We give a general discussion of Sturm-Liouville Eigenvalue Problems (SL Problem).

Background: Suppose we want to solve

$$\begin{cases} w(x)\, u_t = (p(x)\, u_x)_x - q(x)\, u, & 0<x<l \\ u(x,0)=\phi(x) \\ u_x(0,t)-h_1 u(0,t)=0, & u_x(l,t)+h_2 u(l,t)=0 \end{cases}$$

Using the method of separation of variables, we have

Step 1 $u = X(x)T(t)$

$$w(x)X(x)T'(t) = \left((p(x)X')' - qX\right)T$$

$$\frac{T'(t)}{T} = \frac{(pX')' - qX}{wX} = -\lambda$$

We obtain an eigenvalue problem

$$\begin{cases} (p(x)X')' - q(x)X + \lambda w(x) X = 0 \\ X'(0)-h_1 X(0)=0, \quad X'(l)+h_2 X(l)=0 \end{cases}$$

$$T' + \lambda T = 0$$

STURM-LIOUVILLE PROBLEMS

THE SL PROBLEM HAS THE FORM

$$(p(x)\phi')' - q(x)\phi + \lambda w(x)\phi = 0 \quad 0<x<1 \qquad w(x)>0, p(x) > 0$$
$$\phi'(0) - h_1\phi(0) = 0, \quad \phi'(1) + h_2\phi(1) = 0. \qquad \text{ON } 0<x<1$$

WE WRITE THE EIGENVALUE PROBLEM AS

STURM-LIOUVILLE (*)
$$\begin{cases} \mathcal{L}\phi = \lambda w(x)\phi & \mathcal{L}\phi = -(p(x)\phi')' + q(x)\phi \\ \phi'(0) - h_1\phi(0) = 0, & \phi'(1) + h_2\phi(1) = 0 \end{cases} \qquad w(x)>0,\ p(x)>0 \text{ ON } 0<x<1$$

WE WILL DERIVE MANY PROPERTIES OF (*). TO DO SO WE FIRST DERIVE LAGRANGE'S IDENTITY:

$$(1) \quad \int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = -p(x)u'v\Big|_0^1 + p(x)uv'\Big|_0^1.$$

PROOF WE WRITE $\int_0^1 v\mathcal{L}u\,dx = \int_0^1 [-v(pu')' + vqu]\,dx$

$$= -pu'v\Big|_0^1 + \int_0^1 ((pv')u' + vqu\,dx)$$
$$= -pu'v\Big|_0^1 + pv'u\Big|_0^1 - \int_0^1 [(pv')'u - vqu]\,dx$$

THIS YIELDS THAT $\int_0^1 v\mathcal{L}u\,dx = -pu'v\Big|_0^1 + pv'u\Big|_0^1 + \int_0^1 u\mathcal{L}v\,dx$ □

NOW SUPPOSE THAT u, v SATISFY THE BOUNDARY CONDITIONS IN (*) SO THAT $u'(0) - h_1u(0) = 0$, $u'(1) + h_2u(1) = 0$, $v'(0) - h_1v(0) = v'(1) + h_2v(1) = 0$. THEN WE CAN ADD AND SUBTRACT IN (1):

$$\int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = p(1)u(1)\big(v'(1) + h_2v(1)\big) - p(1)v(1)\big(u'(1) + h_2u(1)\big)$$
$$+ p(0)v(0)\big(u'(0) - h_1u(0)\big) - p(0)u(0)\big(v'(0) - h_1v(0)\big)$$
$$= 0$$

THEREFORE, WE OBTAIN

$$\int_0^1 v \mathcal{L} u \, dx = \int_0^1 u \mathcal{L} v \, dx \quad \text{WHENEVER } u, v \text{ SATISFY THE B.C.}$$

IF WE DEFINE $(a, b) \equiv \int_0^1 a\, b \, dx$ THEN WE CAN WRITE $(v, \mathcal{L}u) = (u, \mathcal{L}v)$

WE NOW DERIVE (OR STATE) MANY OF THE KEY PROPERTIES OF THE STURM-LIOUVILLE PROBLEM.

