}: x = 0$

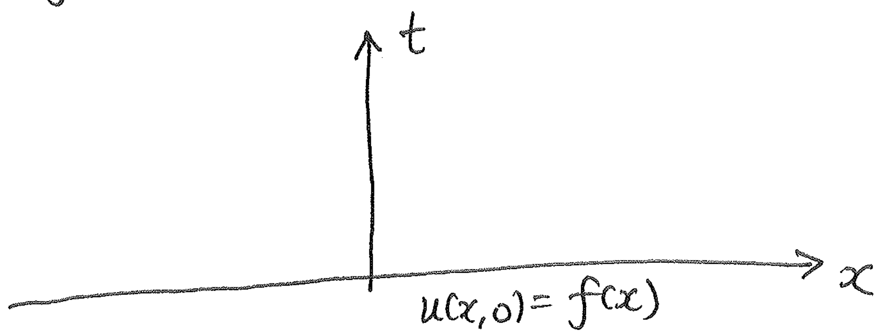
How can we find these "parallel curves"?

Example 2. Solve

$$u_t + c u_x = 0, \quad -\infty < x < +\infty, \quad t > 0 \qquad -(1.8)$$

with $u(x,0) = f(x)$ as DATA

where $c =$ constant



Let $x = x(t)$ be an arbitrary curve.

Then the value of $u(x, t)$ on the curve is

$u(x(t), t)$

By the chain rule

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

$$= u_x \frac{dx}{dt} + u_t$$

Now if we choose $\boxed{\frac{dx}{dt} = c}$ $\qquad -(1.9)$

then it follows that (1.8) is solved if

on $\frac{dx}{dt} = c$ we have $\frac{d\,u(x(t),t)}{dt} = 0$

Thus the original PDE problem becomes two ODE problems

$$u_t + c u_x = 0 \iff \begin{cases} \dfrac{dx}{dt} = c \\ \dfrac{du}{dt} = 0 \end{cases} \qquad (1.10)$$

Initial Condition for (1.8) obtained by parametrizing x-axis by ξ so that

DATA Curve : $\rightarrow$ $x = \xi$, $U = f(\xi)$ at $t=0$, $-\infty < \xi < +\infty$

we are left to solving

$$\text{ODE} \quad \begin{cases} \dfrac{dx}{dt} = c \\ \dfrac{du}{dt} = 0 \end{cases} \qquad (1.11)$$

$$\text{IC} \quad \begin{cases} x(0) = \xi \\ u(0) = f(\xi) \end{cases} \qquad (1.12)$$

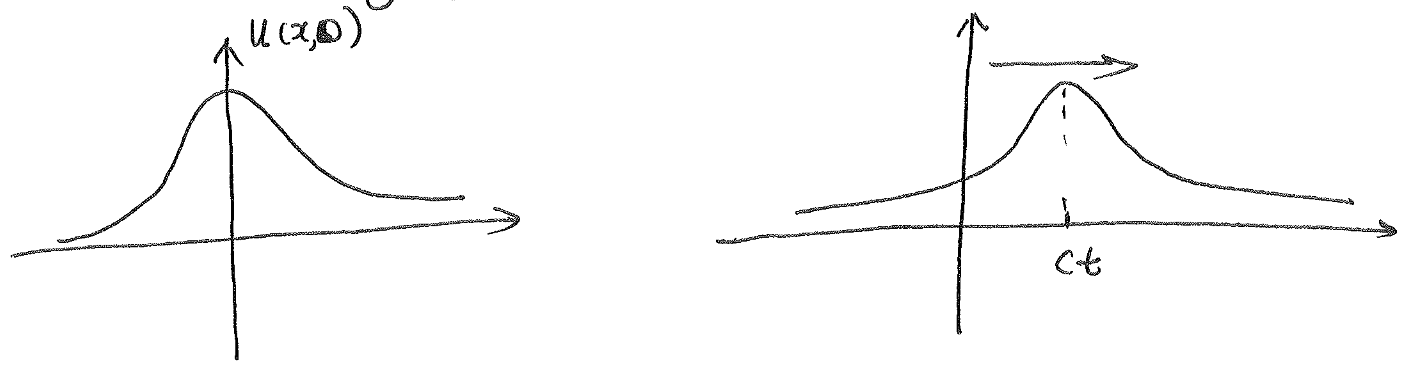
Let us now solve (1.11)–(1.12) :

$$\frac{dx}{dt} = c, \; x(0) = \xi \quad \Rightarrow \quad x = ct + \xi$$

$$\frac{du}{dt} = 0, \; u(0) = f(\xi) \quad \Rightarrow \quad u = f(\xi)$$

Eliminating ξ we obtain $\quad u(x,t) = f(x - ct)$. $\qquad$ (1.13)

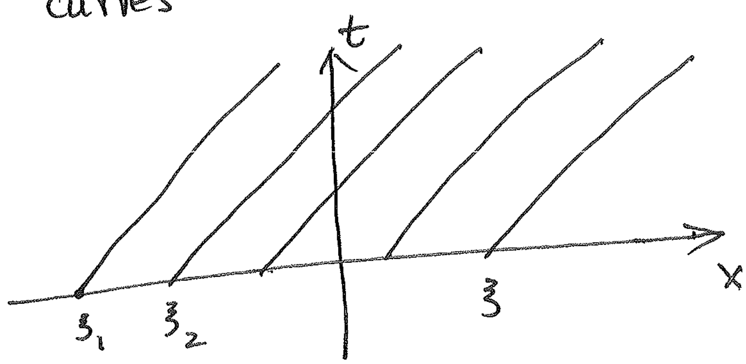
Geometric meaning of (1.13):



$t=0 : u(x,0)$ $\qquad$ $u(x,t)$

waves traveling to the right at speed c.

These curves $x=ct+\xi$



The curves $x=ct+\xi$, $-\infty<\xi<\infty$ are called the characteristic curves.

