FULLY NONLINEAR EQUATIONS : Lecture Note 3

WE CONSIDER $F(x,y,u,p,q)=0$ WITH $p=\partial u/\partial x$, $q=\partial u/\partial y$.

FOR INSTANCE $F=p^2+q^2-1$ IS the EIKONAL EQUATION.

WE DIFFERENTIATE WRT x:

$$F_x+F_u u_x+F_p p_x+F_q q_x=0$$

BUT $u_x=p$ AND $q_x=p_y$. HENCE

$$\frac{\partial p}{\partial x}F_p+\frac{\partial p}{\partial y}F_q=-F_x-pF_u$$

THIS IS A QUASI-LINEAR EQUATION FOR p.

SO ON (1) $\frac{dx}{ds}=F_p$, $\frac{dy}{ds}=F_q$, THEN $\frac{dp}{ds}=-F_x-pF_u$.

NOW REPEAT THE PROCEDURE BY DIFFERENTIATING WRT y:

$$F_y+F_u u_y+F_p p_y+F_q q_y=0$$

BUT $u_y=q$ AND $p_y=q_x$. SO

$$F_p\frac{\partial q}{\partial x}+F_q\frac{\partial q}{\partial y}=-F_y-F_u q.$$

$$\boxed{\begin{aligned}\frac{du}{ds}&=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}\\&=pF_p+qF_q\end{aligned}}$$

THIS IS QUASI-LINEAR EQUATION FOR q:

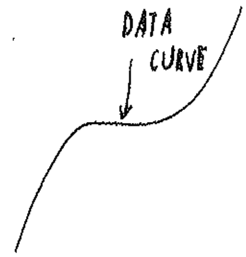
HENCE ON (2) $\frac{dx}{ds}=F_p$, $\frac{dy}{ds}=F_q$, THEN $\frac{dq}{ds}=-F_y-F_u q$.

COMBINING (1) AND (2) WE ARE LEFT WITH A 5 x 5 SYSTEM OF ODE'S:

(CHARPIT'S EQUATIONS)

$$\left\{\begin{aligned}&dx/ds=F_p \qquad dy/ds=F_q\\&dp/ds=-F_x-pF_u, \quad dq/ds=-F_y-qF_u\\&du/ds=pF_p+qF_q\end{aligned}\right.$$

THEY ARE SOLVED WITH THE INITIAL DATA



$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau) \quad \text{AT } s = 0$$

THE INITIAL CONDITION FOR p_0, q_0 (i.e. $p(0), q(0)$) ARE OBTAINED FROM

$$F(x_0, y_0, u_0, p_0, q_0) = 0$$

together WITH $\dfrac{du_0}{ds} = p_0 \dfrac{dx_0}{ds} + q_0 \dfrac{dy_0}{ds}$.

BECAUSE F IS NON-LINEAR IT MAY BE POSSIBLE THAT MORE THAN ONE CHARACTERISTIC PASSES THROUGH EACH POINT ON THE DATA CURVE.

<u>EXAMPLE 1</u> CONSIDER $u_y + u_x^2 = 0$ WITH INITIAL DATA

$$u(x, 0) = ax \quad \text{ON} \quad -\infty < x < \infty.$$

<u>SOLUTION</u> HERE $F(x, y, u, u_x, u_y) = u_y + u_x^2 = p^2 + q$.

SO CHARPIT'S EQUATION BECOME (WITH $F_p = 2p$, $F_q = 1$)

$$\frac{dx}{ds} = 2p, \quad x(0) = \tau \qquad \frac{dp}{ds} = 0, \quad p(0) = a$$

$$\frac{dy}{ds} = 1, \quad y(0) = 0 \qquad \frac{dq}{ds} = 0, \quad q(0) = -a^2$$

$$\frac{du}{ds} = 2p^2 + q, \quad u(0) = a\tau$$

$p(x,0) = a$ $\quad p = \dfrac{du(0)}{ds}$ $\quad \dfrac{du}{ds}\Big|_{s=0} \cancel{= a}$, $\cancel{\dfrac{dx_0}{ds}\Big|_{s=0} = 1}$, $\cancel{\dfrac{dy_0}{ds}\Big|_{s=0} = 0}$