PROPERTIES

(i) THE EIGENVALUES HAVE THE PROPERTIES

a) λ IS REAL

b) THERE ARE AN INFINITE NO. OF EIGENVALUES λ_j WITH

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad \text{AND} \quad \lambda_j \to +\infty \text{ AS } j \to \infty.$$

c) $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, AND $q(x) \geq 0$, AND $h_1 h_2 > 0$

(ii) THE EIGENFUNCTIONS $y = \phi_j(x)$ FOR $j = 1, 2, 3, \ldots$ HAVE THE PROPERTIES

a) $\phi_j(x)$ ARE REAL AND CAN BE NORMALIZED $\int_0^1 w \phi_j^2 \, dx = 1$

b) $\int_0^1 \phi_j(x) \phi_k(x) w(x) \, dx = 0 \quad j \neq k$

(iii) EXPANSION PROPERTY

ANY FUNCTION $f(x)$ WITH $\int_0^1 (f(x))^2 dx < \infty$ CAN BE EXPANDED AS

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

WHERE BY ORTHOGONALITY

$$c_n = \frac{\int_0^1 f(x) \phi_n(x) w(x) \, dx}{\int_0^1 (\phi_n(x))^2 w(x) \, dx}$$

PROOF OF SOME OF THE PROPERTIES

(i) <u>EIGENVALUES ARE REAL</u>

LET $\mathcal{L}\phi = \lambda w \phi$ WITH $\phi'(0) - h_1 \phi(0) = 0$ AND $\phi'(1) + h_2 \phi(1) = 0$.

TAKE THE CONJUGATE $\mathcal{L}\bar{\phi} = \bar{\lambda} w \bar{\phi}$. NOW USE LAGRANGE'S IDENTITY

$$\int_0^1 (\phi \mathcal{L} \bar{\phi} - \bar{\phi} \mathcal{L} \phi)\, dx = 0 \longrightarrow (\phi, \mathcal{L}\bar{\phi}) - (\bar{\phi}, \mathcal{L}\phi) = 0.$$

THIS YIELDS THAT $0 = (\phi, \bar{\lambda}\bar{\phi}) - (\bar{\phi}, \lambda\phi) = (\bar{\lambda} - \lambda)(\bar{\phi}, \phi) = (\bar{\lambda} - \lambda)\int_0^1 w \phi \bar{\phi}\, dx$

HENCE $(\bar{\lambda} - \lambda)\int_0^1 w|\phi|^2\, dx = 0 \longrightarrow \lambda = \bar{\lambda} \longrightarrow \lambda$ IS REAL.

(ii) <u>SHOW $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, $q(x) \geq 0$, $h_1 h_2 > 0$</u>

WE WRITE $\mathcal{L}\phi = \lambda w \phi$

MULTIPLY BY ϕ AND INTEGRATE $\int_0^1 \phi \mathcal{L} \phi\, dx = \lambda \int_0^1 w \phi^2\, dx$

NOW INTEGRATE BY PARTS: $-p(x)\phi'(x)\phi(x)\Big|_0^1 + \int_0^1 (p\phi'^2 + q\phi^2)\, dx$
$= \lambda \int_0^1 w\phi^2\, dx.$

NOW $\phi'(1) = -h_2 \phi(1)$ AND $\phi'(0) \neq h_1 \phi(0)$, WHICH YIELDS

$$p(1) h_2 (\phi(1))^2 + p(0) h_1 (\phi(0))^2 + \int_0^1 (p\phi'^2 + q\phi^2)\, dx = \lambda \int_0^1 w\phi^2\, dx.$$

NOW SINCE $q(x) \geq 0$ (BY ASSUMPTION) AND $p(x) > 0$, $w(x) > 0$ ON $0 < x < 1$ FOR STURM-LIOUVILLE THEN WE HAVE $\lambda > 0$.

<u>REMARK</u> IF $q \equiv 0$ FOR $0 \leq x \leq 1$ AND $h_1 = h_2 = 0$ THEN WE HAVE $(p(x)\phi')' + \lambda w(x)\phi = 0$ WITH $\phi'(0) = \phi'(1) = 0$.

THIS HAS AN EIGENVALUE $\lambda = 0$ AND EIGENFUNCTION $\phi = 1$.

(ii) EIGENFUNCTIONS CORRESPONDING TO DIFFERENT EIGENVALUES ARE ORTHOGONAL

WE WRITE $\mathcal{L}\phi_j = \lambda_j w \phi_j$ WITH ϕ_j AND ϕ_K

$\mathcal{L}\phi_K = \lambda_K w \phi_K$ SATISFYING THE BOUNDARY CONDITIONS.

THEN $(\phi_K, \mathcal{L}\phi_j) = (\phi_j, \mathcal{L}\phi_K)$ BY LAGRANGE'S IDENTITY.

