LAPLACE AND POISSON'S EQUATION

Lecture Note 13

LAPLACE AND POISSON'S EQUATION ARISES AS THE STEADY-STATE OF

$$U_t = \Delta U + F \quad \text{IN } \Omega$$

$$\partial_n U + K(U - U_b) = 0 \quad \text{ON } \partial\Omega$$

THE STEADY-STATE IS

$$\Delta U = -F \quad \text{IN } \Omega$$

$$\partial_n U + K(U - U_b) = 0 \quad \text{ON } \partial\Omega$$

- POISSON WHEN $F \neq 0$ • LAPLACE WHEN $F = 0$.

WE NOW CONSIDER A CIRCULAR DOMAIN, $0 \le r \le R$, $0 \le \varphi \le 2\pi$ AND DERIVE POISSON'S INTEGRAL FORMULA FOR THE SOLUTION TO

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \qquad 0 \le r \le R, \ 0 \le \varphi \le 2\pi$$

$$u(R, \varphi) = f(\varphi), \quad u \text{ bounded } r \to 0$$

$$u, u_\varphi \text{ ARE } 2\pi \text{ periodic IN } \varphi.$$

WE SEPARATE VARIABLES $u(r,\varphi) = R(r)\,\Phi(\varphi)$ TO OBTAIN

$$\left(R'' + \frac{1}{r} R'\right)\Phi + \frac{1}{r^2} R\,\Phi'' = 0 \longrightarrow \frac{r^2(R'' + 1/r\, R')}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

THEN $\Phi'' + \lambda\Phi = 0$ WITH $\Phi(0) = \Phi(2\pi)$ AND $\Phi'(0) = \Phi'(2\pi)$.

THEN $\lambda_n = n^2$ AND $\Phi_n(\varphi) = \begin{cases} A_n \cos n\varphi + B_n \sin n\varphi, & n \ge 1 \\ A_0, & n = 0 \end{cases}$

THEN $r^2 R'' + rR' + n^2 R = 0$ LET $R = r^\beta \rightarrow \beta(\beta-1) + \beta + n^2 = 0$

SO THAT $\beta = \pm n$. WE CALCULATE

$$R_n(r) = \begin{cases} c_n r^n + d_n r^{-n}, & n \ge 1 \\ c_0 + d_0 \log r, & n = 0 \end{cases}$$

FOR BOUNDEDNESS WE TAKE $d_0 = d_n = 0$ FOR $n = 1, 2, \ldots$.

THEN SET $C_0 = C_n = 1$ $\forall n$ SO THAT $R_0 = 1$ AND $R_n = r^n$, $n = 1, 2, \ldots$

BY SUPERPOSITION $u_n = (A_n \cos(n\varphi) + B_n \sin(n\varphi))\, r^n$ IS A SOLUTION FOR $n = 1, 2, \ldots$

AND $u_0 = A_0$ IS A SOLUTION.

$$\text{SO} \quad (*) \quad u(r, \varphi) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) r^n$$

THEN

$$f(\varphi) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) R^n$$

WE INTEGRATE $\int_0^{2\pi}$ TO OBTAIN $\int_0^{2\pi} f(\varphi)\, d\varphi = A_0 (2\pi) \rightarrow A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi)\, d\varphi$.

NOW WE MULTIPLY BY $\cos(m\varphi)$ AND INTEGRATE $\int_0^{2\pi}$ GIVES

$$A_n R^n \int_0^{2\pi} \cos^2(n\varphi)\, d\varphi = \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi \quad \rightarrow \quad A_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi.$$

SIMILARLY, $B_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\varphi) \sin(n\varphi)\, d\varphi$.

REPLACE $\varphi \rightarrow w$ IN INTEGRALS AND SUBSTITUTE IN $(*)$

THIS GIVES:

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(w)\, dw + \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{r}{R}\right)^n \int_0^{2\pi} f(w) \left[\cos(n\varphi)\cos(nw) + \sin(n\varphi)\sin(nw)\right] dw$$

THIS CAN BE WRITTEN AS

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(w) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos\left(n(w - \varphi)\right)\right] dw.$$

NOW DEFINE $z = \frac{r}{R} e^{in(w-\varphi)}$ WITH $|z| < 1$ WHEN $r < R$.

NOW WE OBTAIN: NOTE $1+z+z^2+\dots = \frac{1}{1-z} \longrightarrow z+z^2+\dots = \frac{1}{1-z} - 1$.

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos\left(n(w-\varphi)\right) = 1 + 2\,\mathrm{RE}\left(\sum_{n=1}^{\infty} z^n\right) = 1 + 2\left(\frac{1}{1-z} - 1\right)$$

$$= 1 + \frac{2z}{1-z} = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2}$$

$$= \frac{1 - |z|^2 + (z-\bar{z})}{|1-z|^2}.$$

THEN $1 + 2\,\mathrm{RE}\left(\sum_{n=1}^{\infty} z^n\right) = \mathrm{RE}\left(1 + 2\sum_{n=1}^{\infty} z^n\right) = \mathrm{RE}\left(\frac{1-|z|^2+z-\bar{z}}{|1-z|^2}\right) = \frac{1-|z|^2}{|1-z|^2}$

$$= \frac{1 - r^2/R^2}{\left(1 - \frac{r}{R}\cos(w-\varphi)\right)^2 + \frac{r^2}{R^2}\sin^2(w-\varphi)} = \frac{1 - r^2/R^2}{1 + r^2/R^2 - \frac{2r}{R}\cos(w-\varphi)}$$

THEN $1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos\left(n(w-\varphi)\right) = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(w-\varphi)}$

THIS YIELDS THAT

POISSON'S INTEGRAL FORMULA $u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(w)\left(\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(w-\varphi)}\right) dw$.