$\cancel{p(0) = a}$, $\cancel{p_0^2 + q_0 = 0}$

WE SOLVE THIS ODE SYSTEM:

$$p = a, \quad q = -a^2$$

So $\dfrac{dx}{ds} = 2a, \quad x(0) = \tau \longrightarrow x = 2as + \tau$

$\dfrac{dy}{ds} = 1, \quad y(0) = 0 \longrightarrow y = s$

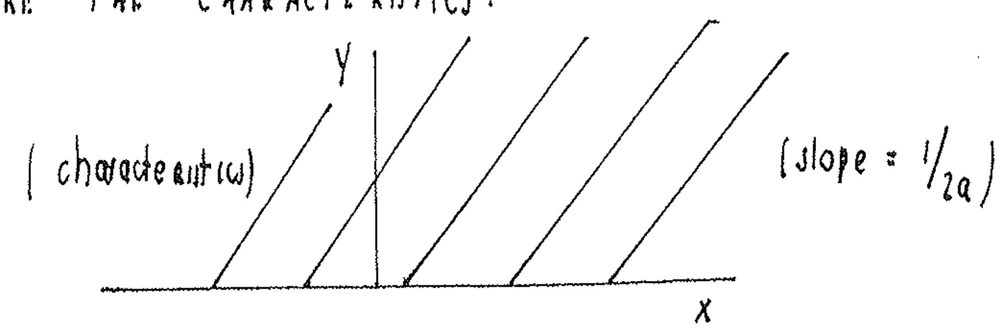
$\dfrac{du}{ds} = 2a^2 - a^2, \quad u(0) = a\tau \longrightarrow u = a^2 s + a\tau.$

NOW ELIMINATE s AND τ: $\quad s = y, \quad \tau = x - 2ay.$

So $\quad u = a^2(y) + a(x - 2ay) \longrightarrow u = a(x - ay)$

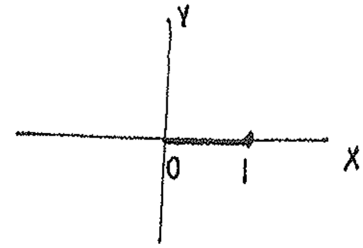
$\longrightarrow u = a(x - ay)$ IS THE SOLUTION AND $x = 2ay + \tau$,

ARE THE CHARACTERISTICS:



So IF y IS time, this IS a wave moving to the Right.

EXAMPLE 2



SOLVE $u - x u_x - \frac{1}{2} u_y^2 + x^2 = 0$

WITH $u(x,0) = x^2 - \frac{1}{6} x^4$ FOR $0 < x < 1$.

NOW WE WRITE $F(x,y,u,u_x,u_y) = u - px - \frac{1}{2} q^2 + x^2 = 0$ $\quad p = u_x,\ q = u_y$

CHARPIT'S SYSTEM BECOMES:

$$\frac{dx}{ds} = F_p = -x$$

$$\frac{dy}{ds} = F_q = -q$$

$$\frac{du}{ds} = pF_p + qF_q = -px - q^2$$

$$\frac{dp}{ds} = -F_x - F_u\, p = -(-p+2x) - p = -2x$$

$$\frac{dq}{dx} = -F_y - F_u\, q = -q$$

NOW parameterize data CURVE BY $x = \tau$, $y = 0$, $u = \tau^2 - \frac{1}{6}\tau^4$, $p = 2\tau - \frac{2}{3}\tau^3$. WE THEN SET $F = 0$ TO FIND q AT $s = 0$:

$$u - px - \frac{1}{2} q^2 + x^2 = \tau^2 - \frac{1}{6}\tau^4 - \left(2\tau - \frac{2}{3}\tau^3\right)\tau - \frac{1}{2} q^2 + \tau^2 = 0.$$

THIS YIELDS $-\frac{1}{6}\tau^4 + \frac{2}{3}\tau^4 - \frac{1}{2} q^2 = 0 \longrightarrow \frac{\tau^4}{2} - \frac{1}{2} q^2 = 0.$

