Method of separation of variables for inhomogeneous PDE

Let us apply the method of separation of variables to

$$\begin{cases} u_t = k u_{xx} + f(x,t), & 0<x<l,\ t>0 \\ u(0,t)=h(t), \quad u(l,t)=j(t) \\ u(x,0)=\phi(x) \end{cases}$$

Suppose $f=\phi=0$ first. Let

$$u = \sum_{n=1}^{+\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

Formally

$$\frac{du}{dt} = \sum_{n=1}^{\infty} \frac{du_n}{dt} \sin\left(\frac{n\pi x}{l}\right) \quad \text{———} (*)$$

$$k u_{xx} = \sum_{n=1}^{\infty} u_n \left(-\left(\frac{n\pi}{l}\right)^2\right) \sin\left(\frac{n\pi x}{l}\right) \text{—} (**)$$

$$u_t - k u_{xx} = \sum \left(\frac{du_n}{dt} + k\left(\frac{n\pi}{l}\right)^2 u_n\right) \sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{du_n}{dt} + k\left(\frac{n\pi}{l}\right)^2 u_n = 0 \Rightarrow u_n = c_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

$$u = \sum c_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

BC $u(0,t)=h(t)$, $u(l,t)=j(t)$ $\Longrightarrow$ always not satisfied.

The problem is: $\displaystyle \frac{d^2}{dx^2} \sum_{n=1} u_n \sin\left(\frac{n\pi x}{l}\right) \neq \sum_{n=1}^{\infty} u_n \left(-\left(\frac{n\pi}{l}\right)^2\right) \sin\left(\frac{n\pi x}{l}\right)$

You can't differentiate Fourier expansion term by term.

<u>The right way</u>: Expand

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right), \quad u_n(t) = \frac{2}{l}\int_0^l u(x,t) \sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{du}{dt} = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi}{l}x\right), \quad v_n(t) = \frac{2}{l}\int_0^l \frac{du}{dt} \sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{+\infty} w_n(t) \sin\left(\frac{n\pi}{\ell}x\right), \quad w_n(t) = \frac{2}{\ell}\int_0^\ell \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t) \sin\left(\frac{n\pi}{\ell}x\right)$$

Then we have

$$v_n(t) = k w_n(t) + f_n(t)$$

$$v_n(t) = \frac{d}{dt}\left(\frac{2}{\ell}\int_0^\ell u \sin\left(\frac{n\pi x}{\ell}\right) dx\right) = \frac{d}{dt} u_n(t)$$

$$w_n(t) = \frac{2}{\ell}\int_0^\ell \left[\frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) - u \frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) - \left(\frac{n\pi}{\ell}\right)^2 \sin\left(\frac{n\pi x}{\ell}\right) u(x,t)\right]$$

$$= \frac{2}{\ell}\left(u_x \sin\frac{n\pi x}{\ell} - u\left(\frac{n\pi}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\right)\Big|_0^\ell - \left(\frac{n\pi}{\ell}\right)^2 \frac{2}{\ell} u_n(t)$$

$$= \frac{2}{\ell}\cdot\frac{n\pi}{\ell}\left(u(0,t) - u(\ell,t)\cos n\pi\right) - \left(\frac{n\pi}{\ell}\right)^2 \frac{2}{\ell} u_n(t)$$

$$= \frac{2}{\ell}\frac{n\pi}{\ell}\left(h(t) - (-1)^n j(t)\right) - \left(\frac{n\pi}{\ell}\right)^2 u_n(t)$$

Thus

$$\frac{du_n}{dt} + k\left(\frac{n\pi}{\ell}\right)^2 u_n = \frac{2k n\pi}{\ell^2}\left(h(t) - (-1)^n j(t)\right) + f_n(t) \quad \text{— (1)}$$

Now initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{+\infty} \phi_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$$u_n(0) = \phi_n \quad \text{— (2)}$$

Solving (1)-(2) together, we obtain u.

$$u_n(t) = \phi_n e^{-\lambda_n k t} + \frac{2n\pi}{\ell^2} k \int_0^t e^{-\lambda_n k(t-s)}\left[h(s) - (-1)^n j(s)\right] ds$$

As a second case, we can solve inhomogeneous wave equation ③

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(0,t) = h(t), \quad u(l,t) = k(t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

$$u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad u_n(t) = \frac{2}{l}\int_0^l u \sin\left(\frac{n\pi x}{l}\right) dx$$

$$u_{tt} = \sum v_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad v_n(t) = \frac{2}{l}\int_0^l u_{tt} \sin\left(\frac{n\pi}{l}x\right) dx$$

$$u_{xx} = \sum w_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad w_n(t) = \frac{2}{l}\int_0^l u_{xx} \sin\left(\frac{n\pi}{l}x\right) dx$$

$$f(x,t) = \sum f_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

$$\begin{cases} \dfrac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n(t) = \dfrac{2n\pi}{l^2}\left[h(t) - (-1)^n k(t)\right] + f_n(t) \\ u_n(0) = \phi_n, \quad u_n'(0) = \psi_n \end{cases}$$

Method of Shifting Data:

Suppose
$$\begin{cases} u_t = k u_{xx} + f(x), \quad 0 < x < l \\ u(0,t) = u_0, \quad u(l,t) = u_l \\ u(0,t) = \phi(x) \end{cases} \quad (3)$$

① Solve the steady-state problem first:

$$\begin{cases} k u^0_{xx} + f(x) = 0, \quad 0 < x < l \\ u^0(0) = u_0, \quad u^0(l) = u_l \end{cases} \quad — (4)$$

② $V(x,t) = u(x,t) - u^0(x)$ Then $\begin{cases} V_t = k V_{xx} \\ V(x,0) = \phi(x) - u^0(x) \\ V(0,t) = V(l,t) = 0 \end{cases}$

Problem (3) is called the steady-state problem of (3). So the solution to (3) is given by

$$u(x,t) = u^0(x) + \sum_{n=1}^{+\infty} a_n e^{-\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right)$$

with $a_n = \frac{2}{\ell}\int_0^\ell (\phi(x) - u^0(x)) \sin\left(\frac{n\pi x}{\ell}\right) dx$

Conclusion: As $t \to +\infty$, $u(x,t) \longrightarrow u^0(x)$.

<u>Example 1</u>: Solve

$$\begin{cases} u_t = k u_{xx}, & 0<x<\ell \\ u(0,t) = e^t, \ u(\ell,t) = 0 \\ u(x,0) = 1 \end{cases}$$

Solution: $u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin\left(\frac{n\pi}{\ell}x\right)$, $u_n(0) = \frac{2}{\ell}\int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right)dx = \frac{2}{n\pi}(1-(-1)^n)$.

So we have

$$\begin{cases} \frac{du_n}{dt} + \lambda_n k\, u_n = \frac{2kn\pi}{\ell} e^t \\ u_n(0) = \frac{2}{n\pi}(1-(-1)^n) \end{cases}$$

$$u_n = u_n(0)\, e^{-\lambda_n k t} + \frac{2n\pi}{\ell^2} k \int_0^t e^{-\lambda_n k(t-s)} e^{s}\, ds$$

$$= \frac{2}{n\pi}(1-(-1)^n) e^{-\lambda_n k t} + \frac{2n\pi}{\ell^2} k \cdot e^{-\lambda_n k t} \frac{1}{1+\lambda_n k}\left[e^{(\lambda_n k+1)t} - 1\right]$$

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