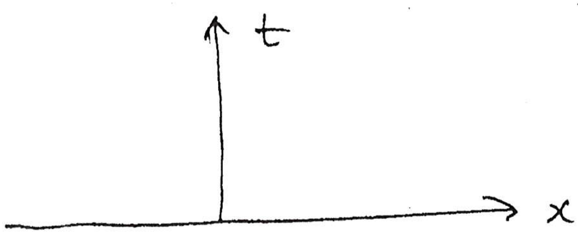
Chapter 2 Wave Equation in the infinite line (Lecture Note 5)

In this chapter, we solve the simplest wave equation on a line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$



Here: $f(x,t)$ – source term

$\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}$ – initial conditions

Idea: First we solve $u_{tt} - c^2 u_{xx} = 0$

- general solution
- plug in initial conditions
- add source terms later

2.1. Wave Eqn

$$u_{tt} - c^2 u_{xx} = 0.$$

Find general solution: use method of Characteristics

Coordinate method.

Let
$$\left.\begin{aligned} \partial_\xi &= \partial_t - c\,\partial_x \\ \partial_\eta &= \partial_t + c\,\partial_x \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \partial_t &= \tfrac{1}{2}\left(\partial_\xi + \partial_\eta\right) \\ \partial_x &= \tfrac{1}{2c}\left(\partial_\eta - \partial_\xi\right) \end{aligned}\right.$$

$$\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2c} & \frac{1}{2c} \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2c} \\ +\frac{1}{2\cancel{c}} & \frac{1}{2c} \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\Rightarrow \left\{\begin{aligned} \xi &= \tfrac{1}{2c}(ct - x) \\ \eta &= \tfrac{1}{2c}(ct + x) \end{aligned}\right.$$

Then $u_{tt} - c^2 u_{xx} = \left(\partial_t^2 - c^2\partial_x^2\right)u = \left(\partial_t - c\,\partial_x\right)\left(\partial_t + c\,\partial_x\right)u = 0$

$$\partial_\xi \cdot \partial_\eta u = 0 \Rightarrow u = f(\xi) + g(\eta)$$

$$\Rightarrow u = f(x+ct) + g(x-ct), \quad \forall \text{ fcns } f, g$$

<u>Check</u>: $\forall f, g$, $\quad u = f(x+ct) + g(x-ct)$ is a sol'n.

Geometric meaning

Factorization Technique for hyperbolic problem

$$a_{11}\partial_t^2 + a_{12}\partial_t\partial_x + a_{22}\partial_x^2 = (\partial_t + \cdot\,\partial_x)(\partial_t + \cdot\,\partial_x) = \partial_\xi \cdot \partial_\eta$$

$$a_{12}^2 - 4a_{11}a_{22} > 0 \qquad \Rightarrow u = f(\xi) + g(\eta)$$

Ex. 1

$$\begin{cases} u_{tt} - u_{tx} - 2u_{xx} = 0 \\ u(x,0) = \cos x, \quad u_t(x,0) = e^x \end{cases}$$

Sol'n:

$$\partial_t^2 - \partial_{tx} - 2\partial_x^2 = (\partial_t - 2\partial_x)(\partial_t + \partial_x)$$

$$u = f(x+2t) + g(x-t)$$

$$f(x) + g(x) = \cos x$$

$$2f'(x) - g'(x) = e^x \quad \rightarrow \quad 2f(x) - g(x) = e^x$$

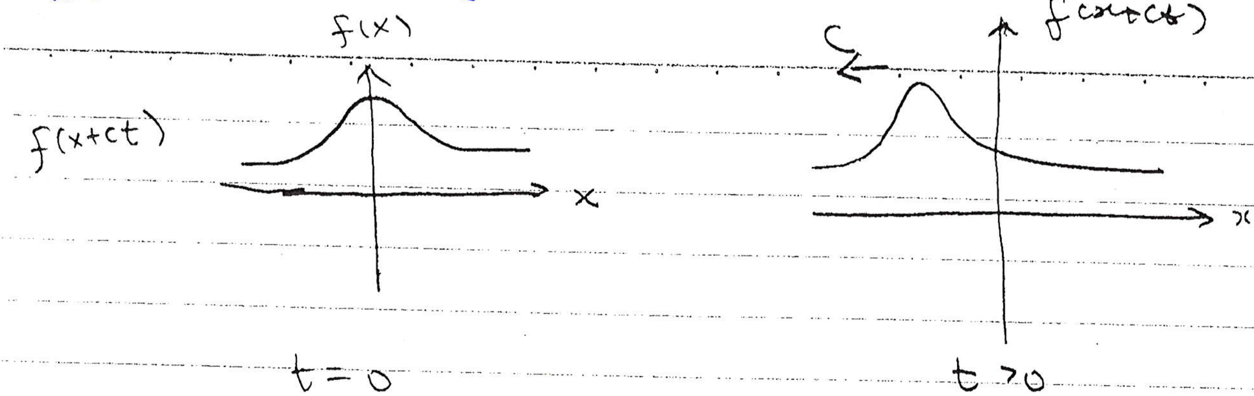
$$f(x) = \frac{1}{3}\cos x + \frac{1}{3}e^x$$

$$g(x) = \frac{2}{3}\cos x - \frac{1}{3}e^x$$

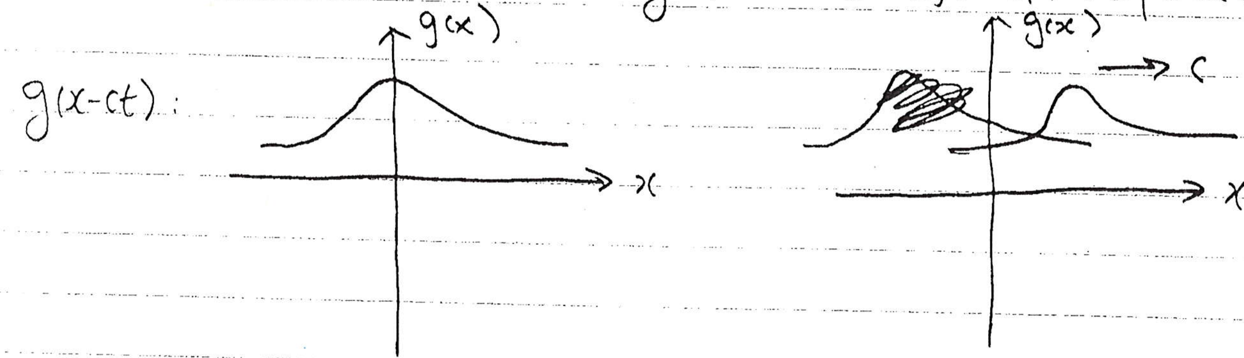
$$u = \frac{1}{3}\cos(x+2t) + \frac{1}{3}e^{x+2t}$$

$$+ \frac{2}{3}\cos(x-t) - \frac{1}{3}e^{x-t}$$

Geometric Meanings



wave moving to the left with speed c



wave moving to the right with speed c

General sol'n = combination of waves moving to the left and to the right with speed c