On these curves, the PDE reduces to the ODE

$$\frac{du}{dt}=0.$$

Similar methods work for

Example 3: $a u_x + b u_y = 0$

Solution: Let $\{(x(s), y(s))\}$ be a given curve. Then

$$\frac{d}{ds} u(x(s), y(s)) = u_x \frac{dx}{ds} + u_y \frac{dy}{ds}$$

Now choose

$$\begin{cases} \dfrac{dx}{ds} = a \\ \dfrac{dy}{ds} = b \end{cases}$$

then $\dfrac{du}{ds} = 0$

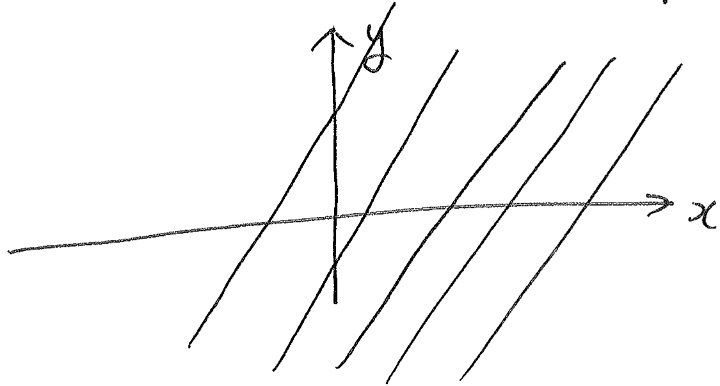
So we just need to solve

$$\begin{cases} \dfrac{dx}{ds} = a & \rightarrow x = as + \xi_1 \\ \dfrac{dy}{ds} = b & \rightarrow y = bs + \xi_2 \\ \dfrac{du}{ds} = 0 & \rightarrow u = \text{Constant on } \begin{cases} x = as + \xi_1 \\ y = bs + \xi_2 \end{cases} \end{cases}$$

eliminating s: $by - ax = b\xi_2 - a\xi_1 = \xi$

$u = f(\xi) = f(by - ax)$

Characterstic curve: $\dfrac{dx}{a} = \dfrac{dy}{b} \Rightarrow y = \dfrac{a}{b}x + c$



In general, we can solve problems

$$a(x,y)\,u_x + b(x,y)\,u_y = 0$$

u is given on DATA Curve

by the following steps:

Step 1: Solve the ODE

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}$$

$\Rightarrow$ We obtain either $y = y(x;\xi)$ (explicit)

or $F(x,y;\xi) = 0$ (implicit)

Step 2: Solve the ξ in terms of x, y

Step 3: General solutions are given by

$$u = f(\xi) = f(\xi(x,y))$$

Step 4: Partial solution: Substitute the initial condition

Example 4: Solve

$$\begin{cases} u_x + x^2 u_y = 0 \\ u(x,0) = e^{x} \end{cases}$$

Solution: Characteristics:

$$\frac{dx}{1} = \frac{dy}{x^2}$$

$$\Rightarrow \quad x^2 dx = dy$$

$$\frac{x^3}{3} = y + \xi_1$$

$$x^3 = 3y + \xi$$

$$\xi = x^3 - 3y$$

general solution: $u = f(\xi)$

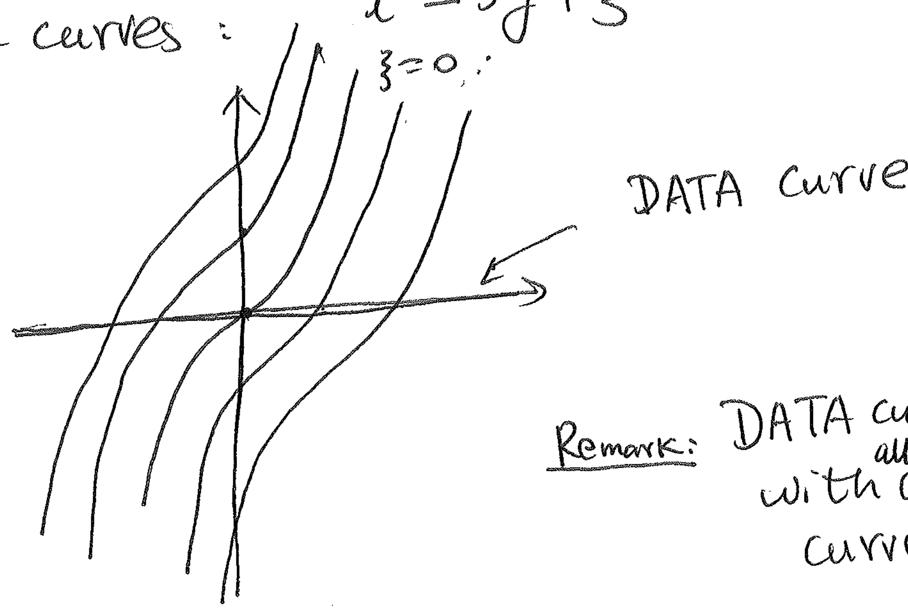
Particular solution: $u = f(x^3 - 3y)$.

Now $f(x,0) = e^x \quad \Rightarrow \quad f(x^3) = e^x$

$$\Rightarrow f(\xi) = e^{\xi^{\frac{1}{3}}}$$

$$u = e^{(x^3 - 3y)^{\frac{1}{3}}}$$

Characteristic curves : $x^3 = 3y + \xi$



Remark: DATA curve intersects with all characteristic curve

Example 5 Solve

$$\begin{cases} u_x + x u_y = 0 \\ u(x,0) = e^x \end{cases}$$

Sol'n: Characteristic Curves:

$$\frac{dx}{1} = \frac{dy}{x}$$

$$\Rightarrow \quad x\,dx = dy$$

$$\frac{x^2}{2} = y + C$$

$$\Rightarrow \quad x^2 = 2y + \xi$$

$$\xi = x^2 - 2y$$

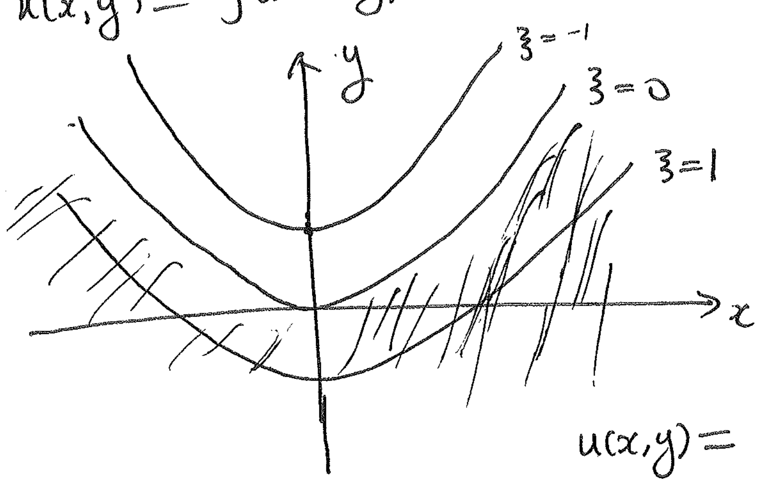
$$u = f(\xi) = f(x^2 - 2y)$$

DATA Curve: $\cancel{u}(x,0) = e^x \Rightarrow f(x^2) = e^x$

$$\xi = x^2 \Rightarrow x = \pm\sqrt{\xi}, \quad f(\xi) = e^{\sqrt{\xi}} \text{ or } e^{-\sqrt{\xi}}$$

