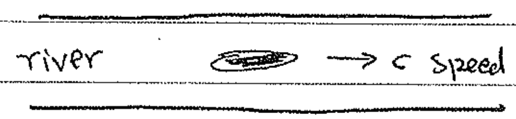
Lecture Note 4

1.3 Flows, Diffusions and Vibrations (From Strauss's Book)

We discuss the physical background of ~~the eqns~~ partial differential equations.

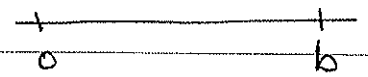
Example 1. Simple Transport (Traffic Flow Model)



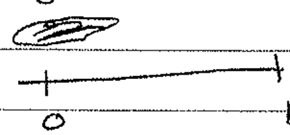
media: water

u = concentration density of the chemicals $u(x,t)$

We look at one segment



At time t



total mass $M = \int_0^b u(x,t)\,dx$

At time $t+h$ $\quad \int_{ch}^{b+ch} u(x,t+h)\,dx = \int_0^b u(x,t)\,dx \qquad \frac{d}{db}$

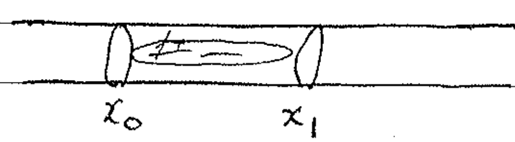
$$u(b,t) = u(b+ch,\ t+h) \qquad \frac{d}{dh}$$

$$0 = c u_x(b,t) + u_t(b,t) \ \rightarrow\ c u_x + u_t = 0$$

We know from section 2, $u(x,t) = f(x-ct)$

Suppose initial distribution is $f(x)$, after time t, it becomes $f(x-ct)$

Example 2: Transport with diffusion (Diffusion Equation



$$M = \int_{x_0}^{x_1} u(x,t)\,dx \quad - \text{ total Mass}$$

$$\frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx$$

Fick's Law : the rate of motion is proportional to the concentratio gradient

$$\frac{dM}{dt} = \text{flow in} - \text{flow out}$$

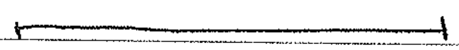
$$= k\,u_x(x_1,t) - k\,u_x(x,t)$$

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = k\,u_x(x_1,t) - k\,u_x(x_0,t) \qquad \frac{d}{dx_1}$$

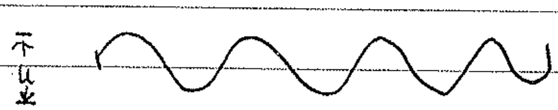
$$\Rightarrow \quad u_t = k\,u_{xx} \quad - \text{ diffusion equation}$$

k - diffusion coefficient

Example 3. Vibrating String (Wave Equation

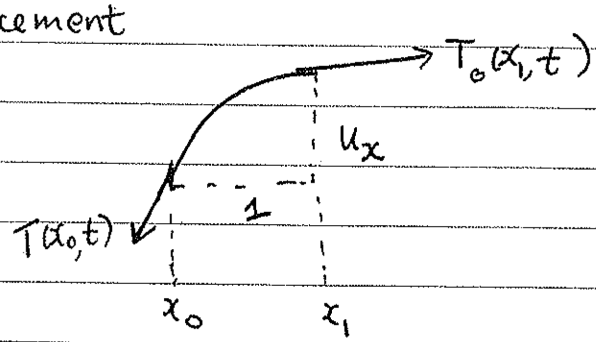


Guitar String



Add some force

u - displacement



Slope : $u_x = \tan\theta$

$$T(x_1,t)\cos\theta_1 - T(x_0,t)\cos\theta_0 = 0 \qquad \text{horizonal}$$

$$T(x_1,t)\sin\theta_1 - T(x_0,t)\sin\theta_0 = \int_{x_0}^{x_1} \rho u_{tt}$$

$$\cos\theta_1 = \frac{1}{\sqrt{1+u_x^2(x_1,t)}}, \quad \cos\theta_0 = \frac{1}{\sqrt{1+u_x^2(x_0,t)}}$$

$$\sin\theta_1 = \frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}}, \quad \sin\theta_0 = \frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}$$