Now we plug into initial conditions

Initial Value Problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

$u(x,0) = \phi(x) \Rightarrow f(x) + g(x) = \phi(x)$

$u_t(x,0) = \psi(x) \Rightarrow c(f'(x) - g'(x)) = \psi(x)$

$\Rightarrow f(x) - g(x) = \frac{1}{c} \int_{x_0}^{x} \psi(x) + f(x_0) - g(x_0)$

$$f(x) = \frac{1}{2}\left[\phi(x) + \frac{1}{c}\int_{x_0}^{x} \psi(s)\,ds + A\right]$$

$$g(x) = \frac{1}{2}\left[\phi(x) - \frac{1}{c}\int_{x_0}^{x} \psi(s)\,ds - A\right]$$

$$u(x,t) = f(x+ct) + g(x-ct)$$

$$= \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

d'Alembert's formula

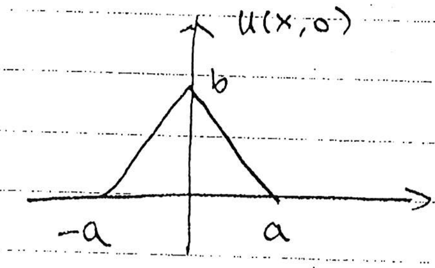
Example 1. $\phi(x) = 0, \quad \psi(x) = \cos x$

$$u(x) = \frac{1}{c}\cos x \sin ct$$

Example 2. $\phi(x) = \cos x, \quad \psi(x) = a$

$$u(x) = \cos x \cos ct$$

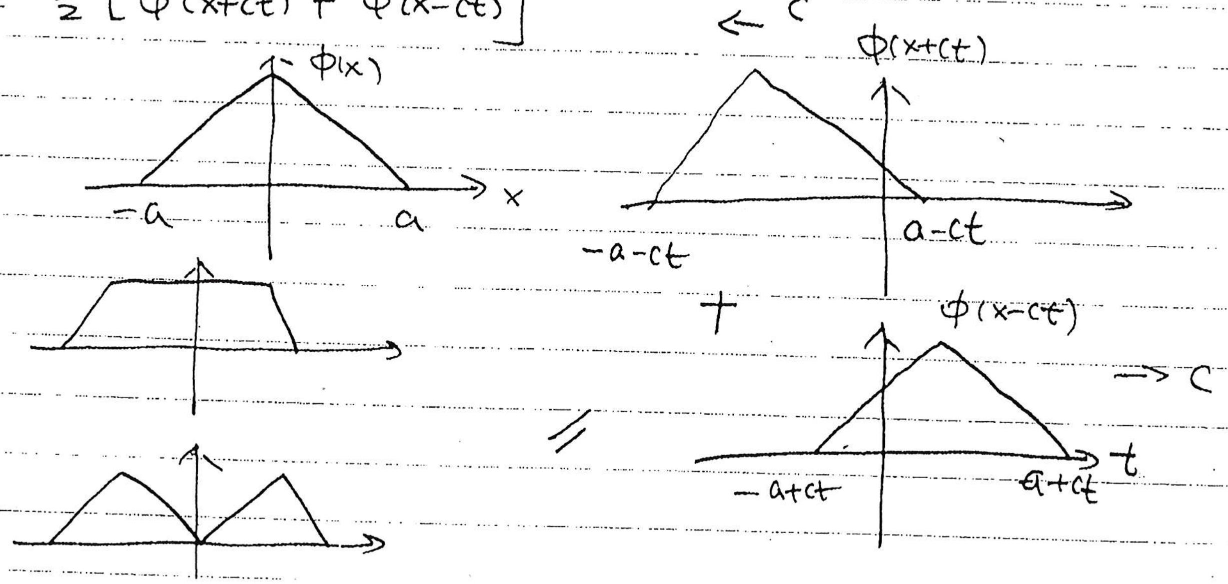
Example 3. $\phi(x) = \begin{cases} b - \frac{b|x|}{a}, & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \qquad \psi(x) = 0$



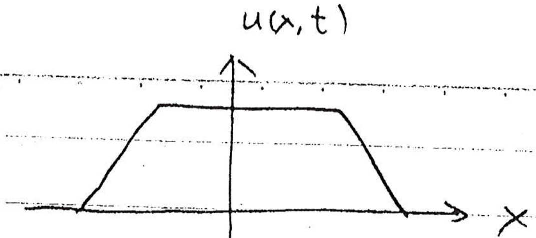
"plucked string"

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

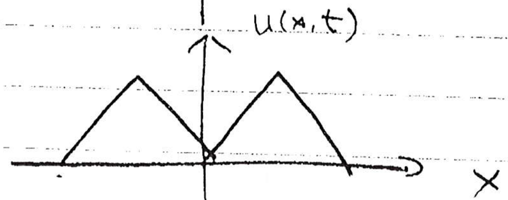
Geometric



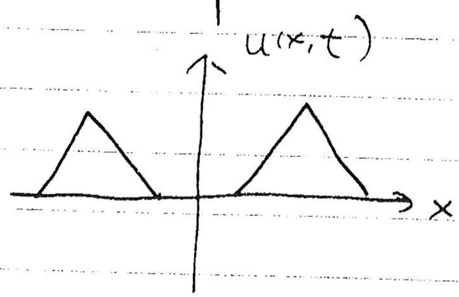
$t = \frac{a}{2c}$



$t = \frac{a}{c}$



$t > \frac{a}{c}$



~~$x < -\frac{3a}{2}$, or~~

Let us compute the case when $t = \frac{a}{2c}$

$x < -\frac{3a}{2}$ or $x > \frac{3a}{2}$, $u(x, \frac{a}{2c}) = 0$

$-\frac{3}{2}a < x < -\frac{a}{2}$, $u(x, \frac{a}{2c}) = \frac{1}{2}\phi(x+ct) = \frac{1}{2}\left(b - b\frac{|x+ct|}{a}\right)$

$= \frac{1}{2}\left(b + \frac{b}{a}\left(x + \frac{a}{2}\right)\right) = \frac{3}{4}b + \frac{bx}{2a}$

