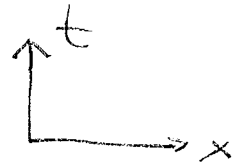
Lecture 9

In summary: we solved the IVP

$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \end{cases} \qquad \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

For BVP we just did

$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \end{cases}$$



We have formula!!!

Chapter 4: we will study BVP.

1-D: $D = (0, l)$, $\partial D = \{0, l\}$.

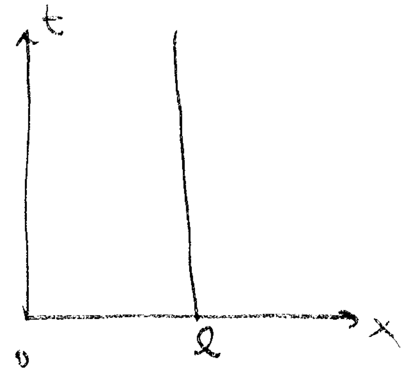
Method: Separation of Variables (against ODE)

3 kinds of BCs:

4.1 Separation of variables — Dirichlet Problem

Model Problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < l \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$



Idea: separate t and x.

Look for sol'ns of the type

$u(x,t) = X(x)\,T(t)$

Substitute into eqn:

$$X(x)\,T''(t) = c^2 X''(x)\,T(t)$$

$$-\frac{X''}{X} = -\frac{T''}{c^2 T} = \lambda. \qquad \lambda \text{ constant}$$

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = X(l) = 0 \end{cases} \qquad T'' + c^2 \lambda T = 0.$$

B.C.s:

Typical Sturm-Liouville Eigenvalue problem (do it) $\leftarrow$ e.v.s

$$\lambda = \left(\frac{n\pi}{l}\right)^2, \quad n=1,2,\cdots; \qquad X_n(x) = \sin\frac{n\pi x}{l} \leftarrow \text{e.fs}$$

$$T(t) = A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}, \quad A_n, B_n \text{ arbitrary}$$

We have obtained a sequence of functions $X_n(x)T_n(t)$: satisfies eqn + BCs.

IVBC.

Summation : $\sum_{n=1}^{+\infty} X_n(x)\,T_n(t) = u(x,t)$, hopefully, by choosen A_n, B_n approriately, $u(x,t)$ satisfies the ICs.

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$u_t(x,t) = \ldots$$

$$\phi(x) = \sum_{n=1}^{+\infty} A_n\sin\frac{n\pi x}{l} \qquad (1)$$

$$\psi(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\sin\frac{n\pi x}{l} \qquad (2)$$

How do we obtain A_n, B_n from (1) & (2).

$$\int_0^l \sin\frac{m\pi x}{l}\,\phi(x) = \sum_{n=1}^{+\infty} A_n\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}$$

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = l\int_0^1 \sin n\pi t\,\sin m\pi t = \begin{cases} 0 & m \neq n \\ \frac{l}{2}, & m = n\end{cases}$$

$$\boxed{\frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,\phi(x) = A_m}$$

$$\frac{2}{m\pi c}\int_0^l \psi\,\sin\frac{n\pi x}{l}\,dx = B_m$$

Conclusion: if we take

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin\frac{n\pi x}{\ell}\,dx$$

$$B_n = \frac{2}{n\pi c} \int_0^{\ell} \psi(x) \sin\frac{n\pi x}{\ell}\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi ct}{\ell} + B_n \sin\frac{n\pi ct}{\ell}\right) \sin\frac{n\pi x}{\ell}$$

then $u(x,t)$ is a sol'n of B.C.

$$\boxed{f(x) = \sum_{n=1}^{+\infty} A_n \sin\frac{n\pi x}{\ell} \quad \text{— Fourier series of } f(x)} \Rightarrow \text{chapter 5}$$

Thm: ① $f(0) = f(\ell) = 0$

② f is cts, piecewise C^1

then

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}, \quad A_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell}\,dx.$$

Separation of Variables:

Step 1: Set $u(x,t) = X(x)\,T(t)$ ⇒ two ODEs, one eigenvalue p & one ODE

Step 2: Solve the e.v. & ODE ⇒ $u_n(x,t) = X_n(x)\,T_n(t)$

Step 3: Summation $\sum X_n(x)\,T_n(t)$. then put into ICs to obtain eqns for coefficients.

