Reflections of Waves:

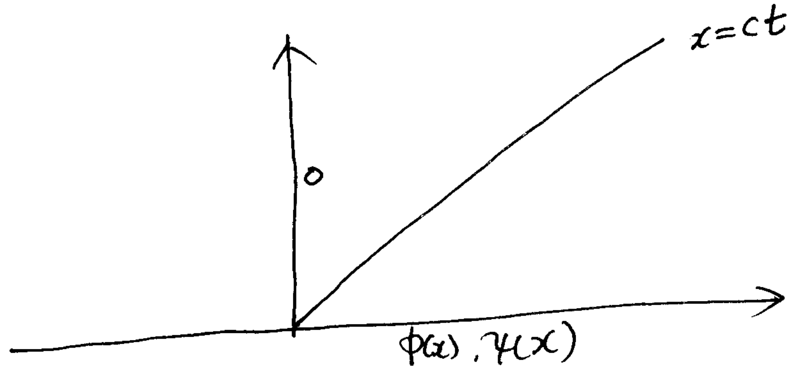
Lecture 6

Another way to solve wave equation

DE: $v_{tt} - c^2 v_{xx} = 0$, $0 < x < +\infty$, $-\infty < t < \infty$

IC: $v(x,0) = \phi(x)$, $v_t(x,0) = \psi(x)$ for $t = 0$

BC: $v(0,t) = 0$ for $x = 0$ $-\infty < t < \infty$



Extend v, ϕ, ψ oddly through $x = 0$

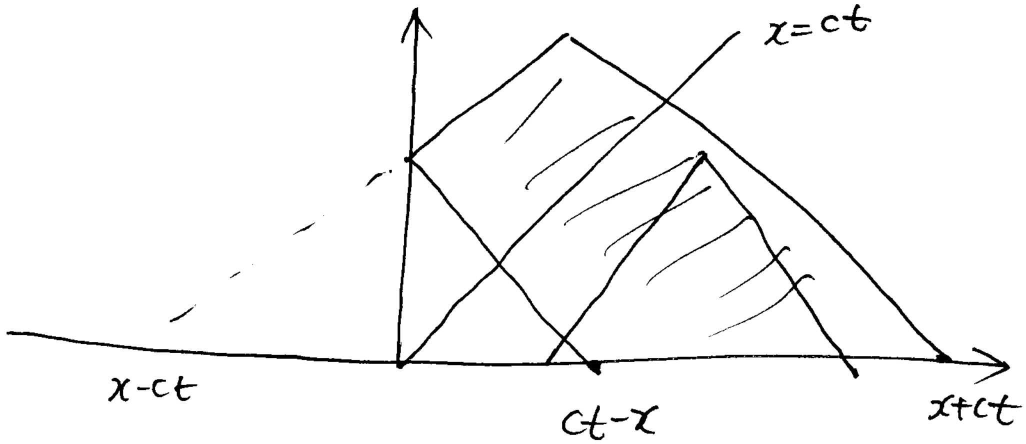
$$\phi_{odd}(x) = \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases}$$

$$\psi_{odd}(x) = \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases}$$

$$v(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy$$

If $x > ct$, then $v = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$

If $x < ct$, then $v = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy$



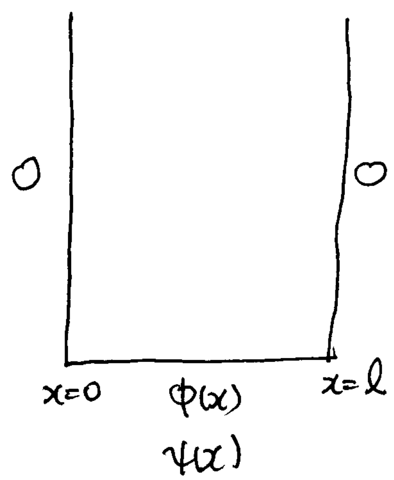
Reflection induces a change of sign

Reflection of Finite Intervals

Let us consider the guitar string with fixed ends:

$$v_{tt} = c^2 v_{xx}, \quad v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \quad 0<x<l$$