Sol'ns are

$$u(x,y) = f(x^2 - 2y) = e^{\sqrt{x^2-2y}} \quad \text{or} \quad e^{-\sqrt{x^2-2y}}$$



Domain of definition

$$x^2 \geq 2y$$

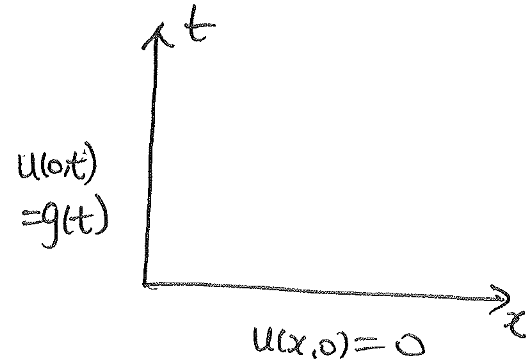
$$u(x,y) = \begin{cases} e^{\sqrt{x^2-2y}}, & x > 0 \\ e^{-\sqrt{x^2-2y}}, & x < 0 \end{cases}$$

Example 4': $\begin{cases} u_x + y u_y = 0 \\ u(0, y) = y^3 \end{cases}$

4'': $u_x + 2xy^2 u_y = 0$

Example 6. Find the solution to the signalling problem

$$\begin{cases} u_t + c u_x = 0, & 0 < x < \infty, \quad 0 < t < \infty \\ u(0,t) = g(t), \quad u(x,0) = 0, & x \geq 0 \end{cases}$$



Sol'n: Characteristic curve

$$\frac{dt}{1} = \frac{dx}{c} \Rightarrow x - ct = \xi$$

General sol'n: $u = f(\xi) = f(x - ct)$

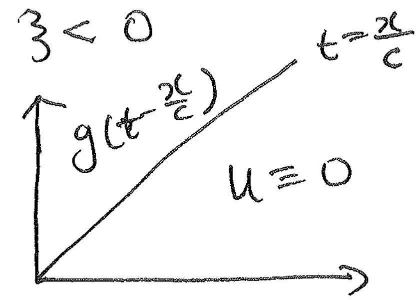
Initial condition: $u(x,0) = 0 = f(x)$; $x \geq 0$.

$$f(\xi) \equiv 0, \text{ when } \xi \geq 0$$

$$u(0,t) = g(t) \Rightarrow f(0 - ct) = g(t)$$

$$\Rightarrow f(-ct) = g(t),$$

$$f(\xi) = g\left(-\frac{\xi}{c}\right), \quad \xi < 0$$



$$f(\xi) = \begin{cases} 0, & \xi \geq 0 \\ g\left(-\frac{\xi}{c}\right), & \xi < 0 \end{cases}$$

$$u(x,t) = \begin{cases} 0, & x - ct \geq 0 \\ g\left(-\frac{x-ct}{c}\right) = g\left(t - \frac{x}{c}\right), & x - ct < 0 \end{cases}$$

- Finite speed of propagation
- If $g(0) \neq 0$ singularity propagated

Next we use the method of characteristic to solve more general first-order PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

On the curve: $\begin{cases} \dfrac{dx}{ds} = a(x,y) \\ \dfrac{dy}{ds} = b(x,y) \end{cases}$

we have $\dfrac{du}{ds} = c(x,y,u)$

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \iff \begin{cases} \dfrac{dx}{ds} = a(x,y) \\ \dfrac{dy}{ds} = b(x,y) \\ \dfrac{du}{ds} = c(x,y,u) \end{cases}$$

in short: $\dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$

Example 7. $u_t + e^{-t}u_x = -u$

with $u(x,0) = f(x)$, $-\infty < x < \infty$

sol'n: Characteristic Curve:

$$\frac{dt}{1} = \frac{dx}{e^{-t}} = \frac{du}{-u}$$

$$\begin{cases} \dfrac{dx}{dt} = e^{-t}, & x(0) = \xi \\ \dfrac{du}{dt} = -u, & u(0) = f(\xi) \end{cases}$$

$x = 1 - e^{-t} + \xi$

u $\frac{du}{dt} = -u \Rightarrow u = f(\xi)\, e^{-t}$

$u = f(x - 1 + e^{-t})\, e^{-t}$

Example 8. $x u_t + u_x + t u = 0$

$u(0,t) = f(t), \quad 0 \le t \le 1$

Sol'n: ~~Characteristic cur~~ Rewrite the PDE

$x u_t + u_x = -t u$

$$\frac{dt}{x} = \frac{dx}{1} = \frac{du}{-tu}$$

$\frac{dt}{dx} = x \quad , \quad t(0) = \xi \quad \Rightarrow \quad t = \frac{x^2}{2} + \xi$