Step 4: Fourier expansion

e.vs, e.fs.

Diffusion Eqn:

DE: $u_t - k u_{xx} = 0, \quad 0<x<\ell, \ 0<t<\infty$

BC: $u(0,t) = u(\ell,t) = 0$

IC: $u(x,0) = \phi(x)$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases} \qquad T' + \lambda k T = 0.$$

$$X_n(x) = \sin\frac{n\pi x}{\ell}, \quad T_n(t) = e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\frac{n\pi x}{\ell}.$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}.$$

BVPs. {

S1. $u(x,t) = X(x)T(t)$
two ODEs. one eigenvalue Prob

S2: solve e.v.p. and ODE

S3: sum up. Use ICs to compute the coefficients

4.2. Neumann Problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

S1: $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\ell) = 0 \end{cases}, \quad T'' + \lambda c^2 T = 0.$

S2: $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \ n = 0, 1, 2, \ldots$

$X_n(x) = \cos\frac{n\pi x}{\ell}, \ n = 0, 1, 2, \ldots$

$$u(x,t) = \frac{1}{2}A_0 + \sum (A_n + B_n t)\cos\frac{n\pi x}{\ell} + \frac{1}{2}B_0 t$$

$$\begin{cases} \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell} = \phi(x) \\ \frac{1}{2}B_0 + \sum^{\infty} \frac{n\pi}{\ell} B_n \cos\frac{n\pi x}{\ell} = \psi(x) \end{cases}$$

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell}$$

$$\int_0^\ell \phi(x) \cdot \; = \frac{\ell}{2} A_0 \qquad A_0 = \frac{2}{\ell}\int_0^\ell \phi(x)$$

$$\int_0^\ell \phi(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{\ell}{2}A_n, \qquad A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\frac{n\pi x}{\ell}\,dx$$

Formula: $A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\frac{n\pi x}{\ell}\,dx, \quad n = 0, 1, 2, \ldots$

Diffusive: $u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{\ell})^2 kt}\cos\frac{n\pi x}{\ell}$.

Mixed B.C.

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(\ell) = 0 \end{cases}$$

$$X_n = \sin\left[\frac{(n+\frac{1}{2})\pi x}{\ell}\right] \quad n = 1, 2, \ldots$$

4.3. Robin B.Cs.

$\cancel{X'' + \lambda X = 0}$

$$\frac{\partial u}{\partial n} + au\Big|_{\partial D} = 0$$

$$\begin{cases} -u_x(0,t) + a_0 u(0,t) = 0 \\ u_x(\ell,t) + a_1 u(\ell,t) = 0 \end{cases}$$

E.P.
$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(\ell) + a_1 X(\ell) = 0 \end{cases}$$

E.Vs.:

Case 1: $\lambda = \beta^2$, $\quad X = A\cos\beta x + B\sin\beta x$.

$$X' = -\beta A\sin\beta x + \beta B\cos\beta x$$

$$x'(0) - a_0 x(0) = \beta B - a_0 A = 0$$

$$x'(0) + a_1 x(l) = 0 \Rightarrow -\beta A \sin\beta l + \beta B \cos\beta l$$
$$+ a_1 (A \cos\beta l + B \sin\beta l) = 0$$

$$(a_0 + a_l)\cos\beta l + \left(-\beta + \frac{a_l a_0}{\beta}\right)\sin\beta l = 0$$

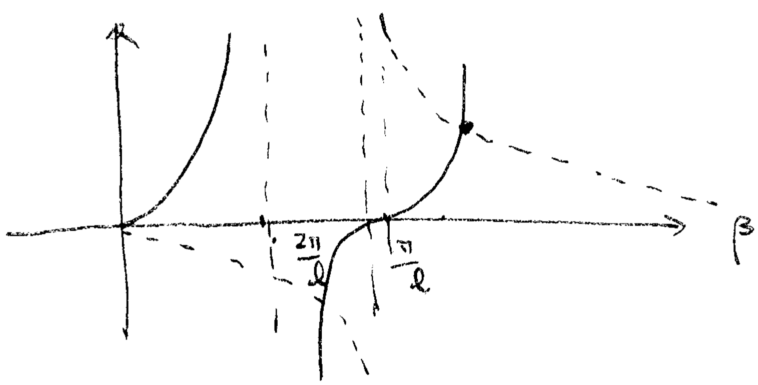
$$(\beta^2 - a_0 a_l)\sin\beta l = (a_0 + a_1)\,\beta \cos\beta l, \quad \beta > 0$$

