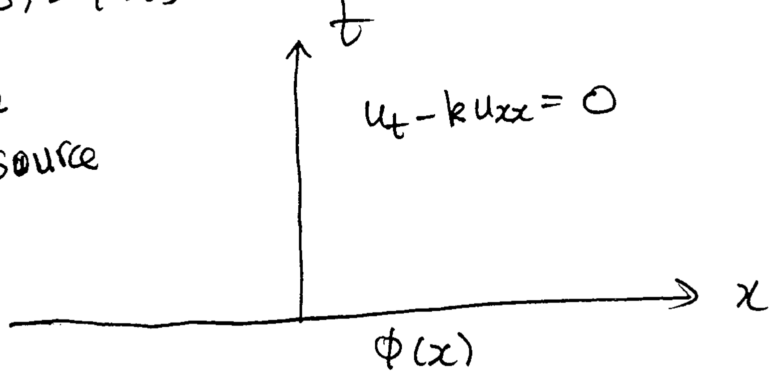
Chapter 2.4 Diffusion Equation on the whole line

<u>AIM:</u> Obtain explicit solution formula for diffusion equation on the whole line $\mathbb{R}$

$$(\text{IVP}) \begin{cases} u_t = k\,u_{xx} + \underbrace{f(x,t)}_{\text{source}}, & -\infty < x < \infty, \quad 0 < t < +\infty \\ u(x,0) = \phi(x) \end{cases}$$

First we assume that $\exists$ no source



Recall: for wave equation, we can factorize the equation.

For diffusion eqn, we can't factorize.

Idea: $u_t = k u_{xx}$ $\longrightarrow$ ① Find a particular solution

② use properties of $u_t - k u_{xx}$ to get general solutions

Properties of diffusion eqn: $u_t - k u_{xx} = 0$

(a) $u(x-y,t)$ is another sol'n (translation invariance)

(b) u_x, u_t, u_{xx}, etc is again a solution

(c) linear combination of sol'ns is again a sol'n

(d) An integral is a sol'n

$$v(x,t) = \int_{-\infty}^{+\infty} u(x-y,t)\, g(y)\, dy \quad \text{is a sol'ns}$$

$$\left(\because \approx \sum u(x-y_i,t)\, g(y_i)\, \Delta y_i\right)$$

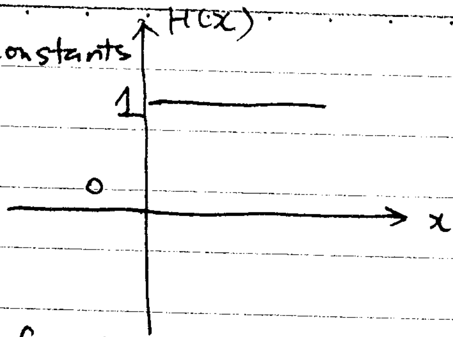
$\boxed{(c)}$ (Dilation Invariance) If $u(x,t)$ is a solution, then the dilation function

$u(\sqrt{a}\,x,\ at)$ is also a sol'n, $\forall\, a>0$

$$\bar{x} = \sqrt{a}\,x,\quad \bar{t} = at, \qquad \frac{\bar{x}^2}{\bar{t}} = \frac{x^2}{t} \quad (\text{invariance}).$$

We will use property (c) to obtain a special solution. First we also need to choose an initial data $\phi(x)$ (different from constant) which has the dilation invariance.

Next to Constants



$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Heaviside function

Solve:
$$\begin{cases} Q_t - kQ_{xx} = 0 \\ Q(x,t) = H(x) \end{cases}$$

Since $H(x)$ is dilation invariant, we look for sol'ns of the following form:

$$Q(x,t) = g(p), \qquad p = \frac{x}{\sqrt{4kt}}$$

This reduces the problem to an ODE

Let us compute

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}\, g'(p)$$

$$Q_x = \frac{1}{\sqrt{4kt}}\, g'(p), \qquad Q_{xx} = \frac{1}{4kt}\, g''(p)$$

$$g_{pp} + 2p\, g_p = 0$$

$$g(p) = c_1 \int_0^p e^{-p^2}\, dp + c_2$$

$$Q(x,t) = c_1 \int_0^p e^{-p^2}\, dp + c_2$$

$$= c_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp + c_2$$

Je now decide the two constants

If $x > 0$, $\quad 1 = \lim_{t \to 0} Q = c_1 \frac{\sqrt{\pi}}{2} + c_2$

If $x < 0$, $\quad 0 = \lim_{t \to 0} Q = -c_1 \frac{\sqrt{\pi}}{2} + c_2$

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp$$

Now we differentiate

$S(x,t) = \frac{\partial Q}{\partial x}$ is again a sol'n

$$= \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}} \quad \text{for } t > 0$$

Theorem: $u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\, \phi(y)\, dy$, for $t > 0$