$\frac{du}{dx} = -tu \quad , \quad u(0) = f(\xi)$

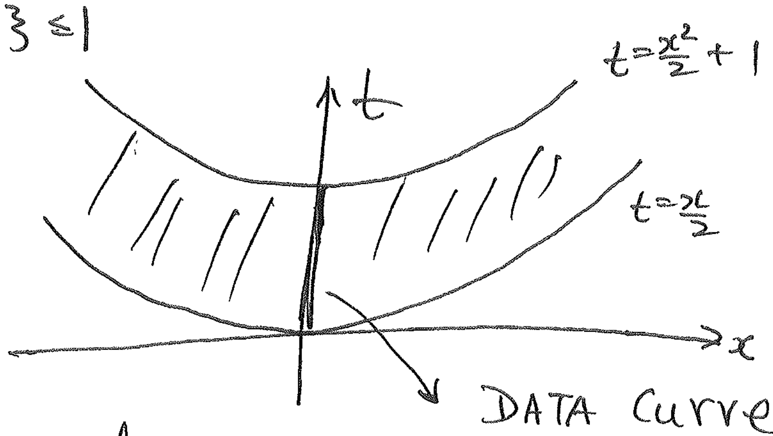
$$\frac{du}{dx} = -\left(\frac{x^2}{2} + \xi\right) u$$

$$u = c\, e^{-\left(\frac{x^3}{6} + \xi x\right)}$$

$$u = f(\xi)\, e^{-\left(\frac{x^3}{6} + \xi x\right)}$$

The characteristic are the set of curves

$$t = \frac{x^2}{2} + \xi, \qquad 0 \le \xi \le 1$$



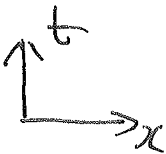
In the shaded region, the solution is (elimiting ξ)

$$u = e^{-\left[\frac{x^3}{6} + \left(t - \frac{x^2}{2}\right)x\right]} f\left(t - \frac{x^2}{2}\right)$$

$$= e^{\frac{x^3}{3} - xt} f\left(t - \frac{x^2}{2}\right)$$

The solution is undefined outside this region

Method of Characteristics: General Theory



Consider the linear 1st order PDE

$$a(x,t)\, u_x + b(x,t)\, u_t = c(x,t)\, u + d(x,t)$$

with u = given on some curve C in x-t plane

consider some curves $t = t(s)$, $x = x(s)$, parametrized by s (e.g. s as arclength along the curve).

Then on $t = t(s)$, $x = x(s)$ we have

$$\frac{du}{ds} = \frac{d}{ds} u(x(s), t(s)) = u_x \frac{dx}{ds} + u_t \frac{dt}{ds}$$

Now choose

$$\frac{dx}{ds} = a(x,t), \qquad \frac{dt}{ds} = b(x,t)$$

Then on these curves the PDE is solved if we impose $\frac{du}{ds} = cu + d \rightarrow$ PDE reduces to ODE

on characteristics.

Let C be the DATA curve. We parametrize C by $x = x_0(\xi)$, $t = t_0(\xi)$ for $0 \le \xi \le 1$ (WLOG).

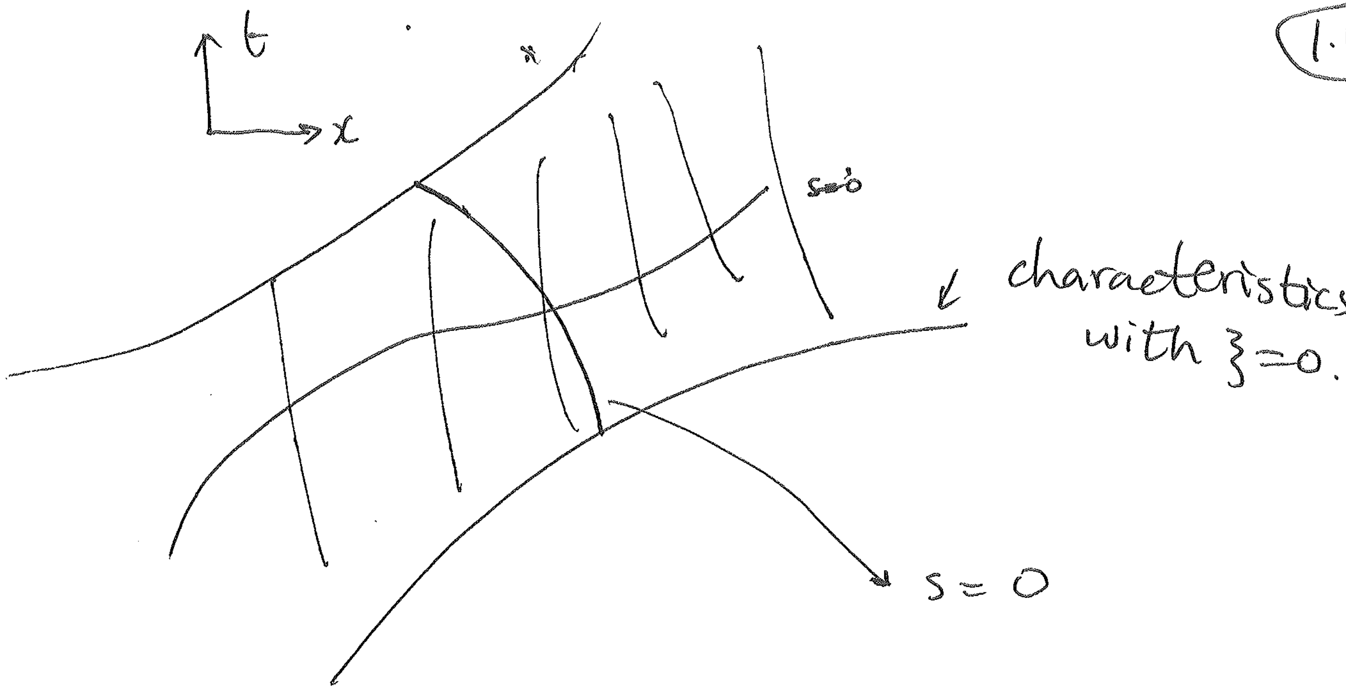
On the Data curve C we have the data $u = u_0(\xi)$

Suppose that $s = 0$ corresponds to the intersection of characteristics and the data curve.