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right).$$

Eqn for β:

$$\tan\beta l = \frac{(a_0 + a_1)\beta}{\beta^2 - a_0 a_l}$$

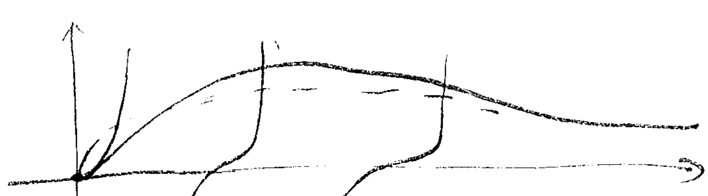
$$Y_1(\beta) = \tan\beta l$$
$$Y_2(\beta) = \frac{(a_0 + a_1)\beta}{\beta^2 - a_0 a_l}.$$



Case 1: $a_0, a_l > 0$. $\quad a_0 + a_1 > 0, \; a_0 a_l > 0.$

$$n^2 \frac{\pi^2}{l^2} < \lambda_n < (n+1)^2 \frac{\pi^2}{l^2}, \quad n = 0, 1, 2, 3, \dots$$

Case 2: $a_0 < 0, \; a_l > 0, \; a_0 + a_l > 0; \; a_0 a_l < 0.$



$$\left.\frac{dY_1}{d\beta}\right|_{\beta=0} = l$$

$$\left.\frac{dY_2}{d\beta}\right| = \frac{a_0 + a_l}{[illegible]}$$

$\exists$ an eigenvalue $0 < \lambda_0 < \left(\frac{\pi}{2\ell}\right)^2$ iff

$$\boxed{a_0 + a_\ell > -a_0 a_\ell \ell.}$$

Zero e.v.

$$a_0 + a_\ell = -a_0 a_\ell \ell.$$

negative e.vs

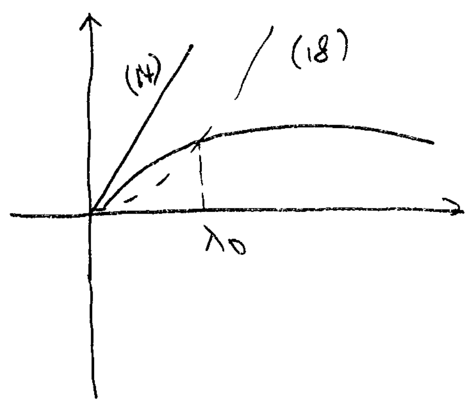
$$\lambda = \gamma^2 < 0$$

$$X(x) = C\cosh \gamma x + D \sinh \gamma x$$

$$\tanh \gamma\ell = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}$$

Case 1. No. $\quad a_0 + a_\ell > 0, \; a_0 a_\ell > 0.$

Case 2. $\quad a_0 < 0, \; a_\ell > 0, \; a_0 + a_\ell > 0.$



If $\quad a_0 + a_\ell < -a_0 a_\ell \ell \qquad \exists 1$ negative e.v. $\lambda_0 < 0$

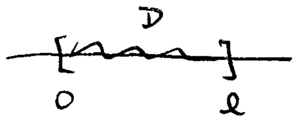
In conclusion

$a_0 > 0, \; a_\ell > 0$: only positive e.vs

$a_0 < 0, \; a_\ell > 0, \quad a_0 + a_\ell > 0$

$$\begin{cases} a_0 + a_\ell < -a_0 a_\ell \ell : & \exists 1 \text{ one negative, all positive} \\ a_0 + a_\ell = -a_0 a_\ell \ell : & 0, \text{ positive} \end{cases}$$

4.3 Robin BVPs



$$\begin{cases} \text{Eqns.} \\ \text{BCs} \quad — \quad \begin{cases} u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(l,t) + a_l u(l,t) = 0 \end{cases} \\ \text{ICs} \end{cases}$$

S1: $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \quad \begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases} \qquad T''(t) + \lambda c^2 T(t) = 0$$

S2: Solve E.V.P.