THIS YIELDS $\lambda_j(\phi_K, w\phi_j) = \lambda_K(\phi_K, w\phi_j)$.

THEREFORE $(\lambda_j - \lambda_K)\int_0^1 w\phi_j\phi_K\,dx = 0$.

IF $\lambda_j \neq \lambda_K$ THEN $\int_0^1 w\phi_j\phi_K\,dx = 0 \longrightarrow$ ORTHOGONALITY.

(iii) IT IS DIFFICULT TO PROVE THAT ANY $f(x)$ WITH $\int_0^1 (f(x))^2\,dx < \infty$ CAN BE EXPANDED AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$.

TO DETERMINE THE COEFFICIENTS, MULTIPLY BY $w(x)\phi_m(x)$ AND GET

$$\int_0^1 f(x)\phi_m(x)w(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_m(x)\phi_n(x)w(x)\,dx$$

BY ORTHOGONALITY $c_m \int_0^1 (\phi_m(x))^2 w(x)\,dx = \int_0^1 f(x)\phi_m(x)w(x)\,dx$,

WHICH DETERMINES c_m.

EXAMPLE 1   FIND THE NORMALIZED EIGENFUNCTIONS FOR

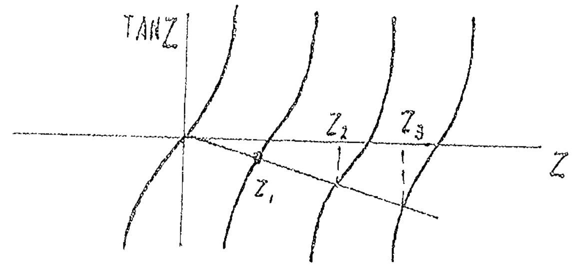
$$\phi'' + \lambda\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0$$

WE KNOW THAT $\lambda > 0$ SINCE $h_1 h_2 > 0$ (property ii).

WE OBTAIN $\phi = A\sin(\sqrt{\lambda}\,x)$. NOW $A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sin(\sqrt{\lambda}) = 0$.

THUS λ SATISFIES $\tan[\sqrt{\lambda}] = -\sqrt{\lambda}$.

WITH $\lambda > 0$. PLOT $\tan z = -z$.



THERE ARE AN INFINITE # OF ROOTS WITH $z_n \approx \frac{(2n+1)}{2}\pi$ AS $n \to \infty$.

NEXT WE WRITE $\phi_n(x) = A_n \sin(\sqrt{\lambda_n}\,x)$. THEN TO NORMALIZE WE WRITE $\int_0^1 A_n^2 \sin^2(\sqrt{\lambda_n}\,x)\,dx = 1 = A_n^2 \int_0^1 \left(\frac{1-\cos(2\sqrt{\lambda_n}\,x)}{2}\right)dx$

THIS YIELDS
$$\frac{A_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n})\right] = 1$$

OR
$$\frac{A_n^2}{2}\left[1 - \frac{\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\right] = \frac{A_n^2}{2}\left[1 + \cos^2(\sqrt{\lambda_n})\right] = 1 \to A_n = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]^{1/2}}$$

(HERE WE USED $\sin(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}\cos(\sqrt{\lambda_n})$ FROM THE EIGENVALUE RELATION).

THIS YIELDS,
$$\phi_n(x) = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]^{1/2}}\sin(\sqrt{\lambda_n}\,x)$$

NOW IF WE EXPAND $f(x) = x$ AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$. WE HAVE
$$c_n = \int_0^1 x\,\phi_n(x)\,dx = A_n\int_0^1 x\sin(\sqrt{\lambda_n}\,x)\,dx = A_n\left[\frac{-x}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}\,x)\Big|_0^1 + \frac{1}{\sqrt{\lambda_n}}\int_0^1 \cos(\sqrt{\lambda_n}\,x)\,dx\right]$$

THIS YIELDS THAT
$$c_n = A_n\left[-\frac{1}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}) + \frac{1}{\lambda_n}\sin(\sqrt{\lambda_n})\right]$$

NOW REPLACE $\cos(\sqrt{\lambda_n}) = -\frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n})$, WHICH GIVES $C_n = \frac{2}{\lambda_n}\sin(\sqrt{\lambda_n})\,A_n$.

