Lecture Note 9.5

Summary of Robin Boundary Condition Eigenvalue Problems:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l \\ X'(0) - a_0 X(0) = 0, & X'(l) + a_l X(l) = 0 \end{cases}$$

→ Equation for negative eigenvalues

$$\lambda = -\gamma^2 < 0$$

$$\tanh \gamma l = \frac{(a_0 + a_l)\gamma}{(-a_0 a_l) \gamma^2}, \quad X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x$$

→ Equation for zero eigenvalue

$$\lambda = 0 \iff a_0 + a_l + (a_0 a_l) l = 0$$

$$X(x) = 1 + \frac{a_0}{\gamma} x$$

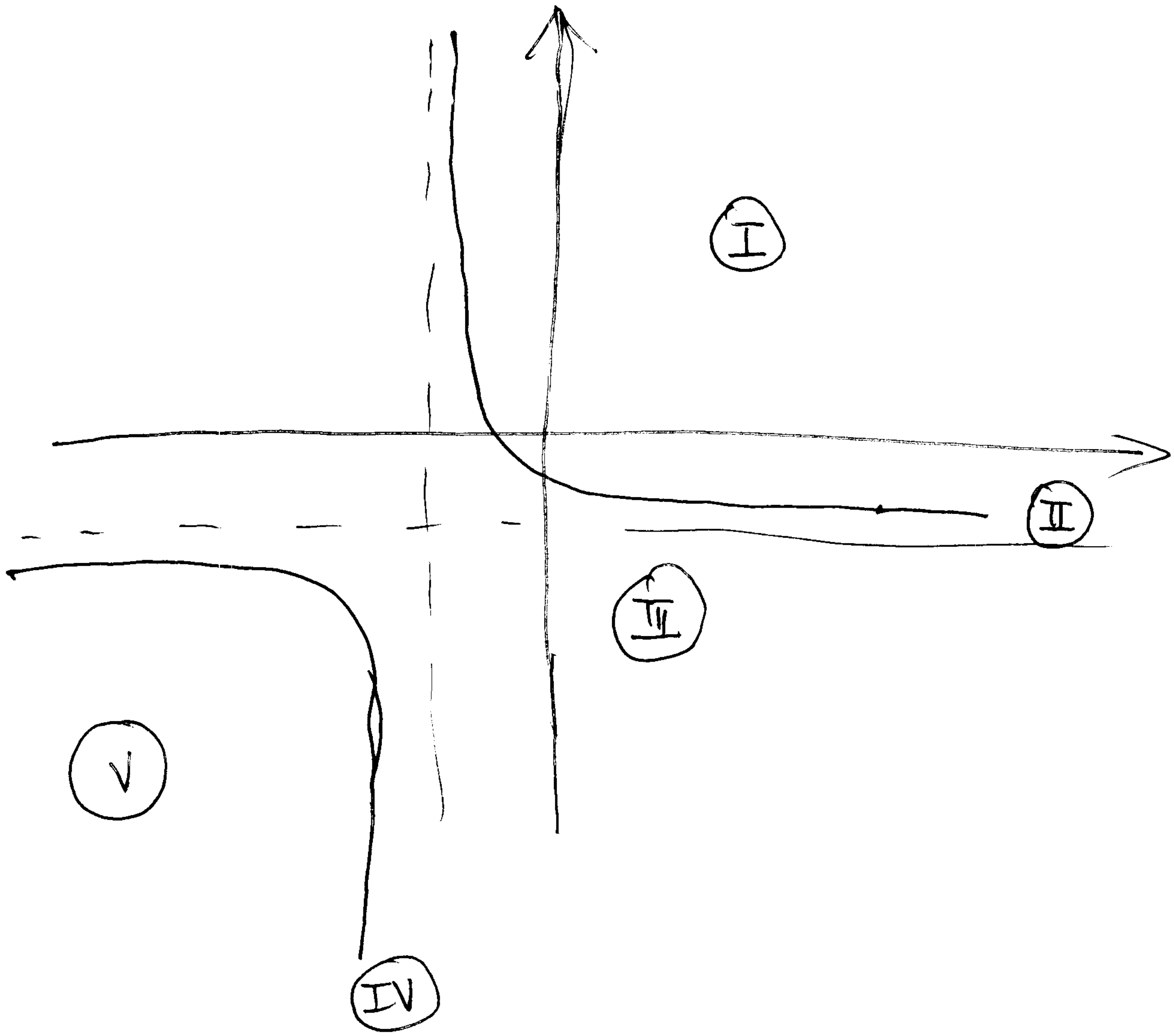
→ Equation for positive eigenvalues

$$\lambda = \beta^2 > 0, \quad \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}$$

$$X(x) = \cos \beta x + \frac{a_0}{\beta} \sin \beta x$$

Number of Negative eigenvalues : Look at the parameter space (a_0, a_ℓ) and hyperbola

$$\left(a_0+\frac{1}{\ell}\right)\left(a_\ell+\frac{1}{\ell}\right)=\frac{1}{\ell^2} \iff a_0+a_\ell+(a_0\cdot a_\ell)\ell=0$$