THIS GIVES $q = \pm \tau^2$

LET'S TAKE THE + ROOT SO THAT (THE MINUS ROOT GIVES ANOTHER SOLUTION)

$$\frac{dx}{ds} = -x \ , \quad x(0) = \tau$$

$$\frac{dy}{ds} = -q \ , \quad y(0) = 0$$

$$\frac{du}{ds} = -px - q^2 , \quad u(0) = \tau^2 - \frac{1}{6}\tau^4$$

$$\frac{dp}{ds} = -2x \ , \quad p(0) = 2\tau - \frac{2}{3}\tau^3$$

$$\frac{dq}{ds} = -q \ , \quad q(0) = \tau^2 .$$

IT IS PERHAPS EASIER TO NOTICE THAT

$$\frac{dq}{dx} = \frac{q}{x} \ , \quad q = \tau^2 \text{ when } x = \tau \longrightarrow q = \tau x .$$

$$\frac{dy}{dx} = \frac{q}{x} = \tau , \quad y = 0 \text{ when } x = \tau \longrightarrow y = \tau x - \tau^2$$

$$\frac{dp}{dx} = 2 , \quad p = 2\tau - \frac{2}{3}\tau^3 \text{ when } x = \tau \rightarrow p = 2x - \frac{2}{3}\tau^3 .$$

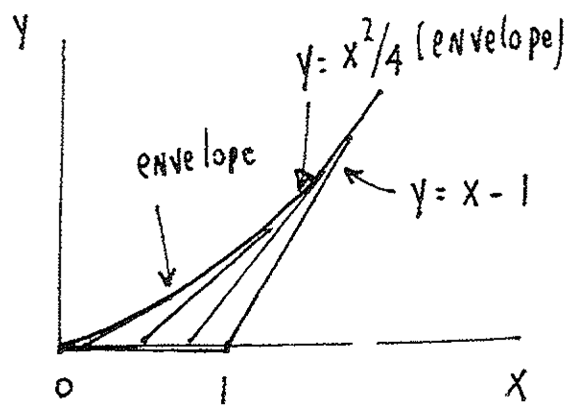
NOW FINALLY $u = px + q^2/2 - x^2$.

THIS GIVES $u = (2x - \frac{2}{3}\tau^3)x + \frac{\tau^2 x^2}{2} - x^2$.

OR $u = x^2 + \frac{\tau^2 x^2}{2} - \frac{2}{3} x \tau^3$

WITH $y = \tau x - \tau^2$ IS IMPLICIT FORM

FOR THE SOLUTION. $0 < \tau < 1$



characteristics are

$$y = \tau X - \tau^2.$$

LET'S CALCULATE THE ENVELOPE OF THE CHARACTERISTICS.

WE WRITE $\quad G(y, X, \tau) = 0 \qquad G = y - \tau X + \tau^2$

$$G_\tau(\cancel{y}, X, \tau) = 0$$

SO $\quad G_\tau = 0 \longrightarrow -X + 2\tau = 0 \longrightarrow \tau = X/2$

HENCE $\quad y - \dfrac{X^2}{2} + \dfrac{X^2}{4} = 0 \longrightarrow y = X^2/4$

IS ENVELOPE

EIKONAL EQUATION

WE BEGIN WITH THE WAVE EQUATION $\phi = \phi(X, Y, t)$

$$\phi_{tt} = c^2(\phi_{XX} + \phi_{YY}).$$

LET $\phi = e^{-i\omega t}\,\psi(X, Y)$ TO GET $\psi_{XX} + \psi_{YY} + K^2\psi = 0$

WHERE $K = \omega/c$. NOW IF WE NON-DIMENSIONALIZE BY SETTING $x = X/L$, $y = Y/L$ WE OBTAIN

$$\psi_{xx} + \psi_{yy} + \kappa^2\psi = 0 \qquad \kappa = L^2 K.$$

ASSUME THAT $\kappa \gg 1$ (HIGH SPATIAL FREQUENCY $\rightarrow$ LOW WAVELENGTH).