THE FINAL RESULT IS $f(x) = x = \sum_{n=1}^{\infty} \frac{2}{\lambda_n}\sin(\sqrt{\lambda_n})\,A_n^2 \sin(\sqrt{\lambda_n}\,x)$

FINALLY $$f(x) = x = 4\sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n})}{\lambda_n(1+\cos^2(\sqrt{\lambda_n}))}\sin(\sqrt{\lambda_n}\,x).$$

USE THIS EXPANSION TO SOLVE THE HEAT CONDUCTION PROBLEM GIVEN BY

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < 1 \\ u(0,t) = 1, & u_x(1,t) + u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

WE FIRST CALCULATE THE STEADY-STATE SOLUTION $u_s(x)$ WHICH SATISFIES

$$u_s'' = 0$$

$$u_s(0) = 1, \quad u_s'(1) + u_s(1) = 0.$$

WE GET $u_s(x) = Ax + B$ THEN $u_s(0) = 1$ GIVES $u_s(x) = Ax + 1$.

THIS YIELDS $A + (A+1)$ OR $A = -1/2$. $\longrightarrow$ $u_s(x) = -x/2 + 1$.

FINALLY, WE WRITE $u = u_s + v$. THIS YIELDS THAT

$$(*) \begin{cases} v_t = \alpha^2 v_{xx} \\ v(0,t) = 0, \quad v_x(1,t) + v(1,t) = 0 \\ v(x,0) = f(x) - u_s(x) \end{cases}$$

WE WRITE $v = XT$ SO THAT $\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda$

$T' = -\alpha^2 \lambda T$

$X'' + \lambda X = 0$

$T = e^{-\alpha^2 \lambda t}$

THIS YIELDS THE EIGENVALUE PROBLEM

$$\phi'' + \lambda\phi = 0, \qquad 0 < x < 1$$

$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0$$

THE NORMALIZED EIGENFUNCTIONS ARE $\phi_n(x) = \dfrac{\sqrt{2}}{[1 + \cos^2(\sqrt{\lambda_n})]^{1/2}} \sin(\sqrt{\lambda_n}\, x)$

WITH $\tan(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}$. WE EXPAND

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x)$$

NOW $\quad v(x,0) = f(x) - u_s(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$

BY ORTHOGONALITY $\quad c_n = \int_0^1 (f(x) - u_s(x))\, \phi_n(x)\, dx.$

AND THEN $\quad u(x,t) = u_s(x) + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x).$

FOR $t \to \infty$, $\quad u(x,t) \sim u_s(x) + c_1 e^{-\alpha^2 \lambda_1 t} \phi_1(x) + \ldots$

EXAMPLE FIND THE NORMALIZED EIGENFUNCTION FOR

$$\begin{cases} (x^2\phi')' + \lambda\phi = 0, & 1 < x < 2 \\ \phi(1) = 0, \quad \phi(2) = 0 \end{cases}$$

WE EXPAND OUT TO OBTAIN $x^2\phi'' + 2x\phi' + \lambda\phi = 0$.

WE LET $\phi = x^r$ TO OBTAIN $r(r-1) + 2r + \lambda = 0$.

THIS YIELDS $r = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$.

FOR AN EIGENVALUE WE NEED $1 - 4\lambda < 0$ OR $\lambda > 1/4$.

THIS YIELDS $r = -\dfrac{1}{2} \pm \dfrac{i}{2}\sqrt{4\lambda - 1}$.

OUR SOLUTION IS

$$\phi(x) = C_1 x^{-1/2} \sin\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right) + C_2 x^{-1/2}\cos\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right)$$

NOW $\phi(1) = 0$ GIVES $C_2 = 0$.

$\phi(2) = 0$ GIVES $\sin\left(\dfrac{\sqrt{4\lambda-1}}{2}\log 2\right) = 0$ OR $\dfrac{\sqrt{4\lambda-1}}{2}\log 2 = n\pi$

$n = 1, 2, 3, \ldots$

THIS YIELDS $\lambda_n = \dfrac{1}{4} + \dfrac{n^2\pi^2}{(\log 2)^2}$ $\quad n = 1, 2, 3, \ldots$

AND $\phi_n(x) = C x^{-1/2} \sin\left(\dfrac{n\pi \log x}{\log 2}\right)$

THEN WE NORMALIZE WITH $\int_1^2 \phi_n^2(x)\,dx = 1$. THIS YIELDS THAT

$$C^2 \int_1^2 \frac{1}{x} \sin^2\left(\frac{n\pi\log x}{\log 2}\right) dx = 1.$$

LET $y = \log x / \log 2$ SO THAT $dy = \dfrac{1}{x \log 2}\,dx$

$$\longrightarrow \quad C^2 \log 2 \int_0^1 \sin^2(n\pi y)\,dy = 1 \rightarrow C^2(\log 2)/2 = 1.$$

THIS YIELDS THAT $C = (2/\log 2)^{1/2}$.