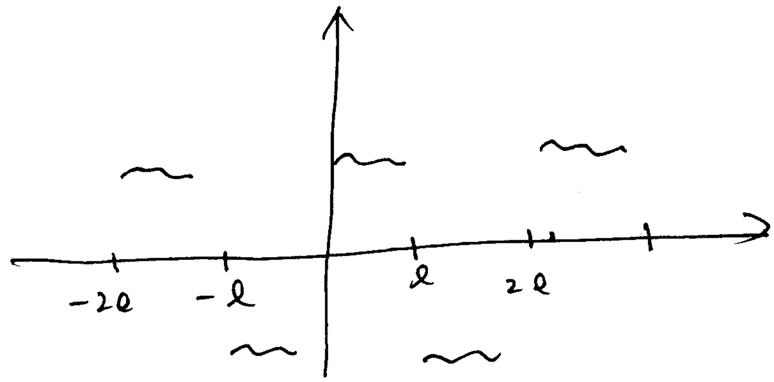
$$v(0,t) = v(l,t) = 0$$



In this problem, there two walls of reflection.

Solution: Extend ϕ, ψ, oddly to $-l<x<0$,

Then extend periodically to the whole line

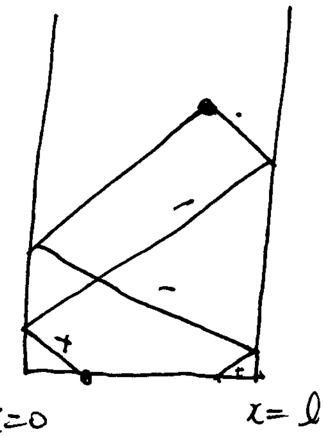


$$\phi_{ext}(x) = \begin{cases} \phi(x), & 0<x<l \\ -\phi(-x), & -l<x<0 \\ \text{extended to be of period } 2l \end{cases}$$

$$\psi_{ext}(x) = \begin{cases} \psi(x), & 0<x<l \\ -\psi(-x), & -l<x<0 \\ \text{extended to be of period } 2l \end{cases}$$

$$v(x,t) = \frac{1}{2}\phi_{ext}(x+ct) + \frac{1}{2}\phi_{ext}(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)\,ds\,ds.$$

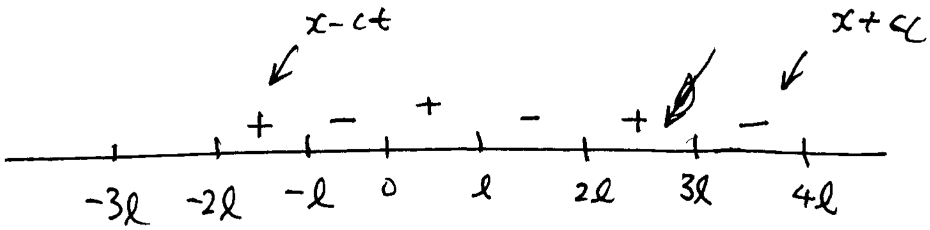
The way to understand the explicit solution is by drawing a space-time diagram.



Each reflection $\Rightarrow$ "$-$" sign

Let us suppose $\quad 3l < x+ct < 4l$

$$-2l < x-ct < -l$$



$$\phi_{ext}(x-ct) = \phi(x-ct+2l)$$

$$\phi_{ext}(x+ct) = -\phi(4l-x-ct)$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{ext} = \frac{1}{2c}\Big[\int_{x-ct}^{-l} + \int_{-l}^{0} -\psi(-y)\,dy + \int_{0}^{l}\psi(y)\,dy$$

$$+ \int_{l}^{2l} -\psi(-y+2l)\,dy + \int_{2l}^{3l}\psi$$

$$+ \int_{3l}^{x+ct} -\psi(-y+4l)\,dy\Big]$$

$$= \frac{1}{2c}\int_{x-ct+2l}^{l}\psi(s)\,ds + \frac{1}{2c}\int_{l}^{4l-x-ct}\psi(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct+2l}^{4l-x-ct}\psi(s)\,ds$$