THEN LET $\psi = A(x,y)\,e^{i\kappa u(x,y)}$ $\qquad u$ = phase of wave.

A = amplitude

WE CALCULATE $\psi_x = i\kappa u_x A e^{i\kappa u} + A_x e^{i\kappa u}$

$$\psi_{xx} = -\kappa^2 u_x^2 A e^{i\kappa u} + i\kappa u_{xx} A e^{i\kappa u} + 2i\kappa u_x A_x e^{i\kappa u} + A_{xx} e^{i\kappa u}.$$

SUBSTITUTING INTO $\psi_{xx} + \psi_{yy} + \kappa^2\psi = 0$ WE OBTAIN

$$\underline{-\kappa^2 A(u_x^2 + u_y^2)} + i\kappa\left[(u_{xx} + u_{yy})A + 2\nabla u\cdot\nabla A\right] + (A_{xx} + A_{yy}) + \underline{\kappa^2 A}$$

THE TWO LARGEST TERMS PROPORTIONAL TO κ^2 BALANCE AND SO

$$-(u_x^2 + u_y^2) = -1 \longrightarrow u_x^2 + u_y^2 = 1 \quad \text{EIKONAL EQUATION.}$$

SPECIAL SOLUTIONS ARE $u = -x$, $A = 1 \longrightarrow \psi = e^{-i\kappa x}$

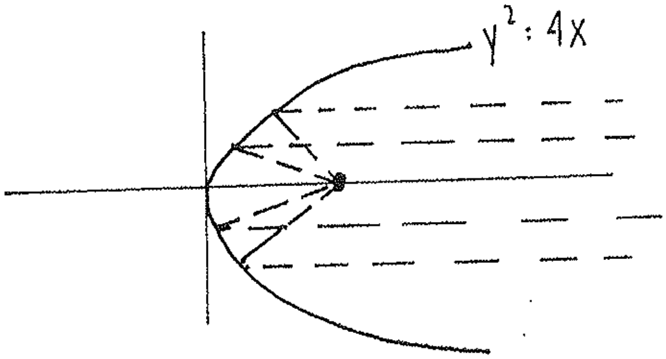
OR $\phi = e^{-i\omega t}e^{-i\kappa x} = e^{-i\kappa(x+ct)}$ $\qquad c = \omega/\kappa$

THIS IS A WAVE PROPAGATING TO THE LEFT.

IF $u = x$, $A = 1$ THEN $\phi = e^{i\kappa(x-ct)}$ A WAVE MOVING TO RIGHT.

EIKONAL EQUATION

EXAMPLE PARABOLIC REFLECTOR.



$p^2 + q^2 = 1 \qquad p = u_x$

$\qquad\qquad\qquad\;\; q = u_y$

$x = \tau^2, \; y = 2\tau, \; -\infty < \tau < \infty$

parametrizes the reflector

WE WANT TO SHOW THAT RAYS OF LIGHT (i.e. THE CHARACTERISTICS) ALL MEET AT THE FOCAL POINT OF THE PARABOLA.

WE ASSUME $u = -x$ on the parabola (i.e. $\phi = e^{-ik(x+ct)}$) A WAVE MOVING TO THE LEFT IS INCIDENT ON THE PARABOLA. WE WANT TO CALCULATE REFLECTED FIELD. WE WANT $u = -x$ ON BOUNDARY OF PARABOLA

NOW THE CHARPIT SYSTEM IS

$$\frac{dx}{ds} = F_p = 2p \;, \qquad x(0) = \tau^2 = x_0(\tau)$$

$$\frac{dy}{ds} = F_q = 2q \;, \qquad y(0) = 2\tau = y_0(\tau)$$

$$\frac{du}{ds} = pF_p + qF_q = 2 \;, \qquad u(0) = -\tau^2 = u_0(\tau)$$

$$\frac{dp}{ds} = -F_x - pF_u = 0 \;, \qquad p(0) = p_0$$

$$\frac{dq}{ds} = -F_y - qF_u = 0 \;, \qquad q(0) = q_0$$

TO CALCULATE (p_0, q_0) WE NOTICE THAT ON T

$$\frac{du_0}{d\tau} = \frac{\partial u_0}{\partial x}\frac{dx_0}{d\tau} + \frac{\partial u_0}{\partial y}\frac{dy_0}{d\tau} \longrightarrow -2\tau = 2p_0\tau + 2q_0$$