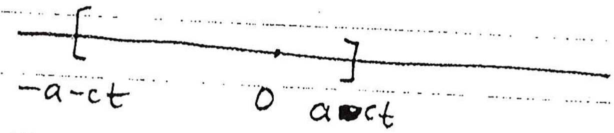
$\frac{a}{2} < x < \frac{a}{2}$, $u(x, \frac{a}{2c}) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$

$= \frac{1}{2}\left(b - b\frac{|x+\frac{a}{2}|}{a} + b - b\frac{|x-\frac{a}{2}|}{a}\right) = \frac{1}{2}b$

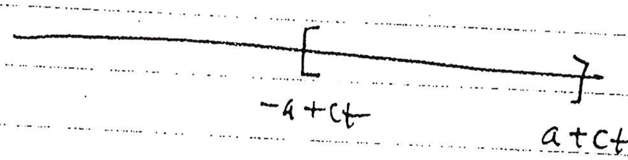
$\frac{1}{2} < x < \frac{3a}{2}$ $u(x, \frac{a}{2c}) = \frac{1}{2}\phi(x-ct)$

You can also compute other cases

$$\phi(x+ct) = \begin{cases} b - \dfrac{b|x+ct|}{a}, & |x+ct| < a \\ 0, & |x+ct| > a \end{cases}$$



$$\phi(x-ct) = \begin{cases} b - b\dfrac{|x-ct|}{a}, & |x-ct| < a \\ 0 & |x-ct| > a \end{cases}$$



Example 4 : $\phi(x) \equiv 0$, $\psi(x) = 1$ for $|x| < a$, $\psi(x) = 0$ for $|x| \geq a$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$= \frac{1}{2c}\left|\text{length of } (x-ct, x+ct) \cap (-a,a)\right|$$

Let $t = \frac{a}{2c}$

So $x < -\frac{3a}{2}$ or $x > \frac{3a}{2}$, $\Rightarrow u(x, \frac{a}{2c}) = 0$

$|x| < \frac{a}{2}$, $\frac{a}{2} < x < \frac{3a}{2}$, $-\frac{3a}{2} < x < -\frac{a}{2}$, u takes different values

Ex. 3, Ex. 4 : If both $u(x,0)$, $u_t(x,0)$ vanish outside $|x| < a$, then $u(x,t)$ vanishes outside $|x| < a + ct$

Example 3. Solve

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = \phi(x) = \begin{cases} b\left(1 - \frac{|x|}{a}\right), & |x| \le a \\ 0, & |x| > a \end{cases}$$

Sol'n: By d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

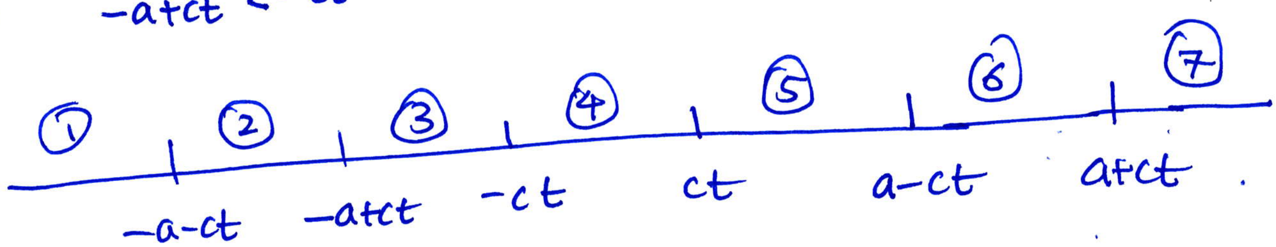
where

$$\phi(x+ct) = \begin{cases} b\left(1 + \frac{x+ct}{a}\right), & -a-ct < x < -ct \\ b\left(1 - \frac{x+ct}{a}\right) & -ct < x < a-ct \\ 0, & x < -a-ct \text{ or } x > a-ct \end{cases}$$

$$\phi(x-ct) = \begin{cases} b\left(1 + \frac{x-ct}{a}\right), & -a+ct < x < ct \\ b\left(1 - \frac{x-ct}{a}\right), & ct < x < a+ct \\ 0 & x < -a+ct \text{ or } x > a+ct \end{cases}$$

Case 1. $t < \frac{a}{2c}$

So $-a+ct < -ct < ct < a-ct$



①&⑦
$x < -a-ct$ or $x > a+ct$: $\quad \phi(x+ct) = \phi(x-ct) = 0$

②: $-a-ct < x < -a+ct \quad \Rightarrow \quad -a < x+ct < 0$

$x - ct < -a$

$$u = \frac{1}{2}\phi(x+ct) = \frac{b}{2}\left(1 + \frac{x+ct}{a}\right)$$

③: $-a+ct < x < -ct \quad \Rightarrow \quad -a < x+ct < 0$

$-a < x-ct < 0$

$$u = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

$$= \frac{b}{2}\left(1 + \frac{x+ct}{a}\right) + \frac{b}{2}\left(1 + \frac{x-ct}{a}\right) = b + \frac{x}{a}$$

④ $-ct < x < ct \quad \Rightarrow \quad x-ct < 0$

$0 < x+ct < a$

$$u = \frac{b}{2}\left(1 - \frac{x+ct}{a}\right) + \frac{b}{2}\left(1 + \frac{x-ct}{a}\right)$$