Then we must solve ODE system

$$\left.\begin{aligned} \frac{dx}{ds} &= a(x,t), \quad x(0) = x_0(\xi) \\ \frac{dt}{ds} &= b(x,t), \quad t(0) = t_0(\xi) \end{aligned}\right\} \text{characteristic}$$

$$\frac{du}{ds} = c(x,t)u + d(x,t), \quad u(0) = u_0(\xi)$$



shaded region: where U will be defined

we integrate the ODEs to obtain

$$\begin{cases} x = x(s, \xi) \\ t = t(s, \xi) \\ u = u(s, \xi) \end{cases}$$

Each curve with ξ ~~fixed~~ is a characteristic

If we can eliminate ξ to find $s = s(x,t)$, $\xi = \xi(x,t)$

then $u(x, t) = u(s(x,t), \xi(x,t))$

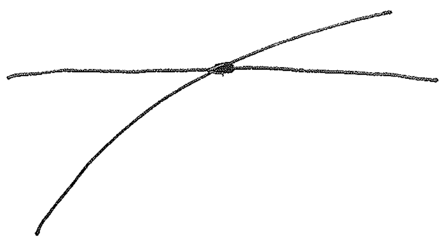
The region spanned by characteristics intersecting data curve is where u is defined.

Difficulties arise when

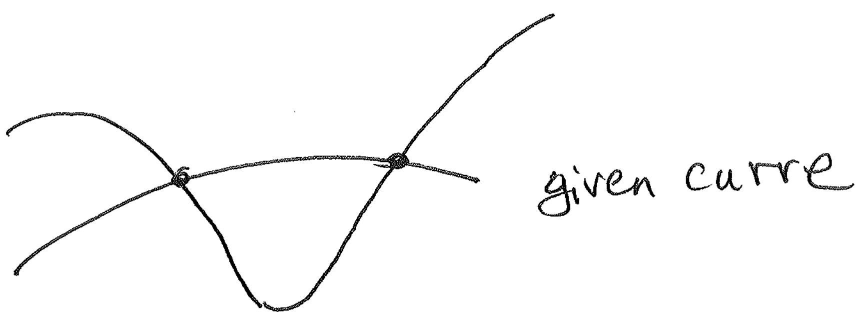
(i) characteristic curves cross at some points (s)
(two)

→ can't eliminate for $s = s(x,t)$
$\xi = \xi(x,t)$

→ u develops a singularity



(ii) characteristic hits data curve twice → no solution in general



(iii) data cure is a characteristic → either no sol'n or infinite # of solutions.

Example 8. $x u_t + u_x + t u = 0$
$u(0,t) = f(t), 0 < t \le 1$

(Example 8 again)

$$\begin{cases} \dfrac{dx}{ds} = 1, \quad x(0) = 0 & \longrightarrow \quad x = s \\[2ex] \dfrac{dt}{ds} = x, \quad t(0) = \xi & \longrightarrow \quad t = \dfrac{s^2}{2} + \xi \\[2ex] \dfrac{du}{ds} = -tu, \quad u(0) = f(\xi) & \longrightarrow \quad \dfrac{du}{ds} = -\left(\dfrac{s^2}{2} + \xi\right) u \rightarrow u = f(\xi)\, e^{-\left(\frac{s^3}{6} + \xi s\right)} \end{cases}$$

eliminating s, ξ: $\quad s = x, \quad \xi = t - \dfrac{x^2}{2}$

$$u = f\left(t - \frac{x^2}{2}\right) e^{\frac{x^3}{3} - tx}$$

Example 9 $\quad x u_x - t u_t = 2x - u$

$u = 0$ on $t = 1$ for $-\infty < x < +\infty$

Solution: $$\frac{dx}{x} = \frac{dt}{-t} = \frac{du}{2x - u}$$

$$\cancel{\frac{dx}{dt} = \frac{x}{-t}, \quad x(0) = \xi} \qquad \cancel{\frac{du}{dt} = \frac{2x-u}{-t}}$$

$$\begin{cases} \dfrac{dx}{ds} = x, \quad x(0) = \xi \\[2ex] \dfrac{dt}{ds} = -t, \quad t(0) = 1 \\[2ex] \dfrac{du}{ds} = 2x - u, \quad u(0) = 0. \end{cases}$$

$$x = \xi e^s, \quad t = e^{-s}, \quad \frac{du}{ds} = 2\xi e^s - u, \quad u(0) = 0$$

$$\frac{d}{ds}(e^s u) = 2\xi e^{2s} \Rightarrow e^s u = \xi e^{2s} + C, \quad u(0) = 0, \quad C = -\xi$$

$$u = \xi(e^s - e^{-s}), \quad x = \frac{\xi}{t} \Rightarrow \xi = xt$$

$u = xt\left(\frac{1}{t} - t\right) = x - xt^2$ with characteristics

$x = \dfrac{\xi}{t}, \quad -\infty < \xi < \infty$

Method 2:

$$\left.\begin{aligned} \frac{dx}{dt} &= -\frac{x}{t} \\ x(1) &= \xi \end{aligned}\right\} \quad x = \frac{\xi}{t}$$

$$\frac{du}{dt} = -\frac{2x}{t} + \frac{u}{t}, \quad u(1) = 0.$$

$$\frac{du}{dt} = -\frac{2}{t^2}\xi + \frac{u}{t}, \quad u(1) = 0$$

$$\frac{d}{dt}\left(\frac{u}{t}\right) = -\frac{2\xi}{t^3} \qquad \frac{1}{t}u = \frac{2\xi}{2t^2} + c \;\Rightarrow\; u = \frac{\xi}{t} + ct$$

$$t = 1 \;\Rightarrow\; u = 0, \qquad u = \frac{\xi}{t} - t\xi \quad \text{or} \quad u = x - xt^2$$

Example 10.
$$\begin{cases} x^2 u_x + xy\, u_y = u^2 \\ u = 1 \text{ on } x = y^2 \end{cases}$$

sol'n:
$$\begin{cases} \dfrac{dx}{ds} = x^2, & x(0) = \xi^2 \\ \dfrac{dy}{ds} = xy, & y(0) = \xi \\ \dfrac{du}{ds} = u^2, & u(0) = 1 \end{cases}$$

$$x^{-2}dx = ds \;\Rightarrow\; -\frac{1}{x} = s - \frac{1}{\xi^2} \;\Rightarrow\; x = \frac{\xi^2}{1-\xi^2 s}$$

$$\frac{dy}{ds} = \frac{\xi^2}{1-\xi^2 s}y, \quad \therefore\; y = \frac{\xi}{1-\xi^2 s} \qquad \frac{x}{y} = \xi$$