Suppose vibrating very small, u_x very small

$$1+u_x^2 \approx 1$$

$$T(x_1,t) - T(x_0,t) \approx 0 \Rightarrow T \text{ constant}$$

$$T\,u_x(x_1,t) - T\,u_x(x_0,t) = \int_{x_0}^{x_1} \rho u_{tt} \qquad \frac{d}{dx_1}$$

$$\rho u_{tt} = (T u_x)_x = T u_{xx}$$

$$\Rightarrow u_{tt} = c^2 u_{xx}, \qquad c^2 = \sqrt{\frac{T}{\rho}}$$

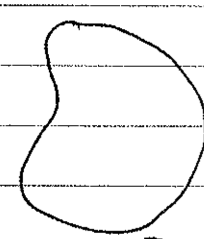
Wave Equation

Ex. 2 & Ex. 3 can be extended to higher-dimensions

$N=2$, $N=3$

Notations & Thms: D - bounded domain

∂D - boundary of D



D

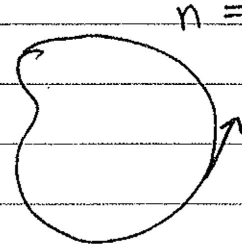
In two-dimensional case

Green's Thm: $\iint_D (q_x - P_y)\,dx\,dy = \int_C p\,dx + q\,dy$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad F = (q, -P)$$

$$\nabla\cdot F = q_x - P_y$$

$$\iint_D \nabla\cdot F = \int_{\partial D} \vec{F}\cdot\vec{n}$$



$\vec{n}$ = outward normal

$X = (x(t), y(t))$

$n = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$, $\vec{T} = \left(\frac{dx}{ds}, \right.$

$$\vec{F}\cdot\vec{n} = q\frac{dy}{ds} + p\frac{dx}{ds}$$

In three-dimensional case, we have Stokes formula

$$\iiint_D \nabla\cdot\vec{F} = \iint_{\partial D} \vec{F}\cdot\vec{n}$$

Using these notations:

$$\frac{\partial u}{\partial n} = \nabla u\cdot\vec{n} = \text{normal derivative of } u$$

$$\iiint_D \Delta u = \iint_{\partial D} \frac{\partial u}{\partial n}.$$

$$u_t - k u_{xx} = 0 \Rightarrow u_t - k\Delta u = 0$$

$$u_{tt} - c^2 u_{xx} = 0 \Rightarrow u_{tt} - c^2\Delta u = 0.$$

Ex 4 Stationary waves

$$u(x,t) = u(x) \Rightarrow \Delta u = 0.$$

4. Initial and Boundary Value Problems

For eqns with time t (heat eqn, wave eqn):

Initial Conditions: $u(x,0) = \varphi(x)$ for heat eqn

$$\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \quad \text{for wave eqn}$$

Boundary Conditions: for eqns with spatial variable $x, y, z, \dots$

Three kinds of BCs.

Dirichlet BC: $u(x) = g(x)$, $x \in \partial D$

Neumann BC: $\frac{\partial u}{\partial n} = \nabla u\cdot n = g(x)$, $x \in \partial D$

Robin BC: $\nabla u\cdot n + a(x)\, n = g$, $a(x) > 0$

Examples: for Laplace eqn, there are 3 BVPs

for heat eqn: IVP $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \varphi(x) \end{cases}$

IBVP $\begin{cases} u_t - k u_{xx} = 0, & 0 < x < l \\ u(x,0) = \varphi(x) \\ u(0,t) = \psi_1(t), & \psi(l,t) = \psi_2(t) \end{cases}$

1.5 Well-posed Problems

A PDE problem is well-posed if it has the following fundamental properties:

(i) Existence: $\exists$1 at least one solution $u(x)$ or $u(x,t)$

(ii) Uniqueness: $\exists$ at most one sol'n

(iii) Stability: The unique solution $u(x,t)$ depends in a stable manner on the data of the problem. Namely, if the initial conditions or boundary conditions change a little, the solution changes only a little

Example 0: $\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$

Example 1: $\begin{cases} T u_{tt} - \rho u_{xx} = f(x,t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \\ u(0,t) = g(t), \quad u(L,t) = h(t) \end{cases}$

Existence: For given ϕ, ψ, g, h, $\exists$ a sol'n $u(x,t)$

Uniqueness: For given ϕ, ψ, g, h, $\exists!$ sol'n $u(x,t)$

Stability: If $\phi_n \to \phi$, $\psi_n \to \psi$, $g_n \to g$, $h_n \to h$, $f_n \to f$, then $u_n \to u$