$$= b - \frac{ct}{a}$$

⑤ $ct < x < a-ct \quad \Rightarrow \quad a > x-ct > 0, \quad 0 < x+ct < a$

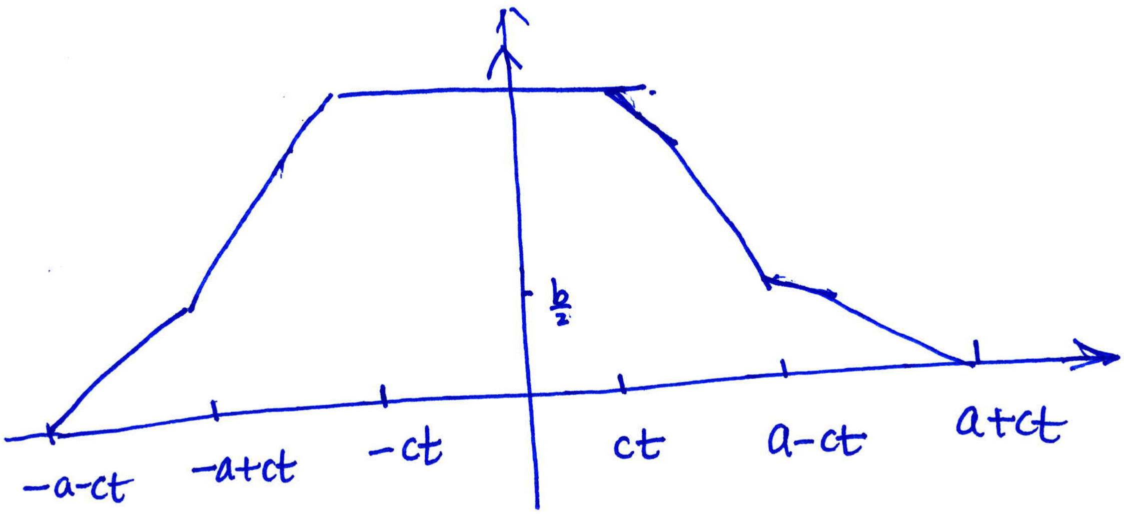
$$u(x,t) = \frac{b}{2}\left(1 - \frac{x+ct}{a}\right) + \frac{b}{2}\left(1 - \frac{x-ct}{a}\right)$$

$$= b - \frac{bx}{a}$$

⑥ $a-ct < x < a+ct \quad \Rightarrow \quad x+ct > a, \quad 0 < x-ct < a$

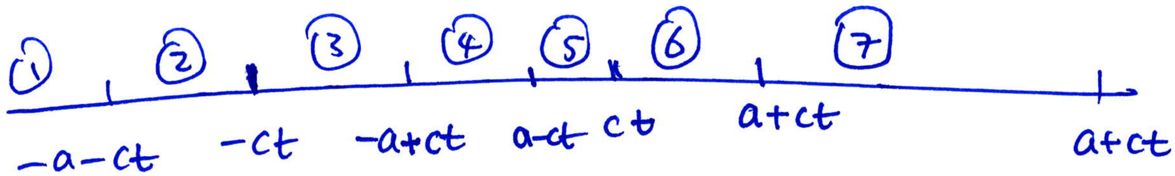
$$u(x,t) = \frac{b}{2}\left(1 - \frac{x-ct}{a}\right)$$

⑦



Case 2. $\frac{a}{2c} < t < \frac{a}{c}$

In this case $a - ct > 0$, $0 > a+ct \cancel{>} -ct$, $a-ct < ct$



(2): $-a-ct < x < -ct \Rightarrow -a < x+ct < 0$

$x-ct < -a$

$$u = \frac{b}{2}\left(1+\frac{x+ct}{2}\right)$$

(3) $-ct < x < -a+ct \Rightarrow 0 < x+ct$, $x-ct < -a$

$$u = \frac{b}{2}\left(1-\frac{x+ct}{2}\right) + \cancel{\frac{b}{2}(1+}$$

(4) $-a+ct < x < a-ct < ct$ $-a < x-ct < 0$, $0 < 2ct-a < x+ct < a$

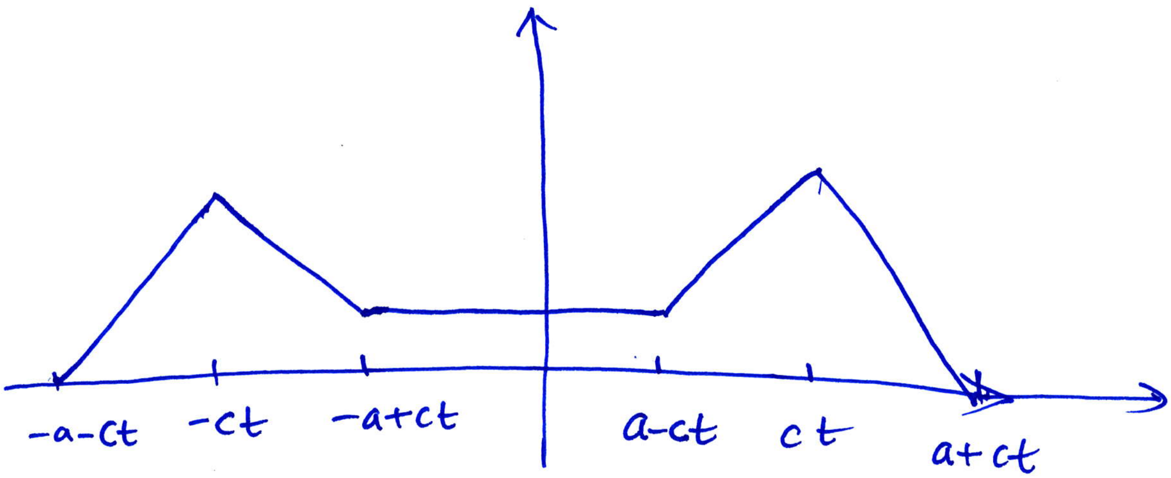
$$u = \frac{b}{2}\left(1-\frac{x+ct}{a}\right) + \frac{b}{2}\left(1+\frac{x-ct}{a}\right) = b - \frac{ct}{a}$$

(5) $a-ct < x < ct \Rightarrow x+ct > a\ ;\ -a < x-ct < 0$

$$u = \frac{b}{2}\left(1 + \frac{x-ct}{2}\right)$$

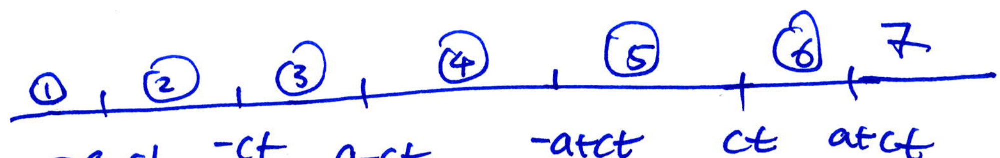
(6) $ct < x < a+ct \Rightarrow 0 < x-ct < a$