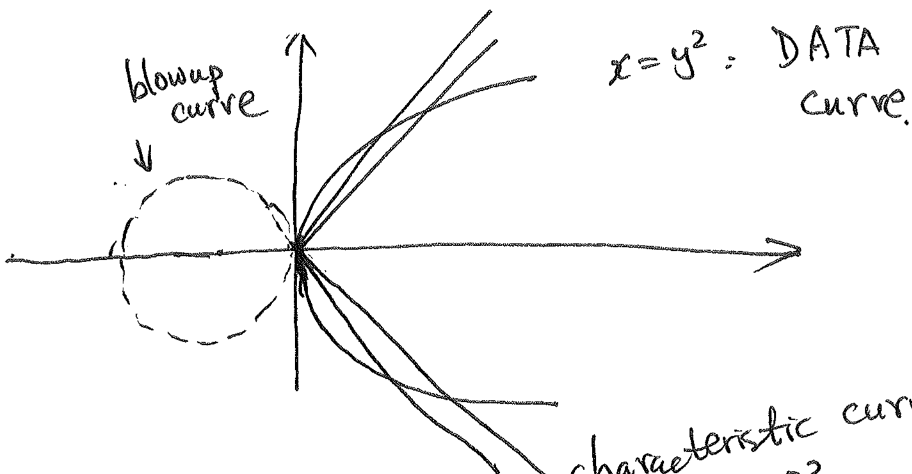
$$y - y\cdot\frac{x^2}{y^2}s = \frac{x}{y} \qquad s = \frac{x}{y^2 - x^2}$$

$$\frac{y^2}{x^2} - \frac{1}{x}$$

$$u^{-2}du = ds \qquad -u^{-1} + 1 = s \Rightarrow u^{-1} = 1 - s$$

$$u = \frac{1}{1-s} = \frac{1}{1 - \frac{x}{y^2 - x^2}} = \frac{y^2 - x^2}{y^2 - x^2 - x}$$

The solution blows up on $y^2 - x^2 = x$ or $(x+\frac{1}{2})^2 - y^2 = \frac{1}{4}$



characteristic curve: $x = \frac{\xi^2}{1-\xi^2 s}$, $y = \frac{\xi}{1-\xi^2 s}$, $y = \frac{1}{\xi} x$

Example 11 $\quad x u_x + (x^2 + y) u_y + (\;)u = \;$, $\quad u = 1$ on $x = y$

Sol'n: $\quad \frac{dx}{x} = \frac{dy}{x^2+y} = \frac{du}{-u(\;)}$

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = x^2 + y$$

$$\frac{du}{ds} = 1 - u(x - \frac{y}{x})$$

~~$\frac{dy}{dx} = x + \frac{y}{x} \Rightarrow \frac{y}{x} - x = \text{constant} = \xi$~~

$$\begin{cases} \frac{dx}{ds} = x, \quad x(0) = \xi & x = \xi e^s \\ \frac{dy}{ds} = y + x^2, \quad y(0) = \xi & y = \xi e^{2s} + (\xi - \xi^2)e^s \\ \frac{du}{ds} = -u(\;), \quad u(0) = 1, & u = e^{-s} \end{cases}$$

$$\xi = \frac{x^2 + x - y}{x} \qquad e^{-s} = \frac{\xi}{x} = \frac{x^2 + x - y}{x^2} = u(x, y).$$

General Solution for 1st-Order Linear PDE:

Let $a(x,t)\, u_x + b(x,t)\, u_t = c(x,t)\, u + d(x,t)$

We want to find a "General solution" in terms of an arbitrary function and then later find this function so that u = given on some curve

Now to find a general solution, let

$$\frac{dx}{dt} = \frac{a(x,t)}{b(x,t)} \rightarrow \text{solve to get } F(x,t) = \lambda = \text{constant}$$

$$F_x \frac{dx}{dt} + F_t = 0$$

Change variables in PDE as

$$x' = x, \quad \lambda = F(x,t), \quad u \rightarrow u(x', \lambda)$$

$$u_x = u_{x'} \frac{dx'}{dx} + u_\lambda \frac{d\lambda}{dx} = u_{x'} + u_\lambda F_x$$

$$u_t = u_\lambda \frac{d\lambda}{dt} = F_t u_\lambda$$

$$a[u_{x'} + u_\lambda F_x] + b F_t u_\lambda = cu + d$$

$$a F_x + b F_t = 0 \quad \text{so} \quad a u_{x'} = cu + d$$

where a, c, d are written in terms of x' and λ

We integrate to obtain an arbitrary function of λ
$\rightarrow$ general solution

Example 1. Find the general solution to

$$2u_x + u_t = e^t$$

Sol'n: $\frac{dx}{2} = \frac{dt}{1}$, $\lambda = x - 2t$

$x' = x$.

$u = u(x', \lambda)$

$$2u_{x'} = e^x = e^{\frac{1}{2}(\lambda - x')} = e^{\frac{1}{2}\lambda} e^{-\frac{1}{2}x'}$$

$$u = \frac{1}{2} e^{\frac{1}{2}\lambda}\left[(-2) e^{-\frac{1}{2}x'} + f(\lambda)\right]$$

$$= -e^{\frac{1}{2}\lambda - \frac{1}{2}x'} + \frac{1}{2} e^{\frac{1}{2}\lambda} f(\lambda)$$

$$= -e^{+t} + \frac{1}{2} e^{\frac{1}{2}(x-2t)} f(x-2t)$$

Example 2. Find general sol'ns to

$$x^2 u_x + 2xt u_t = xu$$

Sol'n: $\frac{dx}{x^2} = \frac{dt}{2xt}$ $\Rightarrow$ $\frac{dx}{dt} = \frac{x}{2t}$ $\Rightarrow$ $\lambda = x t^{-\frac{1}{2}}$

$x' = x$, $\lambda = x t^{-\frac{1}{2}}$, $u \to U$

$$u_x = U_{x'} + t^{-\frac{1}{2}} U_\lambda$$

$$u_t = U_\lambda \left(-\frac{x}{2t^{3/2}}\right) = -\frac{\lambda}{2t} U_\lambda$$

$$\to U_{x'} - \frac{1}{x'} U = 0 \Rightarrow U = x' F(\lambda) \Rightarrow u(x,t) = xF\left(\frac{x}{\sqrt{t}}\right)$$