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

A.

~~Case A~~: $\lambda = \beta^2 > 0, \quad \beta > 0$

$$X = A\cos\beta x + B\sin\beta x$$

$$X' = -\beta A\sin\beta x + \beta B\cos\beta x$$

$$X'(0) - a_0 X(0) = \beta B - a_0 A = 0 \qquad \Rightarrow \qquad B = \beta^{-1} a_0 A$$

$$X'(l) + a_l X(l) = -\beta A\sin\beta l + \beta B\cos\beta l + a_l(A\cos\beta l + B\sin\beta l) = 0$$

$$A\left[-\beta\sin\beta l + a_0\cos\beta l + a_l\cos\beta l + \beta^{-1}a_0 a_l\sin\beta l\right] = 0$$

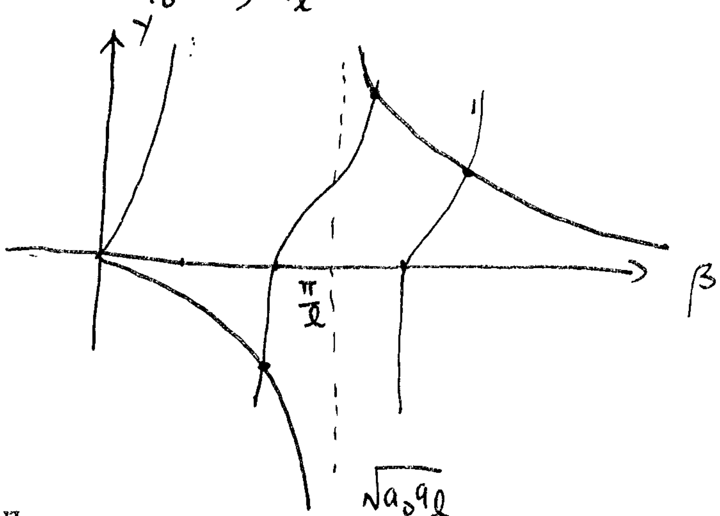
$$\Rightarrow \quad (-\beta + \beta^{-1}a_0 a_l)\sin\beta l + (a_0 + a_l)\cos\beta l = 0$$

$$\Rightarrow \quad \tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}, \quad \beta > 0$$

Sol'ns: intersection of two graphs

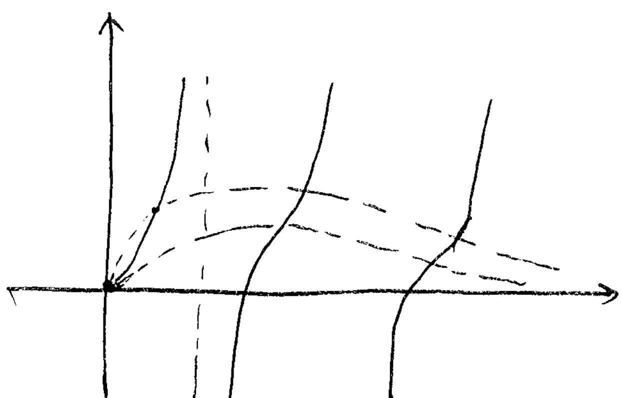
$$y_1(\beta) = \tan \beta \ell$$
$$y_2(\beta) = \frac{(a_0 + a_\ell)\beta}{\beta^2 - a_0 a_\ell}$$

Case 1: $a_0 > 0$, $a_\ell > 0$.



$$\frac{n\pi}{\ell} < \beta_n < (n+1)\frac{\pi}{\ell}$$
$$n^2\frac{\pi^2}{\ell^2} < \lambda_n < (n+1)^2\frac{\pi^2}{\ell^2}, \quad n = 0, 1, 2, 3, \cdots$$

Case 2: $a_0 < 0$, $a_\ell > 0$, $a_0 + a_\ell > 0$, $a_0 a_\ell < 0$.



$$\left.\frac{dy_1}{d\ell}\right|_{\beta=0} = \ell$$

$$\left.\frac{dy_2}{d\ell}\right|_{\beta=0} = \frac{a_0 + a_\ell}{-a_0 a_\ell}$$

If $\ell > \frac{a_0 + a_\ell}{-a_0 a_\ell}$, then No eigenvalue between $(0, \frac{\pi}{2\ell})$