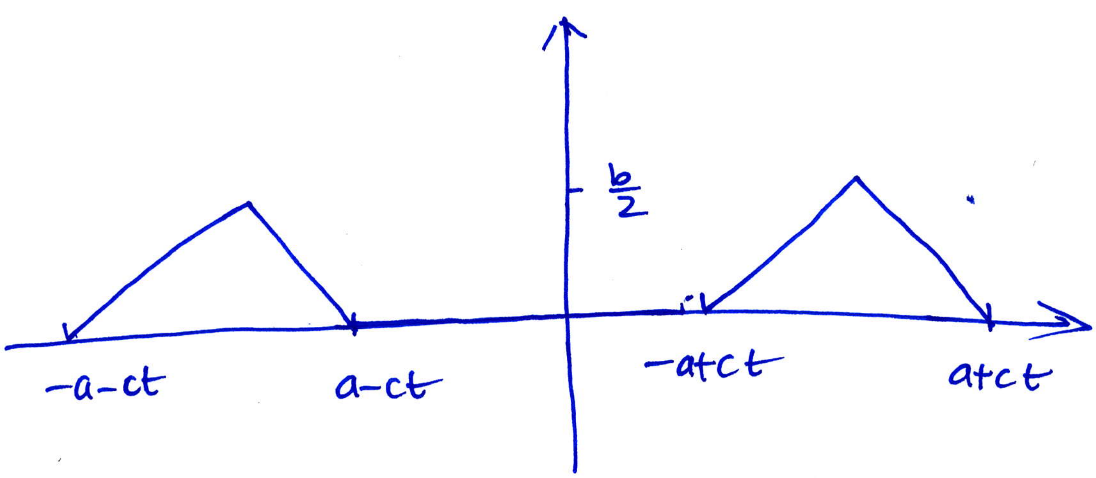
$$u = \frac{b}{2}\left(1 - \frac{x-ct}{2}\right)$$



Case 3 $\quad t > \frac{a}{c}$

In this case,





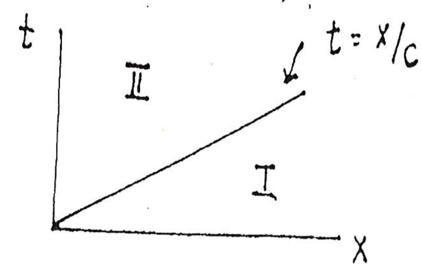
EXAMPLE FIND THE SOLUTION $u(x,t)$ TO THE FOLLOWING WAVE EQUATION IN THE QUARTER PLANE $x>0,\ t>0$.

$$u_{tt} = c^2 u_{xx} \qquad x>0,\ t>0$$

$$u(0,t) = h(t), \quad u(x,0) = g(x), \quad u_t(x,0) = f(x).$$

THE GENERAL SOLUTION IS $u(x,t) = F(x-ct) + G(x+ct)$.

THERE ARE TWO DIFFERENT REGIONS:



SATISFYING THE DATA GIVES

(1) $F(-ct) + G(ct) = h(t), \quad t>0$

(2) $F(x) + G(x) = g(x), \quad x>0$

(3) $-cF'(x) + cG'(x) = f(x), \quad x>0.$

NOW (2) AND (3) YIELD F AND G FOR POSITIVE ARGUMENTS.

WE OBTAIN $\quad -cF(x) + cG(x) = \int_0^x f(\lambda)\,d\lambda$

HENCE $\quad -F(x) + G(x) = \frac{1}{c}\int_0^x f(\lambda)\,d\lambda$

$$F(x) + G(x) = g(x)$$

THIS YIELDS (4) $\quad F(x) = \frac{g(x)}{2} - \frac{1}{2c}\int_0^x f(\lambda)\,d\lambda, \quad x>0$

(5) $\quad G(x) = \frac{g(x)}{2} + \frac{1}{2c}\int_0^x f(\lambda)\,d\lambda, \quad x>0.$

HENCE (6) $\quad F(x-ct) = \frac{1}{2}g(x-ct) - \frac{1}{2c}\int_0^{x-ct} f(\lambda)\,d\lambda \qquad x-ct>0$

(7) $\quad G(x+ct) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} f(\lambda)\,d\lambda$

THIS GIVES $\quad u(x,t) = \frac{1}{2}\left[g(x-ct)+g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f(\lambda)\,d\lambda$ REGION I.
$x-ct>0$

NOW FROM (1) WE OBTAIN

$$F(-ct) = h(t) - G(ct) \qquad t > 0$$

SO $$F(z) = h(-z/c) - G(-z), \quad z < 0$$

THEREFORE

$$(8) \quad F(x-ct) = h(t-x/c) - G(ct-x), \quad x-ct < 0$$

$$(9) \quad G(x+ct) = \frac{1}{2} g(x+ct) + \frac{1}{2c}\int_0^{x+ct} f(\lambda)\,d\lambda$$

HENCE $$(10) \quad F(x-ct) = h(t-x/c) - \frac{1}{2} g(ct-x) - \frac{1}{2c}\int_0^{ct-x} f(\lambda)\,d\lambda$$

NOW ADDING (9) AND (10) WE OBTAIN

$$u(x,t) = h(t-x/c) + \frac{1}{2}[g(x+ct) - g(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} f(\lambda)\,d\lambda$$

THIS IS THE SOLUTION IN REGION II WHERE $t > x/c$.

Ex. Solve

$$\begin{cases} u_{tt} - u_{tx} - 2u_{xx} = 0, & 0 < x < +\infty, \ t > 0 \\ u(x,0) = \cos x, \ u_t(x,0) = e^x, & 0 < x < +\infty \\ u(0,t) = 1, & t > 0 \end{cases}$$

sol'n: Step 1. General sol'ns

$$\partial_t^2 - \partial_t \partial_x - 2\partial_x^2 = (\partial_t - 2\partial_x)(\partial_t + \partial_x)$$

$$u = f(x+2t) + g(x-t)$$

$$f(x) + g(x) = \cos x, \quad x > 0$$

$$2f'(x) - g'(x) = e^x, \quad x > 0 \Rightarrow 2f(x) - g(x) = e^x$$

$$f(x) = \frac{1}{3}\cos x + \frac{1}{3}e^x, \quad x > 0$$

$$g(x) = \frac{2}{3}\cos x - \frac{1}{3}e^x, \quad x > 0$$

$$u(0,t) = 1, \ t > 0 \Rightarrow f(2t) + g(-t) = 1$$

$$g(-t) = 1 - f(2t) = 1 - \left(\frac{1}{3}\cos 2t + \frac{1}{3}e^{2t}\right)$$

$$-t = \bar{x} < 0 \Rightarrow t = -\bar{x} \Rightarrow$$

$$g(\bar{x}) = 1 - \left(\frac{1}{3}\cos(-2\bar{x}) + \frac{1}{3}e^{-2\bar{x}}\right)$$

$$g(x) = \begin{cases} \frac{2}{3}\cos x - \frac{1}{3}e^x, & x > 0 \\ 1 - \left(\frac{1}{3}\cos 2x + \frac{1}{3}e^{-2x}\right), & x < 0 \end{cases}$$

so

$$u(x,t) = f(x+2t) + g(x-t) = \begin{cases} \frac{1}{3}\cos(x+2t) + \frac{1}{3}e^{x+2t} + \frac{2}{3}\cos(x-t) - \frac{1}{3}e^{x-t}, & x > t \\ \frac{1}{3}\cos(x+2t) + \frac{1}{3}e^{x+2t} + 1 - \left(\frac{1}{3}\cos 2(x-t) + \frac{1}{3}e^{-2(x-t)}\right), & x < t \end{cases}$$

IMPEDANCE MATCHING 3-LAYER MATERIAL

WE CONSIDER $U_{tt} = c^2 U_{xx}$ WITH
$$c = \begin{cases} c_1, & x < 0 \\ c_2, & 0 < x < L \\ c_3, & x > L \end{cases}$$

WE WANT NO REFLECTED WAVE IN $x < 0$.