Example 3. Find the solution to

$$t u_x + x u_t = x t^3$$

Sol'n: Characteristic $\frac{dx}{dt} = \frac{t}{x}$ so $x^2 = t^2 - \lambda$, c

Change of variables: $x' = x$, $\lambda = t^2 - x^2$, $u \to U$

$$t U_{x'} + U_\lambda (t \lambda_x + x \lambda_t) = x' t^3$$

$$U_{x'} = x' t^2 = x' (x'^2 + \lambda)$$

$$U = \frac{1}{4} x'^4 + \lambda \frac{x'^2}{2} + F(\lambda)$$

The general sol'n is

$$u(x,t) = \frac{x^4}{4} + (t^2 - x^2)\frac{x^2}{2} + f(t^2 - x^2)$$

(i) Suppose that $u = x^2$ on $t = 0$ for $-1 < x < 1$

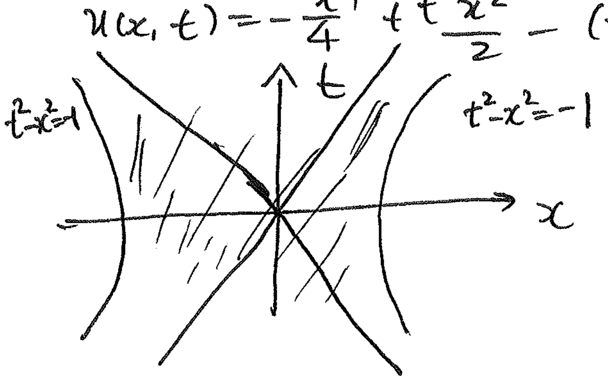
Then characteristics are $t^2 - x^2 = \lambda$, for $-1 < \lambda < 0$

$$x^2 = \frac{x^4}{4} - \frac{x^4}{2} + f(-x^2) \Rightarrow x^2 + \frac{1}{4}x^4 = f(-x^2)$$

so $f(\sigma) = -\sigma + \sigma^2/4$, $\underset{=-x^2}{\sigma} < 0$

$$f(t^2 - x^2) = -(t^2 - x^2) + (t^2 - x^2)^2/4$$

$$u(x,t) = -\frac{x^4}{4} + \frac{t^2 x^2}{2} - (t^2 - x^2) + \frac{(t^2 - x^2)^2}{4}$$



solution defined in the shaded region

ii) Suppose that $u=g(t)$ on $t^2=x^2+1$

Find the allowable form for $g(t)$

Notice: $t^2=x^2+1$ is a characteristic curve!

Thus characteristic curves do not intersect data curve in a transverse way

Method 1: use the general sol'n

$$u(x,t)=\frac{x^4}{4}+(t^2-x^2)\frac{x^2}{2}+f(t^2-x^2)$$

Then on $t^2=x^2+1$

$$g(t)=\frac{x^4}{4}+\frac{x^2}{2}+f(1)=\frac{(t^2-1)^2}{4}+\frac{t^2-1}{2}+f(1)$$

Hence there is no solution unless $g(t)$ has the specific form

$$(*)\quad g(t)=\frac{(t^2-1)^2}{4}+\frac{t^2-1}{2}+B=\frac{t^4}{4}+\text{Constant}$$

where B is an arbitrary constant. If $(*)$ does hold then $\exists\ \infty$ of sol'ns of the form

$$u(x,t)=\frac{x^4}{4}+(t^2-x^2)\frac{x^2}{2}+f(t^2-x^2) \text{ with } f(1)=B$$

Method 2 $u(x,t)=g(t)$ on $t^2=x^2+1 \rightarrow 2t=2xx'$, $x'=\frac{t}{x}$

so $\frac{d}{dt}u(x(t),t)=g'(t)=u_x x'+u_t=\frac{t}{x}u_x+u_t$

$tu_x+xu_t=xt^3 \Rightarrow u_t+\frac{t}{x}u_x=t^3$

$\Rightarrow g'(t)=t^3$ or $g(t)=\frac{t^4}{4}+C$

GENERAL SOLUTIONS OF FIRST ORDER PDE

EXAMPLE LET $u(x,t)$ SOLVE THE FIRST ORDER PDE

$$(*) \qquad x^2 u_x + 2xt\, u_t = xu$$

(i) FIND THE GENERAL SOLUTION TO (*). PLOT THE FAMILY OF CHARACTERISTICS.

(ii) USE THE GENERAL SOLUTION TO FIND $u(x,t)$ SATISFYING THE DATA

$$u = \sin t \quad \text{ON} \quad x = 1 \quad \text{FOR} \quad \pi/2 \le t \le \pi.$$

(iii) SUPPOSE WE PUT $u = 1$ ON $t = x^2$ FOR $-\infty < x < \infty$.
DOES THE PDE HAVE A SOLUTION?

(iv) SUPPOSE WE PUT $u = h(t)$ ON $t = x^2$. SHOW IN TWO WAYS THE CONDITION THAT $h(t)$ MUST SATISFY FOR A SOLUTION TO EXIST. IF THIS CONDITION IS SATISFIED IS THE SOLUTION UNIQUE?

SOLUTION

(i) GENERAL SOLUTION

WE FIND CHARACTERISTICS AS $\dfrac{dx}{dt} = \dfrac{x^2}{2xt} = \dfrac{x}{2t}$.

THUS $\dfrac{dx}{x} = \dfrac{dt}{2t}$ SO THAT $\ln x = \frac{1}{2}\ln t + \text{CONSTANT}$.

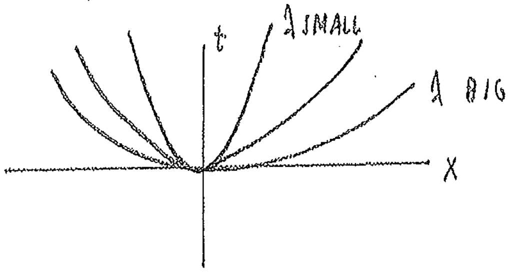
FOR CONVENIENCE WRITE $\text{CONSTANT} = -\ln\lambda$ SO THAT $\ln\left(\frac{x}{\lambda}\right) = \ln(t^{1/2})$.

THIS GIVES $\lambda = x\,t^{-1/2} \equiv F(x,t)$

THE CHARACTERISTICS ARE THE FAMILY OF CURVES

$t = x^2/\lambda^2$ WHICH ARE PARAMETRIZED BY λ.