THUS $p_0^2 + q_0^2 = 1$, AND $q_0 = -\tau(1+p_0)$

WE SOLVE $p_0^2 + (1+p_0)^2\tau^2 = 1$

HENCE $p_0 = \dfrac{-2\tau^2 \pm \sqrt{4\tau^4 - 4(\tau^2-1)(\tau^2+1)}}{2\tau^2+2} = \dfrac{-2\tau^2 \pm 2}{2\tau^2+2}$.

WE WANT + SIGN SO THAT $p_0 \neq -1$ (i.e. $u \neq -x$)

HENCE $p_0 = \dfrac{-2\tau^2+2}{2\tau^2+2}$ $\quad q_0 = -\tau(1+p_0) = -\tau\left(1 + \dfrac{(2-2\tau^2)}{(2+2\tau^2)}\right) = \dfrac{-2\tau}{1+\tau^2}$.

THEN SOLVING CHARPIT'S SYSTEM WE OBTAIN

$$p = \frac{1-\tau^2}{1+\tau^2}, \quad q = \frac{-2\tau}{1+\tau^2}, \quad u = 2s - \tau^2$$

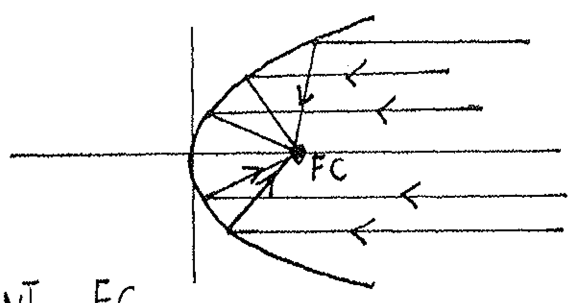
AND $x = 2\left(\dfrac{1-\tau^2}{1+\tau^2}\right)s + \tau^2$, $\quad y = -\dfrac{4\tau}{(1+\tau^2)}s + 2\tau$.

WE THEN ELIMINATE S TO OBTAIN: $x - \tau^2 = 2\dfrac{(1-\tau^2)}{((1+\tau^2)}\dfrac{(y-2\tau)(1+\tau^2)}{(-4\tau)}$.

CLEANING THIS UP ONE OBTAINS THE FAMILY OF CURVES

$$x - \tau^2 = (\tau^2-1)\left(\frac{y}{2\tau} - 1\right) \longrightarrow 2\tau(x-\tau^2) = (\tau^2-1)(y-2\tau)$$

THIS IS A FAMILY OF STRAIGHT LINES FOR EACH τ IN $-\infty < \tau < \infty$



NOTICE THAT FOR ANY τ

$x = 1$ WHEN $y = 0$ $\longleftarrow$ FOCAL POINT FC.

EXAMPLE 2 (CAUSTIC IN A LIQUID SURFACE UNDER AN OBLIQUE LIGHT SOURCE)

please see http://www.ballandclaw.com/Caustic/index.html

↑capital

FOR A Java applet of the phenomenon. $F = p^2 + q^2 - 1$

WE WANT TO SOLVE $p^2 + q^2 = 1$ WITH $u = -x$ ON

THE PORTION OF THE CIRCLE $x^2 + y^2 = 1$ WITH $x_0 = \cos\tau$, $y_0 = \sin\tau$

AND $\pi/2 < \tau < 3\pi/2$.