WE WRITE
$$U = \begin{cases} \exp(i\omega(t - x/c_1)) & x \le 0 \\ T_2 \exp(i\omega(t - x/c_2)) + R_2 \exp(i\omega(t + x/c_2)), & 0 \le x \le L \\ T_3 \exp(i\omega(t - x/c_3)), & x \ge L \end{cases}$$

NOW WE REQUIRE THAT U AND U_x ARE CONTINUOUS ACROSS $x = 0$ AND $x = L$.

ON $x = 0$:
$$T_2 + R_2 = 1, \quad [U] = 0$$
$$T_2 - R_2 = c_2/c_1, \quad [U_x] = 0$$

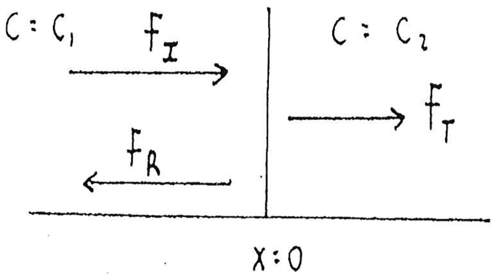
ACROSS $x = L$:
$$T_2 e^{-i\omega L/c_2} + R_2 e^{i\omega L/c_2} = T_3 e^{-i\omega L/c_3}, \quad [U] = 0$$
$$T_2 e^{-i\omega L/c_2} - R_2 e^{i\omega L/c_2} = \frac{c_2}{c_3} T_3 e^{-i\omega L/c_3}, \quad [U_x] = 0$$

YOU WILL OBTAIN THAT
$$\cos\left(\frac{2\omega L}{c_2}\right) = \left(\frac{c_3 - c_2}{c_3 + c_2}\right)\left(\frac{c_1 + c_2}{c_1 - c_2}\right)$$

IF $c_2 = \sqrt{c_1 c_3}$ THEN $\cos\left(\frac{2\omega L}{c_2}\right) = -1$

AND $L = \dfrac{\pi c_2}{2\omega}$.

REFLECTION AT AN INTERFACE



$$u_{tt} = c^2 u_{xx} \qquad c^2 = T/\rho$$

IF $c_1 > c_2$ THEN SECOND string FOR $X > 0$ IS heavier

ASSUME AN INCIDENT WAVE ON THE LEFT $F_I(x - c_1 t)$.

THEN, WE HAVE

$$u(x,t) = \begin{cases} F_I(x - c_1 t) + F_R(x + c_1 t), & x \le 0 \\ F_T(x - c_2 t), & x \ge 0. \end{cases}$$

NOW u AND u_x ARE CONTINUOUS ACROSS THE INTERFACE $X = 0$

$$F_I(-c_1 t) + F_R(c_1 t) = F_T(-c_2 t)$$

$$F_I'(-c_1 t) + F_R'(c_1 t) = F_T'(-c_2 t)$$

INTEGRATE WRT t TO GET

$$-\frac{1}{c_1} F_I(-c_1 t) + \frac{1}{c_1} F_R(c_1 t) = -\frac{1}{c_2} F_T(-c_2 t).$$

THIS CAN BE SOLVED TO OBTAIN

$$F_T(-c_2 t) = \frac{2 c_2}{c_1 + c_2} F_I(-c_1 t)$$

$$F_T(x - c_2 t) = \frac{2 c_2}{c_1 + c_2} F_I\left(\frac{c_1}{c_2}(x - c_2 t)\right)$$

$$F_R(x + c_1 t) = \left(\frac{c_2 - c_1}{c_1 + c_2}\right) F_I(-(x + c_1 t))$$

IF $c_1 \gg c_2$ TRANSMITTED WAVE AMPLITUDE IS VERY SMALL (string is essentially immobile FOR $X > 0$)

2.2 Causality and Energy

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Thm: If $\phi(x) = 0$ for $|x| > a$, $\psi(x) = 0$ for $|x| > a$, then $u(x,t) = 0$ for $|x| > a + ct$