THESE ARE PARABOLAS IN t VS x PLANE.



NOW CHANGE VARIABLES: $x' = x$, $\lambda = F(x,t)$ $\quad u \to U(x', \lambda)$.

$\lambda_x = t^{-1/2}, \quad \lambda_t = -\frac{1}{2} x t^{-3/2}$

SO $u_x = U_{x'} + U_\lambda \lambda_x, \quad u_t = U_\lambda \lambda_t$.

SO $x^2 u_x + 2xt\, u_t = xu \longrightarrow x^2 (U_{x'} + U_\lambda \lambda_x) + 2xt (U_\lambda \lambda_t) = xU$.

SO $x^2 U_{x'} + U_\lambda (x^2 \lambda_x + 2xt \lambda_t) = xU$.

BUT $x^2 \lambda_x + 2xt \lambda_t = x^2 t^{-1/2} + 2xt \left(-\frac{1}{2} x t^{-3/2} \right) = 0$ AS EXPECTED.

SO $U_{x'} = \frac{1}{x} U$ BUT $x' = x$

HENCE $U_{x'} = \frac{1}{x'} U$ WHICH IS A FIRST ORDER ODE.

$U_{x'} - \frac{1}{x'} U = 0$ INTEGRATING FACTOR IS $\phi = \exp\left(-\int 1/x' \right) = 1/x'$

SO $\left(\frac{1}{x'} U \right)_{x'} = 0 \to \frac{1}{x'} U = g(\lambda) \to U = x' g(\lambda)$

THEN $\lambda = x/\sqrt{t}$, $x' = x$ SO

GENERAL SOLUTION IS $u(x,t) = x\, g\left(x/\sqrt{t} \right)$. } (*)

(ii) NOW USE GENERAL SOLUTION TO PUT $u = \sin t$ ON $x = 1$ FOR $\pi/2 \le t \le \pi$.

THUS $\sin t = g(1/\sqrt{t})$

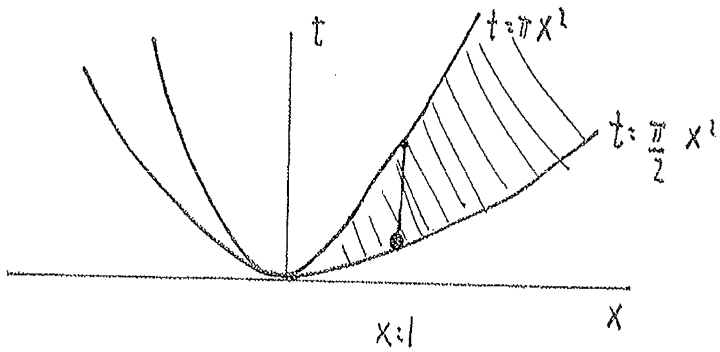
LET $\sigma = 1/\sqrt{t}$, SO $g(\sigma) = \sin(1/\sigma^2)$ FOR $\dfrac{1}{\sqrt{\pi}} \le \sigma \le \sqrt{\dfrac{2}{\pi}}$

IN (*) WE NEED $g(x/\sqrt{t})$. SO

$$u(x,t) = x \sin\left(\left(\sqrt{t}/x\right)^2\right) \quad \text{FOR} \quad \frac{1}{\sqrt{\pi}} \le \frac{x}{\sqrt{t}} \le \sqrt{\frac{2}{\pi}}.$$

THIS GIVES $$u(x,t) = x \sin\left(t/x^2\right) \quad \text{FOR} \quad \frac{\pi}{2} \le \frac{t}{x^2} \le \pi$$

THE SOLUTION IS DEFINED IN SHADED REGION



(iii) SUPPOSE $u = 1$ ON $t = x^2$ FOR $-\infty < x < \infty$. IS THERE A SOLUTION?
WE TRY TO FIT (*). SINCE $t = x^2$ IS ALSO A CHARACTERISTIC CURVE, IN GENERAL WE EXPECT NO SOLUTION. LET'S TRY TO FIT (*)

WE HAVE $1 = x\, g\left(x/\sqrt{x^2}\right) = x\, g(1)$ WHICH MUST HOLD FOR $-\infty < x < \infty$.

THIS IS INCONSISTENT AND SO THERE IS NO SOLUTION!

(iv) NOW PUT $u = h(t)$ ON $t = x^2$. WHAT CONDITION MUST $h(t)$ SATISFY FOR A SOLUTION TO EXIST?

METHOD 1 WE HAVE FROM (*) THAT

$$h(t) = \sqrt{t}\, g(1).$$

THUS $\exists$ SOLUTION ONLY WHEN $h(t) = c\sqrt{t}$ FOR ANY CONSTANT c.

THEN CHOOSE ANY FUNCTION $g(\xi)$ WITH $g(1) = c$.

THE SOLUTION IS $u = x\, g(x/\sqrt{t})$ FOR ANY $g(\xi)$ WITH $g(1) = c$.

THUS NON-UNIQUE SOLUTIONS.

METHOD 2 RECALL PDE $x^2 u_x + 2xt u_t = xu$. (1)

WE WANT $u = h(t)$ ON $t = x^2$.

THUS $u[x, x^2] = h(x^2)$

NOW DIFFERENTIATE WRT x:

$$u_x + u_t\, 2x = h'(x^2)\, 2x.$$

MULTIPLY BY x^2: $x^2 u_x + 2x^3 u_t = 2x^3 h'(x^2)$

BUT ON $t = x^2$ THE PDE (1) GIVES

$$x^2 u_x + 2x^3 u_t = x\, h(x^2)$$

HENCE FOR A SOLUTION TO EXIST WE NEED $2x^3 h'(x^2) = x\, h(x^2)$

OR $h'(x^2) = \frac{1}{2x^2} h(x^2) \longrightarrow \frac{dh}{d\sigma} = \frac{1}{2\sigma} h$ WITH $h = h(\sigma)$.

THE SOLUTION TO THIS FIRST ORDER ODE IS $h = c\sigma^{1/2}$ FOR ANY c.

THUS $h(\sigma) = c\sigma^{1/2}$ OR $h(t) = ct^{1/2}$. SAME CONCLUSION AS WITH METHOD 1