Not all eqns are well-posed

Example 2. $\begin{cases} \Delta u = 1 & D_1 = B_1(0) \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial n} = 1 & \text{on } \partial B_1(0) \end{cases}$

No sol'n

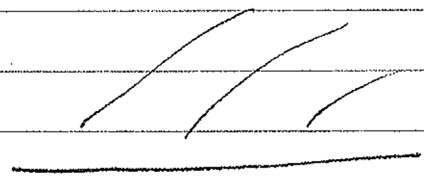
$$\pi = \int 1 = \int_{\partial B_1} \frac{\partial u}{\partial n} = 1 \cdot |\partial B_1(0)| \neq 2\pi$$

No existence

Ex. 3. $\begin{cases} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \end{cases}$

No uniqueness

Ex. 4. $\begin{cases} u_{xx} + u_{yy} = 0 & \\ u(x,0) = 0 & -\infty < x < +\infty, \quad 0 < y < +\infty \\ \frac{\partial u}{\partial y}(x,0) = 0 & -\infty < x < +\infty, \quad 0 < y < +\infty \end{cases}$



$\Delta u = 0$

It has a sol'n $u = 0$ $\qquad$ $u_t + u_{xx} = 0$

Perturbation

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \\ u(x,0) = 0 \\ \frac{\partial u}{\partial y}(x,0) = e^{-\sqrt{n}} \sin x \end{cases} \qquad e^{n^2 t} \sin nx$$

sol'n: $u_n(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin x \sinh ny$

$u_n(x,y) \to \infty$ as $n \to +\infty$ for $y \neq 0$

No stability

1.6 Types of 2nd-Order Eqns

There are three types of eqns of 2nd-order

$u_{xx} + u_{yy} = \Delta u = f$ $\qquad$ elliptic

$u_y - k^2 \Delta u = f$ $\qquad$ parabolic (In applications, $u_y \to u_t$)

$u_{yy} - c^2 \Delta u = f$ $\qquad$ hyperbolic

Main Result: General 2nd Order Linear PDE of two variables can be reduced to one of these three eqns

General 2nd order linear PDE

$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$

$\cancel{= a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1 u_x + b_2 u_y + cu + h(x,y)}$

Idea: Change of variables.

Introduce new variables

$$\begin{cases} \xi = b_{11}x + b_{12}y \\ \eta = b_{21}x + b_{22}y \end{cases} \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \partial x \\ \partial y \end{pmatrix} = B^T\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

$u_x = b_{11}u_\xi + b_{21}u_\eta$

$u_y = b_{12}u_\xi + b_{22}u_\eta$

$$\begin{pmatrix} \partial x \\ \partial y \end{pmatrix} = B^T\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1 u_x + b_2 u_y$

$= [a_{11}\partial_x^2 + 2a_{12}\partial_x\partial_y + a_{22}\partial_y^2]u + (b_1, b_2)\begin{pmatrix} \partial x \\ \partial y \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{12} = a_{21}$

$= (\partial_x, \partial_y)\, A \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} u + (b_1, b_2)B\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$

$= (\partial_\xi, \partial_\eta)\, B^T A B \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} u + (b_1, b_2)B\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$

Since A is symmetric, it can be diagonalized

$$B^T A B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Case 1: Both λ_1, λ_2 are positive or negative

$a_{12}^2 < a_{12}a_{22}$

Then we have

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y$$
$$= \lambda_1 u_{\xi\xi} + \lambda_2 u_{\eta\eta} + (b_1, b_2) B \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} u$$

Introducing

$$\begin{cases} x' = \sqrt{\lambda_1}\, \xi \\ y' = \sqrt{\lambda_2}\, \eta \end{cases}$$

Then

$$\lambda_1 u_{\xi\xi} + \lambda_2 u_{\eta\eta} + \cdots = \underbrace{u_{x'x'} + u_{y'y'}}_{\Delta u} + \cdots = 0$$

Case 2 $\quad \lambda_1 > 0 > \lambda_2$

In this case

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy}$$
$$= \lambda_1 u_{\xi\xi} + \lambda_2 u_{\eta\eta}$$
$$= u_{x'x'} - u_{y'y'} \qquad \text{— hyperbolic}$$