Proof: when $|x| > a+ct$, $x > a+ct$ or $x < -a-ct$

$x > a+ct$, $x+ct > a+2ct > a$, $\phi(x+ct) = 0$

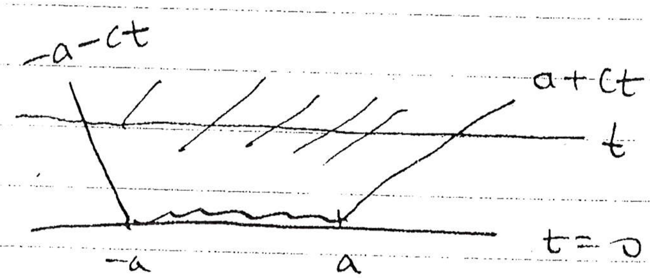
$x - ct > a \Rightarrow \phi(x-ct) = 0$

$\psi(s) = 0$ for $s > x-ct > a$

$$\Rightarrow \int_{x-ct}^{x+ct} \psi(s)\,ds = 0$$

so $u = 0$. Similar case for $x < -a-ct$

#

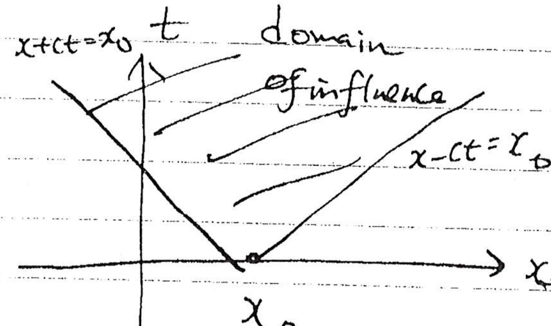


waves traveling in the speed c

no part goes faster than speed c: Principle of Causality

Two more definitions

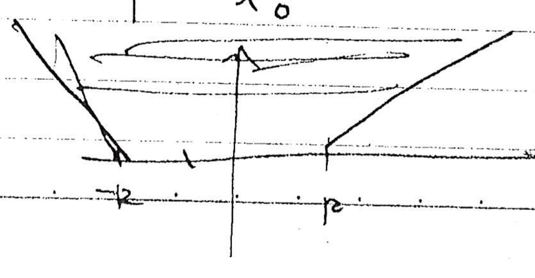
Def 1: domain of influence



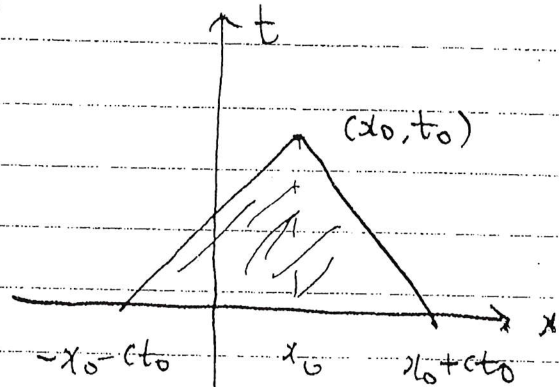
at $(x_0, 0)$:

domain of influence of interval $|x| \le R$

is a sector $|x| \le R + ct$



Def 2. Domain of dependence at (x_0, t_0)



The value of u at (x_0, t_0) depends only on the Initial value of u on the interval $(x_0 - ct_0, \ x_0 + ct_0)$

$(x_0 - ct_0, x_0 + ct_0)$ is called the domain of dependence

Energy: $\rho u_{tt} = T u_{xx}$

Energy = Kinetic Energy + Potential energy

$\frac{1}{2} m v^2 \qquad \frac{1}{2} T u_x^2$

$$= \frac{1}{2} \rho \int u_t^2 + \frac{T}{2} \int u_x^2$$

Thm 2. The energy is preserved, i.e. $E(t) = E(0), \ \forall t > 0$

Proof: $\frac{dE}{dt} = \rho \int u_t u_{tt} + T \int u_x u_{tx}$

$$= \int (T u_t u_{xx} + T u_x u_{tx})$$

$$= \int T (u_t u_x)_x$$

$$= 0$$

$E(t) \equiv E(0)$

Example: goback to Example 3

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a}, & |x| < a \\ 0, & |x| \geq a \end{cases}, \quad \psi \equiv 0$$

Then $E(t) = E(0) = \frac{\rho}{2}\int u_t^2(x,0)\,dx + \frac{T}{2}\int u_x^2(x,0)\,dx$

$$= \frac{T}{2}\int \phi_x^2\,dx$$

$$= \frac{Tb^2}{a}$$

Well-posedness of wave eqns:

- Existence: yes
- uniqueness: yes
- stability: yes

3.4 Wave with a Source

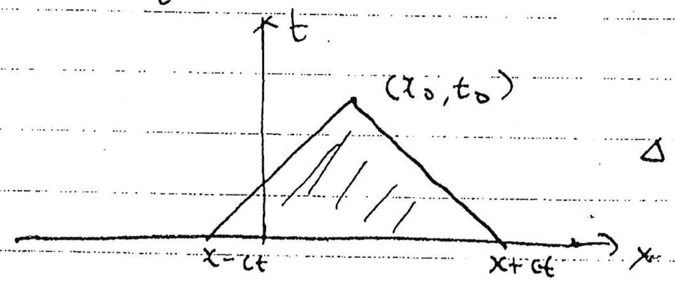
$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(s) \end{cases}$$

As before, we decompose

$$u = u_1 + u_2, \qquad \text{where} \quad u_1 = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

We need to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$



Δ = domain of dependence

$$\Delta = \{ 0 \le s \le t, \quad x - c(t-s) \le y \le x + c(t-s) \}$$

Theorem: $u(x,t) = \frac{1}{2c}\iint_\Delta f$

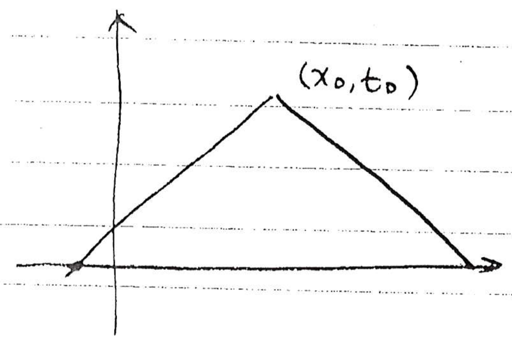
$$= \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-s)}^{x_0+c(t_0-s)} f(y,s)\,dy\,ds$$

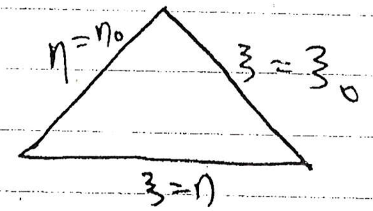
Proof: We prove it ~~in~~ by ~~two methods~~

~~Method I: by Green's formula~~

Me~~thod II~~: by coordinate method

Coordinate Method





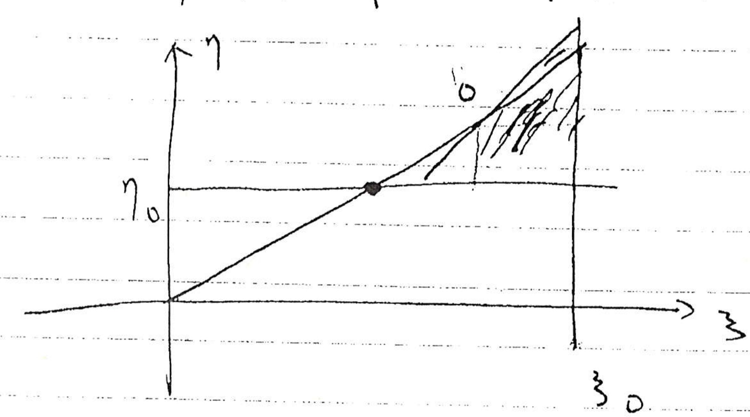
et $\quad \xi = x + ct, \quad \eta = x - ct, \quad x = \dfrac{\xi+\eta}{2}, \quad t = \dfrac{\xi - \eta}{2c}$

$$\Delta_{(y,s)} = \{(y,s) \mid 0 \le s \le t, \quad x_0 - c(t_0 - s) \le y \le x_0 + c(t_0 - s)\}$$

$$\Delta_{(\xi,\eta)} = \left\{(\xi,\eta) \;\middle|\; 0 \le \xi - \eta, \quad x_0 - ct_0 + \frac{\xi-\eta}{2} \le \frac{\xi+\eta}{2} \le x_0 + ct_0 - \frac{\xi-\eta}{2}\right\}$$

$$= \{(\xi,\eta) \mid 0 \le \xi - \eta, \quad \eta \ge x_0 - ct_0 := \eta_0, \quad \xi \le x_0 + ct_0 := \xi_0\}$$

$$= \{(\xi,\eta) \mid 0 \le \cancel{\xi-\eta}, \quad \eta \ge \eta_0, \quad \eta \le \xi \le \xi_0\}$$



$$= \{(\xi,\eta) \mid \eta_0 \le \eta \le \xi, \quad \eta_0 \le \xi \le \xi_0\}$$