FINALLY, WE OBTAIN THAT

$$\phi_n(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log 2)^2}.$$

NOW IF WE EXPAND $f(x)$ IN TERMS OF THIS SERIES WE OBTAIN $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ $\qquad c_n = \int_1^2 f(x)\phi_n(x)\,dx$.

EXAMPLE SOLVE THE HEAT CONDUCTION PROBLEM

$$\begin{cases} u_t = D(x^2 u_x)_x - u & 1 \le x \le 2,\ t > 0 \\ u(1,t) = u(2,t) = 0, & u(x,0) = f(x) \end{cases}$$

SEPARATING VARIABLES WE OBTAIN $u(x,t) = X(x)\,T(t)$

THEN $X T' = D T (x^2 X')' - X T \quad \rightarrow \frac{1}{D}\left(\frac{T'}{T} + 1\right) = \frac{(x^2 X')'}{X} = -\lambda$.

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\left.\begin{array}{ll} (x^2\phi')' + \lambda\phi = 0 & 1 < x < 2 \\ \phi(1) = \phi(2) = 0 & \end{array}\right\} \qquad \phi_k(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right)$$

$$\lambda_k = \frac{1}{4} + \frac{k^2\pi^2}{(\log 2)^2}, \quad k = 1, 2, \ldots$$

WE THEN OBTAIN $T_k' = -(1 + D\lambda_k)T$

THIS YIELDS THAT $T_k(t) = e^{-t} e^{-D\lambda_k t}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(1+D\lambda_n)t} \phi_n(x) \qquad \text{WITH } u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

THUS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ SO THAT $c_n = \int_1^2 f(x)\phi_n(x)\,dx$.

SPECIAL CASES OF STURM LIOUVILLE PROBLEMS

(i) $\phi'' + x\phi' + \lambda\phi = 0$

$\phi(0) = \phi(1) = 0$

MULTIPLY BY $e^{x^2/2}$ SO THAT $(e^{x^2/2}\phi)' + \lambda e^{x^2/2}\phi = 0$

$\phi(0) = \phi(1) = 0.$

THE WEIGHT FUNCTION IS $w = e^{x^2/2}$ SO THAT $\int_0^1 \phi_n \phi_m e^{x^2/2}\,dx = 0$ IF $n \neq m$.

(ii) $\phi'' + \frac{2}{x}\phi' + \lambda\phi = 0$

$\phi(1) = \phi(2) = 0$

IN STURM-LIOUVILLE FORM $(x^2\phi')' + \phi\lambda x^2 = 0$

THE WEIGHT FUNCTION IS $w = x^2$ AND $\int_0^1 \phi_n \phi_m x^2\,dx = 0,\ n \neq m$

WE CAN SOLVE FOR ϕ BY WRITING $\phi(x) = v(x)/x$

SO THAT $v'' + \lambda v = 0$ AND $v = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$.

THIS YIELDS $\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)\right]$

IT IS MORE CONVENIENT TO WRITE

$$\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}\,(x-1)) + B\sin(\sqrt{\lambda}\,(x-1))\right].$$

NOW $\phi(1) = 0 \to A = 0.$ $\phi(2) = 0 \longrightarrow \sin(\sqrt{\lambda}) = 0$

HENCE $\sqrt{\lambda} = n\pi$ OR $\lambda = n^2\pi^2$

AND $\phi_n(x) = \frac{1}{x}\sin(n\pi(x-1))$

WHEN $p(x)$ VANISHES AT ONE OF THE ENDPOINTS, OR WHEN EITHER ENDPOINT IS ∞ WE HAVE A <u>SINGULAR</u> STURM-LIOUVILLE PROBLEM.

(iii) $\phi'' + \frac{1}{x}\phi' + \lambda\phi = 0, \quad 0<x<1; \ \phi(0)$ FINITE, $\phi(1)=0$.

THIS IS BESSEL'S EQUATION WITH $\phi = A J_0(\sqrt{\lambda}x) + B Y_0(\sqrt{\lambda}x)$

AND IN STURM-LIOUVILLE FORM $(x\phi')' + \lambda x\phi = 0$

SO THAT $\int_0^1 x\phi_n\phi_m\,dx = 0$ FOR $n \neq m$.

