Recall: Let $S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$, $t>0$

be the source function. Then the solution to the diffusion equation

$$\begin{cases} u_t = k u_{xx}, & t>0 \\ u(x,0) = \phi(x), & -\infty < x < +\infty \end{cases}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

$$= \int_{-\infty}^{+\infty} S(x-y,t)\, \phi(y)\, dy$$

$$\boxed{u := S(t)\phi}$$

Next we want to solve diffusion with a source

$$(*) \quad \begin{cases} u_t - k u_{xx} = f(x,t), & -\infty < x < +\infty,\ t>0 \\ u(x,0) = \phi(x) \end{cases}$$

Theorem: The solution to $(*)$ is given by

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s) f(y,s)\,dy\,ds$$

$(**)$

Proof: We just need to show that the solution to (*) with $\phi \equiv 0$.

Let $$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} S(x-y, t-s) f(y,s)\, dy\, ds$$

Let us compute

$$u_t = \int_{-\infty}^{+\infty} S(x-y, 0) f(y,t)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial t} S(x-y, t-s) f(y,s)\, dy\, ds$$

$$= f(x,t) + \int_0^t \int_{-\infty}^{+\infty} k \frac{\partial^2 S}{\partial x^2}(x-y, t-s) f(y,s)\, dy\, ds$$

$$= f(x,t) + k \frac{\partial^2}{\partial x^2} \left[\int_0^t \int_{-\infty}^{+\infty} S(x-y, t-s) f(y,s)\, dy\, ds \right]$$

$$= f(x,t) + k \frac{\partial^2}{\partial x^2} u$$

so $u_t - k u_{xx} = f(x,t)$

On the other hand, clearly we have
$$u(x,0) = 0$$

\#

General solution:

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y,s)\, dy\, ds$$

Example 1: Solve

$$\begin{cases} u_t - k u_{xx} = 1 \\ u(x,0) = 0 \end{cases}$$

Solution:

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} dy\, ds$$

$$= \int_0^t 1\, ds = t$$

#

Now we discuss well-posedness of diffusion equation

$$\begin{cases} u_t - k u_{xx} = f(x,t), & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Well-posedness.

(1) Existence: Given $(f(x,t), \phi(x))$, there exists a solution

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s) f(y,s)\, dy\, ds$$

② Uniqueness: Given (f, ϕ), the solution is unique.

discuss later

③ stability: From the formula (**), we have

$$\max_{0\le t\le T} |u(x,t)| \le \int_{-\infty}^{+\infty} S(x-y, t) \, |\max|\phi(y)|| \, dy$$

$$+ \int_0^t \int_{-\infty}^{+\infty} S(x-y, t-s) \max_{0\le s\le T} |f(y,s)| \, dy \, ds$$

$$\le \max|\phi| + T \max_{\substack{-\infty<x<\infty \\ 0\le t\le T}} |f(x,t)|$$

$$\le (1+T)\left(\max|\phi| + \max|f(x,t)|\right)$$

So if ϕ, f are small, then u is small.

Finally, we prove uniqueness: by energy method:

Consider diffusion equation

$$u_t - k u_{xx} = 0, \quad -\infty < x < +\infty, \ t > 0$$

Consider its energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u^2(x,t) \, dx$$

let $E(t)=\frac{1}{2}\int_{-\infty}^{+\infty} v^2(x,t)\,dx$

Then $E(0)=\frac{1}{2}\int_{-\infty}^{+\infty} v^2(x,0)\,dx = 0$

$$\frac{dE}{dt} = -k\int_{-\infty}^{+\infty} v_x^2 \le 0$$

So $0 \le E(t) \le E(0)=0$ for $t \ge 0$

$\Rightarrow$ $E(t)=0,\ \forall t>0 \Rightarrow v(x,t)\equiv 0,\ \forall x,\ \forall t>0$ #

Now we go back to wave equation.

$$\begin{cases} u_{tt}-c^2u_{xx} = f(x) \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x) \end{cases}$$

We can use similar energy method to prove uniqueness:

Let $E(t) = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2(x,t)dx+\frac{1}{2}c^2\int_{-\infty}^{+\infty} u_x^2(x,t)\,dx$

Then
$$\frac{dE}{dt} = \int(u_t u_{tt} + c^2 u_x u_{xt})$$
$$= \int(u_t c^2 u_{xx} + c^2 u_x u_{xt})$$
$$= c^2 \int (u_t u_x)_x\,dx$$
$$=0$$

Then

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_x u_{xt}\, dx$$

$$= k \int_{-\infty}^{+\infty} u_x u_{xx}\, dx$$

$$= -k \int_{-\infty}^{+\infty} u_x^2 \quad + k \int_{-\infty}^{+\infty} (u u_x)_x\, dx$$

$$= -k \int_{-\infty}^{+\infty} u_x^2 \quad + k\, u u_x \Big|_{-\infty}^{+\infty}$$

$$= -k \int_{-\infty}^{+\infty} u_x^2$$

Consequence: $\dfrac{dE}{dt} \leq 0 \Rightarrow$ energy is decreasing

$\Rightarrow$ dissipative

<u>Proof of Uniqueness</u>: Suppose there are two solutions u_1, u_2 to the problem

$$\begin{cases} u_t - k u_{xx} = f(x,t), & t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Then $v(x,t) = u_1 - u_2$ satisfies

$$\begin{cases} v_t - k v_{xx} = 0, & t > 0 \\ v(x,0) = 0 \end{cases}$$

As a consequence, for wave equation, the energy is "conserved"

Conservation of Energy: $E(t) = E(0), \forall t$

Proof of uniqueness: As before, $u = u_1 - u_2$ satisfies

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = 0, \ u_t(x,0) = 0 \end{cases}$$

So $u_x(x,0) = 0$, $u_t(x,0) = 0$

$\Rightarrow E(0) = 0 = E(t) \Rightarrow u_t = u_x \equiv 0 \Rightarrow u \equiv$ Constant

$\Rightarrow u \equiv 0$.

#

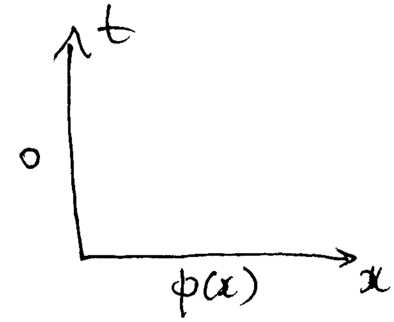
Summary:

	Formula	Energy	Wellposedness	Discontinuity
diffusion Equation $\begin{cases} u_t - k u_{xx} = f \\ u(x,0) = \phi(x) \end{cases}$	$u = \int S(x-y,t)\phi(y)\,dy + \int_0^t \int S(x-y,t-s) f(y,s)\,dy\,ds$	decreasing	Yes	becomes smooth when $t>0$
Wave Eqn $u_{tt} - c^2 u_{xx} = f$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$	$u = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$	conserved	Yes	Propogate along the charateristics

Diffusion Equation with boundaries: Method of Reflection

Ex. 1 Dirichlet BC

$$\begin{cases} u_t - k u_{xx} = f(x,t), & x>0,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \end{cases}$$



Solution: Reflection: extend ϕ, f, oddly to $x<0$

$$f_{ext}(x,t) = \begin{cases} f(x,t), & x>0 \\ -f(-x,t), & x<0 \end{cases}$$

$$\phi_{ext}(x,t) = \begin{cases} \phi(x), & x>0 \\ -\phi(-x), & x<0 \end{cases}$$

Solve $\begin{cases} u_t - k u_{xx} = f_{ext}(x,t) \\ u(x,0) = \phi_{ext} \end{cases}$

$$u_{ext}(x,t) = \int S(x-y,t)\,\phi_{ext} + \int_0^t \int S(x-y,t-s)\, f_{ext}$$

Check: $u_{ext}(0,t) = 0$.