Conclusion:

A: $\lambda = \beta^2 > 0$

Case 1: $a_0 > 0, a_\ell > 0$ $\exists$ a sequence of λ_n

Case 2: $a_0 < 0, a_\ell > 0, a_0 + a_\ell > 0$

2.1. $a_0 + a_\ell > -a_0 a_\ell \ell$, NO

2.2, $a_0 + a_\ell < -a_0 a_\ell \ell$, $\exists 1$

$X_n(x) = \cos\beta_n x + \beta_n^{-1} a_0 \sin\beta_n x$

B: $\lambda = 0$, In this case

$X(x) = A + Bx$

$X' = B$.

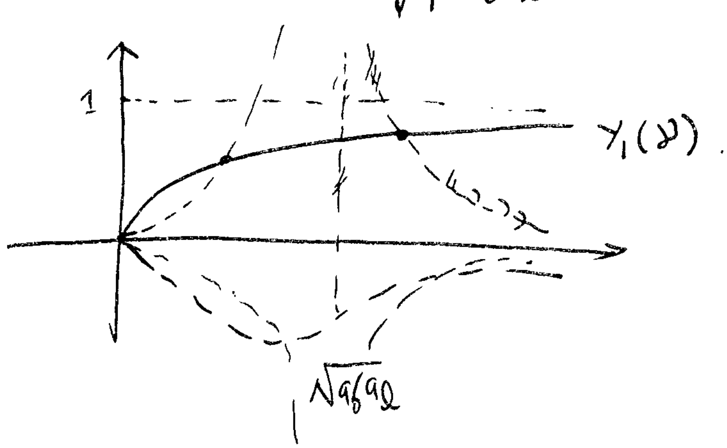
$$\left.\begin{array}{l} B - a_0 A = 0 \\ B + a_\ell (A + B\ell) = 0 \end{array}\right\} \Rightarrow A(a_0 + a_\ell + a_0 a_\ell \ell) = 0$$

$\lambda = 0$ is an eigenvalue if and only if

$a_0 + a_\ell = -a_0 a_\ell \ell$ $\boxed{X_0 = 1 + a_0 x}$

C: $\lambda < 0$, $\lambda = -\gamma^2$, In this case

$$\tanh \gamma\ell = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}, \quad \gamma > 0$$



Case 1: No sol'n

Case 2.1, NO

2.2, $\exists 1$ sol'n λ_c

$X = C\cosh\gamma x + D\sinh\gamma x$

$= \cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x$

Summarize:

~~Case 1~~ $a_0 > 0, a_\ell > 0$: only positive eigenvalues

$a_0 < 0, a_\ell > 0$, $a_0 + a_\ell > 0$

This finishes the e. v. p. $\quad X_n(x) =$

$$T'' + \lambda_n c^2 T = 0$$

Depending on cases:

$a_0 < 0,\ a_\ell > 0,\ a_0 + a_\ell > 0;\ a_0 + a_\ell < -a_0 a_\ell l:$

$$\lambda_0 < 0,\quad \lambda_1 > 0,\quad \lambda_2 > 0,\quad \cdots$$

$$T_0(t) = A_0 e^{-\sqrt{\lambda_0} c t} + B_0 e^{\sqrt{\lambda_0} c t}$$

$$X_0(x) = \cosh \gamma_0 x + \frac{a_0}{\gamma_0} \sinh \gamma_0 x$$

$$T_n(t) = A_n \cos(\sqrt{\lambda_n} c t) + B_n \sin(\sqrt{\lambda_n} c t)$$

$$X_n(x) = \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x.$$

Ex:

$$\begin{cases} u_t - k u_{xx} = 0 \\ u_x - a_0 u = 0 \\ u_x + a_\ell u = 0 \\ u = \phi \end{cases}$$

$$a_0 < 0 < a_0 + a_\ell < -a_0 a_\ell l.$$

Periodic B.Cs:

$$\begin{cases} X(0) = X(\ell) \\ X'(\ell) = X'(\ell) \end{cases}$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 0, 1, 2, 3, \cdots$$

$$X_n = \cos \frac{n\pi}{\ell} x \qquad \sin \frac{n\pi}{\ell} x.$$