(iv) $\phi'' - 2x\phi' + \lambda\phi = 0 \quad -\infty < x < \infty$; HERMITE'S EQUATION

IN STURM-LIOUVILLE FORM $(e^{-x^2}\phi')' + \lambda e^{-x^2}\phi = 0$

SO THAT $\int_{-\infty}^{\infty} e^{-x^2}\phi_n\phi_m\,dx = 0$ FOR $n \neq m$.

USING FROBENIUS SERIES THERE ARE POLYNOMIAL SOLUTIONS TO THIS EQUATION WHEN $\lambda = 2n$, FOR $n = 0,1,2,..$

THEN $\phi_n(x) = H_n(x)$ HERMITE POLYNOMIALS

$\lambda = 2n, \quad n = 0,1,2,..$

$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, ..$

(v) $\phi'' - \frac{2x}{1-x^2}\phi' + \frac{\lambda}{1-x^2}\phi = 0$ IN $-1<x<1$, $\phi(\pm 1)$ FINITE.

THIS IS LEGENDRE'S EQUATION AND IN SL FORM WE GET

$[(1-x^2)\phi']' + \lambda\phi = 0$ SO $\int_{-1}^1 \phi_n(x)\phi_m(x)\,dx = 0$ $\quad n \neq m$

THE SOLUTIONS ARE $\phi = A P_n(x) + B Q_n(x)$ WHEN $\lambda = n(n+1)$.

NOW $P_n(x)$ ARE LEGENDRE POLYNOMIALS OF DEGREE n.

REMARKS IN GENERAL FOR EIGENVALUE PROBLEMS WITH NON-SEPARATED BOUNDARY CONDITIONS SUCH AS

$$(*)\quad \begin{cases} \phi'' + \lambda\phi = 0, \quad 0<x<1 \\ \phi(0) = 0 \\ \phi'(1) = h\,\phi'(0) \end{cases}$$

$\longleftarrow$ HERE THE CONDITION AT $x=1$ DEPENDS ON THAT AT $x=0$ $\rightarrow$ NON-SEPARATED BC

WE CAN EXPECT THE POSSIBILITY OF COMPLEX EIGENVALUES. THIS IS BECAUSE THE PROOF THAT EIGENVALUES ARE REAL FAILS SINCE LAGRANGE'S IDENTITY IN EQUATION (1) ON PAGE 6 DOES NOT GIVE $\int_0^1 (u\mathcal{L}v - v\mathcal{L}u)\,dx = 0$.

AN ANALYSIS OF (*) SHOWING AN INFINITE # OF COMPLEX EIGENVALUES WHEN $h>1$ IS DONE IN THE HOMEWORK.

HOWEVER, THERE IS ONE TYPE OF NON-SEPARATED BOUNDARY CONDITIONS WHICH OCCURS OFTEN AND LEADS TO REAL EIGENVALUES. CONSIDER THE CASE OF PERIODIC BOUNDARY CONDITIONS:

$$\phi'' + \lambda\phi = 0, \quad 0<x<L; \qquad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).$$

FOR THIS PROBLEM WE CALCULATE FOR $\lambda>0$ THAT $\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$.

WE PUT

$$\phi(0) = \phi(L) \longrightarrow A = A\cos(\mu) + B\sin(\mu) \qquad \mu = \sqrt{\lambda}\,L$$

$$\phi'(0) = \phi'(L) \longrightarrow B = -A\sin(\mu) + B\cos(\mu)$$

THIS GIVES

$$\underbrace{\begin{pmatrix} 1-\cos\mu & -\sin\mu \\ \sin\mu & 1-\cos\mu \end{pmatrix}}_{A}\begin{pmatrix} A \\ B \end{pmatrix} = \underset{\sim}{0}$$

WHICH HAS A NON-TRIVIAL SOLUTION WHEN $\det A = 0$.

THUS THE EIGENVALUES CORRESPOND TO $\det \underset{\sim}{A} = 0 \longrightarrow (1-\cos\mu)^2 + \sin^2\mu = 2 - 2\cos\mu = 0$.

THUS, $\cos\mu = 1 \rightarrow \mu = \sqrt{\lambda}\,L = 2n\pi$, $n = 0,1,2,\ldots$ THE EIGENVALUES ARE

$\lambda_n = (2n\pi/L)^2$ FOR $n = 1,2,\ldots$ AND $\phi_n = A\cos\left(\frac{2n\pi x}{L}\right) + B\sin\left(\frac{2n\pi x}{L}\right)$.