By uniqueness $u(x,t) = u_{ext}(x,t)$
(same proof as before)

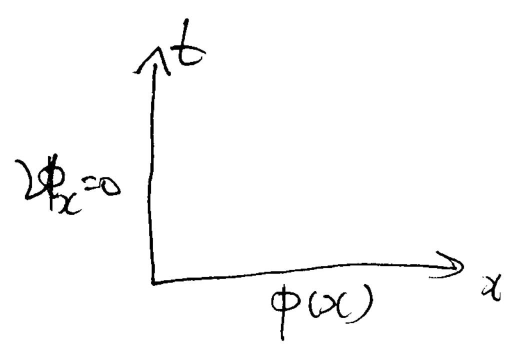
Example 1: Solve $\begin{cases} u_t - k u_{xx} = 0, & x>0 \\ u(x,0) = 1 \\ u(0,t) = 0 \end{cases}$

Solution

$$
\begin{aligned}
u(x,t) &= \int S(x-y,t)\,\phi_{ext}(y)\,dy \\
&= \int_{-\infty}^{+\infty} S(x-y,t)\,dy - \int_{-\infty}^{0} S(x-y,t)\,dy \\
&= \int_{0}^{+\infty} S(x-y,t)\,dy - \int_{0}^{+\infty} S(x+y,t)\,dy \\
&= \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp\Big/\sqrt{\pi} - \int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-q^2}\,dq\Big/\sqrt{\pi} \\
&= \left[\frac{1}{2} + \frac{1}{2}\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right] - \left[\frac{1}{2} - \frac{1}{2}\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right] \\
&= \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) \qquad \#
\end{aligned}
$$

Ex. 2 Neumann BC

$$
\begin{cases}
u_t - k u_{xx} = f(x,t), & x>0, \quad t>0 \\
u(x,0) = \phi(x) \\
u_x(0,t) = 0
\end{cases}
$$



Solution: Reflection: extend ϕ, f evenly to $x>0$

$$f_{ext}(x,t) = \begin{cases} f(x,t), & x>0 \\ f(-x,t), & x<0 \end{cases}$$

$$\phi_{ext}(x,t) = \begin{cases} \phi(x), & x>0 \\ \phi(-x), & x<0 \end{cases}$$

Solve $\begin{cases} u_t - k u_{xx} = f_{ext}(x,t) \\ u(x,0) = \phi_{ext} \end{cases}$

$$u_{ext}(x,t) = \int S(x-y,t)\,\phi_{ext} + \int_0^t \int S(x-y,t-s)\, f_{ext}$$

Check: $u_{ext,x}(0,t) = 0$

By uniqueness, $u(x,t) = u_{ext}(x,t)$

Example 2 Solve $\begin{cases} u_t - k u_{xx} = 0, & x>0 \\ u(x,0) = 1 \\ u_x(0,t) = 0 \end{cases}$

Sol'n: $u(x,t) = \int S(x-y,t)\,\phi_{ext}(y)\,dy = \int S(x-y,t)\,dy = 1$

#

Source on a half-line

Ex. 3
$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(0,t) = h(t) \\ u(x,0) = \phi(x) \end{cases}$$

Sol'n: Let $V(x,t) = u(x,t) - h(t)$ $\Rightarrow$

$$\begin{cases} V_t - k V_{xx} = f(x,t) - h'(t) \\ V(0,t) = 0 \\ V(x,0) = \phi(x) - h(0) \end{cases}$$

Ex. 4
$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u_x(0,t) = h(t) \\ u(x,0) = \phi(x) \end{cases}$$

Sol'n: Let $V(x,t) = u(x,t) - x\,h(t)$

$$\begin{cases} V_t - k V_{xx} = f(x,t) \\ V_x(0,t) = 0 \\ V(x,0) = \phi(x) - x\,h(0) \end{cases}$$

#

What about two boundary conditions
$$\begin{cases} u_t - k u_{xx} = f(x,t), \quad 0 < x < l \\ u(x,0) = \phi(x) \\ u(0,t) = 0, \quad u(l,t) = 0 \end{cases}$$

$\Rightarrow$ Lecture 9