I : $0<\lambda_1<\lambda_2<\cdots$: No negative or zero

II : $\lambda_0=0<\lambda_1<\lambda_2<\cdots$: No negative, one zero

III : $\lambda_{-1}<0<\lambda_1<\lambda_2<\cdots$: One negative

IV : $\lambda_{-1}<\lambda_0=0<\lambda_1<\lambda_2<\cdots$ One negative, one zero

V : $\lambda_{-2}<\lambda_{-1}<\cancel{\lambda_0=}\,0<\lambda_1<\lambda_2<\cdots$, Two negative

Example 1. Determine how many negative eigenvalues for

$$X'' + \lambda X = 0, \quad 0 < x < 2$$

(a) $X'(0) - X(0) = 0, \quad X'(l) - X(l) = 0$

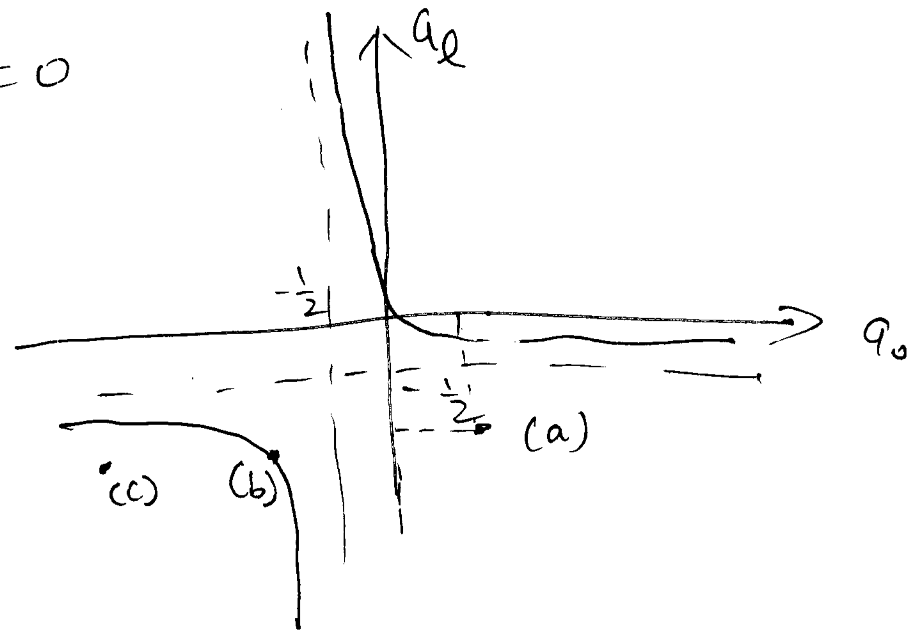
(b) $X'(0) + X(0) = 0, \quad X'(l) - X(l) = 0$

(c) $X'(0) + 2X(0) = 0, \quad X'(l) - X(l) = 0$

Sol'n: $l = 2$, hyperbola is

$$a_0 + a_l + 2a_0 a_l = 0$$

Map the parameters in the parameter Space



(a) $a_0 = 1, a_l = -1$

Region (II).

$\lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots$

$\lambda_{-1} = -\gamma^2, \quad \tanh \gamma l = \frac{0}{1-\gamma^2} \Rightarrow \gamma = 1.$

$\lambda = \beta^2, \quad \tan(2\beta) = 0, \quad \beta = \frac{(2n-1)}{2}\pi$

(b) $a_0 = -1, a_l = -1$, Region (IV). one negative eigenvalue, one zero

$\lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$

$\lambda_{-1} = -\gamma^2, \quad \tanh 2\gamma = \frac{-2\gamma}{1-\gamma^2}, \quad \lambda = \beta^2, \quad \tan\beta = \frac{-2\beta}{\beta^2 - 1}$

(c) $a_0 = -2, a_l = -1$, Region (V), two negative eigenvalues,

$\lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots, \quad \tanh 2\gamma = \frac{-3\gamma}{2-\gamma^2}, \quad \lambda = \beta^2, \quad \tan\beta = \frac{-3\beta}{\beta^2 - 2}$