THE CHARPIT SYSTEM is

$$\frac{dx}{ds} = 2p \quad , \qquad x(0) = \cos\tau$$

$$\frac{dy}{ds} = 2q \quad , \qquad y(0) = \sin\tau$$

$$\frac{du}{ds} = 2 \quad , \qquad u(0) = -\cos\tau$$

$$\frac{dp}{ds} = 0 \quad , \qquad p(0) = p_0(\tau)$$

$$\frac{dq}{ds} = 0 \quad , \qquad q(0) = q_0(\tau)$$

TO DETERMINE p_0, q_0 WE USE $p_0^2 + q_0^2 = 1$

Together WITH $du_0/d\tau = p_0 x_0' + q_0 y_0' \longrightarrow \sin\tau = -\sin\tau\, p_0 + \cos\tau\, q$

BY INSPECTION ONE SOLUTION IS $p_0 = -1$, $q_0 = 0$. THIS

CORRESPONDS TO $u = -x$ INCIDENT WAVE. WE, HOWEVER,

WANT THE REFLECTED WAVE.

THE OTHER SOLUTION IS $q_0 = \sin(2\tau)$, $p_0 = \cos(2\tau)$

SINCE $-\sin\tau \cos(2\tau) + \sin(2\tau)\cos\tau = \sin(2\tau - \tau) = \sin\tau$.

THEN WE OBTAIN

$$p = p_0 = \cos(2\tau)$$
$$q = q_0 = \sin(2\tau)$$

AND

$$\frac{dx}{ds} = 2\cos(2\tau), \qquad x = 2\cos(2\tau)s + \cos\tau$$

$$\frac{dy}{ds} = 2\sin(2\tau), \qquad y = 2\sin(2\tau)s + \sin\tau$$

NOW WE ELIMINATE s: $(x - \cos\tau) = 2\cos(2\tau)\dfrac{(y - \sin\tau)}{2\sin(2\tau)}$.

THIS GIVES $(x - \cos\tau)\sin 2\tau = (y - \sin\tau)\cos 2\tau$

OR EQUIVALENTLY $\sin 2\tau\, x - \cos(\tau)\sin 2\tau = y\cos 2\tau - \sin\tau\cos 2\tau$.

THIS GIVES: $x\sin(2\tau) - y\cos(2\tau) = \sin\tau$, $\quad \pi/2 < \tau < 3\pi/2$

THESE ARE THE CHARACTERISTICS: STRAIGHT LINES!

NOW WHAT IS THE ENVELOPE?

$$F = x\sin(2\tau) - y\cos(2\tau) - \sin\tau = 0$$

$$F_\tau = 2x\cos(2\tau) + 2y\sin(2\tau) - \cos\tau = 0$$

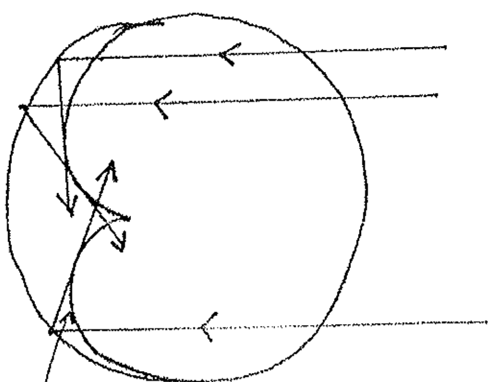
$$\begin{pmatrix} \sin 2\tau & -\cos 2\tau \\ \cos 2\tau & \sin 2\tau \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin\tau \\ \frac{1}{2}\cos\tau \end{pmatrix}$$

RECALLING:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$

THEN
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sin 2\tau & \cos 2\tau \\ -\cos 2\tau & \sin 2\tau \end{pmatrix} \begin{pmatrix} \sin \tau \\ \frac{1}{2}\cos \tau \end{pmatrix}$$

parametric EQUATION OF envelope
$$\left\{ \begin{aligned} X &= \sin\tau \sin 2\tau + \frac{1}{2}\cos\tau \cos 2\tau \\ Y &= -\sin\tau \cos 2\tau + \frac{1}{2}\cos\tau \sin 2\tau \end{aligned} \right. \qquad \pi/2 < \tau < 3\pi/2$$

IF YOU PLOT THIS CURVE THE PICTURE IS AS FOLLOWS



envelope.

THE ENVELOPE IS A NEPHROID OR (KIDNEY)