Case 3 $\quad \lambda_1 > 0 = \lambda_2$ or $\lambda_1 = 0 < \lambda_2$

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + \cdots$$
$$= \lambda_1 u_{\xi\xi} + \cdots \qquad \text{parabolic}$$

Summary: For $F = a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y + \cdots$

$$D = a_{12}^2 - a_{11} a_{12} \begin{cases} > 0, & \text{hyperbolic} \\ = 0, & \text{parabolic} \\ < 0, & \text{elliptic} \end{cases}$$

Examples

(a) $u_{xx} - 5u_{xy} = 0$

$$A = \begin{pmatrix} 1 & -\frac{5}{2} \\ -\frac{5}{2} & 0 \end{pmatrix}$$ hyperbolic

Complete squares

$\partial_x^2 u - 5\partial_x\partial_y u$

$(\partial_x^2 u - 5\partial_x\partial_y u + (\frac{5}{2}\partial_y)^2 - (\frac{5}{2}\partial_y)^2 u$

$= (\partial_x - \frac{5}{2}\partial_y)^2 - (\frac{5}{2}\partial_y)$

$$\begin{cases} \partial_\xi = \partial_x - \frac{5}{2}\partial_y \\ \partial_\eta = \frac{5}{2}\partial_y \end{cases}$$

$$\begin{cases} \partial_x = \partial_\xi + \partial_\eta \\ \partial_y = \frac{2}{5}\partial_\eta \end{cases}$$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & \frac{2}{5} \end{pmatrix}$$

(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

$4\partial_x^2 - 12\partial_x\partial_y + 9\partial_y^2 + \partial_y$

$(2\partial_x - 3\partial_y)^2 + \partial_y = 0$

$$\begin{cases} \partial_\xi = 2\partial_x - 3\partial_y \\ \partial_\eta = -\partial_y \end{cases}$$

$$\begin{cases} \partial_x = \frac{1}{2}\partial_\xi \, \frac{3}{2}\partial_\eta \\ \partial_y = -\partial_\eta \end{cases} = B^T \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} = B$$

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}$

$$B = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix}$$

$\partial_\xi^2 - \partial_\eta$

(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$

$$\left[4\partial_x^2 + 6\partial_x\partial_y + 9\partial_y^2\right]u = (2\partial_x + 3\partial_y)^2 u$$

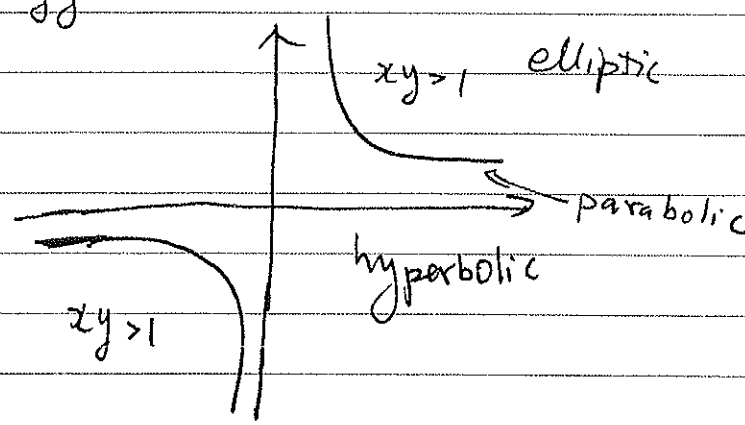
$$\begin{cases} \partial_\xi = 2\partial_x + 3\partial_y \\ \partial_\eta = \partial_x \end{cases}$$

If $a_{11}, a_{12}, a_{22}, \cdots$ depends on x, y variables, then the types of equations depends on x, y.

$D = a_{12}^2 - a_{11}a_{22} = D(x,y)$

Ex. 4. $y u_{xx} - 2u_{xy} + x u_{yy} = 0$

$D = 1 - xy$



Ex. 5. $y\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (Tricomi Eqn)

$D = 1 - y$

3-dimensional case:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + 2a_{13}u_{xz} + \overset{a_{22}}{u_{yy}} + \cdots$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A = \text{symmetric} \quad A \sim \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}$$

If all $d_i > 0, \forall i$, or $d_i < 0, \forall i$, elliptic

If $d_1 < 0$, $d_2, d_3 > 0$, hyperbolic

If one of $d_i = 0$, parabolic