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = g(\xi,\eta)$$

$$u_{\xi\eta} = -\frac{1}{4c^2} g(\xi,\eta)$$

$-u_\xi(\eta_0, \eta_0)$

$$u_\xi(\xi, \eta_0) = -\frac{1}{4c} \int_\xi^{\eta_0} g(\xi, \eta) \, d\eta$$

$$u(\xi_0, \eta_0) - u(\eta_0, \eta_0) = -\frac{1}{4c} \int_{\eta_0}^{\xi_0} \int_\xi^{\eta_0} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\eta \, d\xi$$

$$= \frac{1}{4c^2} \iint_\Delta f \; J \, dx \, dt$$

$$J = \left| \det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \right| = 2c$$

$$u(x_0, t_0) = \frac{1}{4c^2} \iint_\Delta f \cdot J \, dx \, dt = \frac{1}{2c} \iint_\Delta f \, dx \, dt$$

#

In conclusion:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi + \frac{1}{2c} \iint_\Delta f$$

From the formula, we obtain the

① existence

② uniqueness

③ stability

$$|u(x,t)| \le \max|\phi| + \frac{1}{2c} \max|\psi| \cdot 2ct + \frac{1}{2c} \max|f| ct^2$$

$$= \max|\phi| + t \max|\psi| + \frac{t^2}{2} \max|f|$$

Hence if $|\phi_1 - \phi_2| < \delta$, $|\psi_1 - \psi_2| < \delta$, $|f_1 - f_2| < \delta$

then $|u_1 - u_2| \le \delta (1 + T + T^2) < \varepsilon$

Example 1: $\begin{cases} u_{tt} = c^2 u_{xx} + xt \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$$\Rightarrow u(x_0, t_0) = \frac{1}{2c} \iint_{\Delta} f(x,t)\,dx\,dt$$

$$= \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0-s)}^{x_0 + c(t_0-s)} xt \; dx\,dt$$

$$= \frac{1}{2c} \int_0^{t_0} t \left[\frac{1}{2} \left((x_0 + c(t_0-t))^2 - (x_0 - c(t_0-t))^2 \right) \right] dt$$

$$= \frac{1}{2c} \int_0^{t_0} t \, 2(x_0 (t_0-t))\, dt$$

Duhammel's Principle:

We want to solve

$$(1)\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0)=0, \quad u_t(x,0) = 0 \end{cases}$$

Fix $s>0$. Solve $U(x,t;s)$

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0, & t>s \\ U(x,s;s) = 0, \quad U_t(x,s;s) = 0 \end{cases}$$

Then $u(x,t) = \int_0^t U(x,t;s)\,ds$ solves (1)

<u>Proof:</u> $u(x,0)=0$, $u_t(x,t) = U(x,t;t) + \int_0^t U_t(x,t;s)\,ds$

$$= 0 + \int_0^t U_t(x,t;s)\,ds$$

$u_t(x,0) = 0$

$$u_{tt} = U_t(x,t;t) + \int_0^t U_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t U_{xx}$$

$$= f(x,t) + \left(\int_0^t U\right)_{xx}$$

$$= f(x,t) + u_{xx}.$$

#

OTHER EXAMPLES OF WAVE EQUATION

SOUND WAVES CONSIDER

$$\rho_t + \nabla \cdot [\rho \underline{u}] = 0 \qquad p = \text{PRESSURE}$$

$$\underline{u}_t + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p \qquad \rho = \text{density.}$$

WE ASSUME A CONSTITUTIVE LAW THAT

$$\rho = \rho(p) \quad \text{(density depends on pressure).}$$

LET $p = p_0 + \tilde{p}$ ($\tilde{p} \ll 1$ SMALL) $\quad p_0$ = CONSTANT.

THEN $\rho = \rho(p_0 + \tilde{p}) = \rho(p_0) + \rho'(p_0)\tilde{p} + \ldots$

BY TAYLOR'S THEORY.

THEN WE LABEL $\tilde{\rho} \equiv \rho'(p_0)\tilde{p}$

THEN

$$\frac{\partial}{\partial t}\tilde{\rho} + \nabla \cdot [(\rho_0 + \tilde{\rho})\underline{u}] = 0$$

$$\frac{\partial}{\partial t}\underline{u} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho_0 + \tilde{\rho}} \nabla \tilde{p}$$

ASSUMING THAT $\|\underline{u}\| \ll 1$ AND NEGLECTING QUADRATIC TERMS SUCH AS $\tilde{\rho}\,\underline{u}$ AND $\tilde{\rho}\,\nabla \hat{p}$ WE OBTAIN

$$\frac{\partial}{\partial t}\tilde{\rho} + \rho_0 \nabla \cdot \underline{u} = 0, \qquad \frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho_0} \nabla \hat{p}.$$

NOW DIFFERENTIATE WRT t TO OBTAIN $\tilde{\rho}_{tt} + \rho_0 \nabla \cdot [\underline{u}_t] = 0$

THUS

$$\frac{1}{\rho_0}\tilde{\rho}_{tt} = -\nabla \cdot [\underline{u}_t] = \frac{1}{\rho_0} \nabla^2 \hat{p}.$$

THIS YIELDS $\tilde{\rho}_{tt} = \nabla^2 \tilde{p}$.

HOWEVER, $\tilde{\rho} = \rho'(p_0)\,\tilde{p}$

HENCE $\tilde{p}_{tt} = c^2 \nabla^2 \tilde{p}$ $\qquad c = 1/\sqrt{\rho'(p_0)}$

THIS IS A PRESSURE WAVE EQUATION FOR THE PERTURBATION OF THE PRESSURE $\tilde{p}$.