Proof: $u(x,t) = \int_{-\infty}^{+\infty} \frac{\partial Q}{\partial x}(x-y,t)\, \phi(y)\, dy$

$$= -\int_{-\infty}^{+\infty} \frac{\partial}{\partial y}\left[Q(x-y,t)\right] \phi(y)\, dy$$

$$= \int_{-\infty}^{+\infty} Q(x-y,t)\, \phi'(y)\, dy - Q(x-y,t)\,\phi(y)\Big|_{y=-\infty}^{y=+\infty}$$

$$u(x,0) = \int_{-\infty}^{+\infty} S(x-y,0)\, \phi'(y)\, dy = \phi \Big|_{-\infty}^{x} = \phi(x)$$

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$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$ – fundamental sol'ns

– Green's function

– Source Function

– Gaussian Potential

Properties of $S(x,t)$

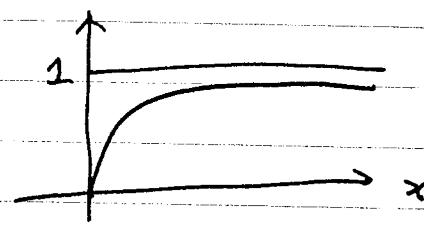
1. $\int_{-\infty}^{+\infty} S(x,t)\,dx = 1 \quad \Rightarrow \int_{-\infty}^{+\infty} S(x-y,t)\,dy = 1, \quad \forall y$

2. $S(x,t) = S(-x,t)$

3. Fix $\delta > 0$, $\max_{|x|\geq\delta} S(x,t) \to 0$ as $t \to 0$

To use the formula, it is convenient to introduce the following notation

$$\mathcal{E}rf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp$$



$$\mathcal{E}rf(0) = 0, \quad \mathcal{E}rf(+\infty) = 1$$

Ex. 1 $\quad Q(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\,\mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)$$

2. $\phi(x) = e^{-x}$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y}\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{y^2-2xy+4kty+x^2}{4kt}}\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+2kt-x)^2+x^2-(2kt-x)^2}{4kt}}\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy \; e^{-\frac{4ktx-4k^2t^2}{4kt}}$$

$$= \int_{-\infty}^{+\infty} S(y-2kt+x, t)\, dy \; e^{-x+kt}$$

$$= e^{kt-x}$$

Ex. 3. $\phi(x) = 1$ for $|x| < a$, $\phi(x) = 0$ for $|x| > a$

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\, \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-a}^{a} e^{-\frac{(x-y)^2}{4kt}} dy \qquad \left(\text{let } y - x = \sqrt{4kt}\; p\right)$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{x-a}^{x+a} e^{-p^2} \sqrt{4kt}\, dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4kt}}}^{\frac{x+a}{\sqrt{4kt}}} e^{-p^2} dp$$

$$= \frac{1}{2}\left(-\mathcal{E}rf\left(\frac{a-x}{\sqrt{4kt}}\right) + \mathcal{E}r\left(\frac{a+x}{\sqrt{4kt}}\right)\right) \neq 0 \quad \forall x, t$$

This shows that diffusion eqn has ∞ speed

Diffusion with a source

$$\begin{cases} u_t - k u_{xx} = f(x,t) & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = \phi(x) \end{cases} \qquad (*)$$

Theorem: The sol'n to $(*)$ is given by

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\,\phi(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds$$

Proof: Let $u_1(x) = \int_{-\infty}^{+\infty} S(x-y,t)\,\phi(y)\,dy$

$$u_2(x) = \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds$$

Let us compute u_2:

$$(u_2)_t = \int_{-\infty}^{+\infty} S(x-y,0)\,f(y,t)\,dy$$

$$= f(x,t) + \int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial t} S(x-y,t-s)\,f(y,s)\,dy\,ds$$

$$= f(x,t) + \int_0^t \int_{-\infty}^{+\infty} k\,\frac{\partial^2 S}{\partial x^2}(x-y,t-s)\,f(y,s)\,dy\,ds$$

$$= f(x,t) + k\,\frac{\partial^2}{\partial x^2}\left[\int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds\right]$$

$$= f(x,t) + k\,\frac{\partial^2}{\partial x^2} u_2$$

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If $\phi = 0$, $u(x,t) = \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}}\, e^{-\frac{(x-y)^2}{4k(t-s)}}\, f(y,s)\,dy\,ds$

general solution:

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t-s)\,\phi(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds$$

~~Idea~~ $u_t - k u_{xx} = 0 \Rightarrow u = e^{-tA}\phi(0)$

$$u_t + Au = f(t) \Rightarrow u = e^{-tA} u(0) + \int_0^t e^{(s-t)A} f(s)\,ds$$

Source on a half-line

$$\begin{cases} v_t - k v_{xx} = f(x,t), & x > 0 \\ v(x,0) = \phi(x) \\ v(0,t) = h(t) \end{cases}$$

Let $V(x,t) = v(x,t) - h(t)$

$$\begin{cases} V_t - k V_{xx} = f(x,t) - h'(t) \\ V(x,0) = \phi(x) - h(0) \\ V(0,t) = 0 \end{cases}$$

method of reflection

$$f_{odd} \to \phi_{odd} \to V_{ext} \to V(x,t)$$

Neumann problem

$$\begin{cases} v_t - k v_{xx} = f(x,t), & x > 0 \\ v(x,0) = \phi(x) \\ v_x(0,t) = h(t) \end{cases}$$

$V(x,t) = \phi(x) - x h(t)$