<u>REMARK</u> (i) $u\big|_{r=0} = \frac{1}{2\pi}\int_0^{2\pi} f(w)\,dw$ TEMPERATURE AT CENTER OF DISC IS THE AVERAGE OF THE TEMPERATURE OVER THE ENTIRE BOUNDARY OF THE DISC.

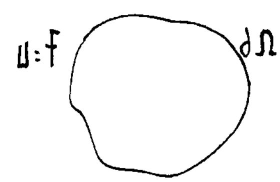
$$\min_{\varphi} f(\varphi) = f_{MIN} \le u\big|_{r=0} \le f_{MAX} = \max_{\varphi} f(\varphi)$$

(ii) $$\lim_{r\to R^-}\frac{1}{2\pi}\left(\frac{R^2-r^2}{R^2+r^2-2rR\cos(\omega-\varphi)}\right)=\delta(\omega-\varphi) \quad \text{so} \quad u(R,\varphi)=\int_0^{2\pi}f(\omega)\,\delta(\omega-\varphi)\,d\varphi$$

GIVES $u(R,\varphi)=f(\varphi)$.

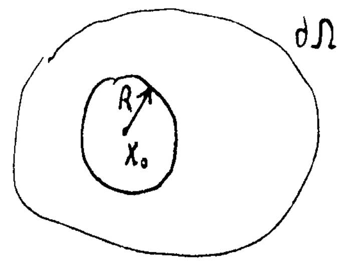
MAX - MIN PRINCIPLE

CONSIDER $\Delta u=0$ IN Ω AND $u=f$ ON $\partial\Omega$ WITH Ω BEING A BOUNDED TWO-DIMENSIONAL DOMAIN.



THEN

$$\min f \le u \le \max f.$$



PROOF SUPPOSE, BY CONTRADICTION THAT u ACHIEVES ITS MAXIMUM AT SOME POINT $\underline{x}_0$ IN Ω. LET R BE ANY VALUE SO THAT THE DISC CENTERED AT $\underline{x}_0$ IS STRICTLY INSIDE Ω.

THEN BY POISSON'S INTEGRAL FORMULA $f(\omega)=u(R,\omega)$

$$u(r,\varphi)=\frac{1}{2\pi}\int_0^{2\pi}f(\omega)\left(\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\omega-\varphi)}\right)d\omega$$

THEN WITH $u(\underline{x}_0)=u(0,\varphi)$ WE OBTAIN

$$u(\underline{x}_0)=\frac{1}{2\pi}\int_0^{2\pi}u(R,\varphi)\,d\varphi$$

SO $$\min_{r=R} u \le u(\underline{x}_0)\le \max_{r=R} u$$

THIS VIOLATES THE ASSUMPTION THAT u ATTAINS ITS MAXIMUM AT $\underline{x}=\underline{x}_0$ (UNLESS OF COURSE $u=$ CONSTANT EVERYWHERE IN Ω).

6.3 6.4

Ex 3

$$D = \{(x,y) \mid x^2+y^2 < a^2\}$$

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

Use polar coordinates

S1:

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0$$

$$\begin{cases} \Theta''(\theta) + \lambda\Theta = 0 \\ \Theta(\theta+2\pi) = \Theta(\theta) \end{cases}$$

S2: $\lambda = n^2$, $n = 0, 1, 2, \cdots$, $\Theta = A\cos n\theta + B\sin n\theta$

$R = r^{\alpha}$, $\alpha = \pm n$,

$R(r) = c\, r^n$

$n = 0$

$R(r) = c$

$\lambda = 0$, $R(r) = c$, $\Theta = 1$

$\lambda = n^2$, $n > 0$, $R(r) = c r^n$, $\Theta = A\cos n\theta + B\sin n\theta$

$$R(r)\Theta = r^n(cA\cos n\theta + cB\sin n\theta)$$
$$= r^n(A_n\cos n\theta + B_n\sin n\theta)$$

S3 Sum up

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

Coefficients: $u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta) = h(\theta)$.

$$\frac{1}{2}A_0 = \int_0^{2\pi} h(a,\theta)\,d\theta$$

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\,d\theta$$

$$a^n A_n = \frac{1}{\pi}\int_0^{2\pi}\cos n\theta\, h(\theta)\,d\theta \qquad a^n B_n = \frac{1}{\pi}\int_0^{2\pi}\sin n\theta\, h(\theta)\,d\theta$$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$= \int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi} + \sum_{n=1}^{\infty}\frac{r^n}{a^n}\int_0^{2\pi} h(\phi)\{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\} d$$

$$= \int_0^{2\pi} h(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right\}\frac{d\phi}{2\pi}$$

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right]$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{-i(\theta-\phi)}\right)^n$$

$$= 1 + \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-\frac{r}{a}e^{i(\theta-\phi)}} + \frac{\frac{r}{a}e^{-i(\theta-\phi)}}{1-\frac{r}{a}e^{-i(\theta-\phi)}}$$

$$= 1 + \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}$$

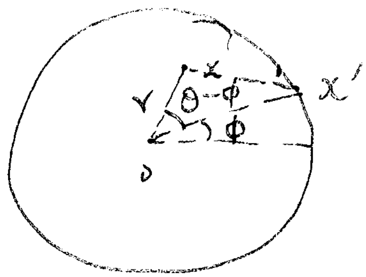
$$= \frac{(a-re^{i(\theta-\phi)})(a-re^{-i(\theta-\phi)}) + re^{i(\theta-\phi)}(a-re^{-i(\theta-\phi)}) + r\cdots}{(a-re^{i(\theta-\phi)})(a-re^{-i(\theta-\phi)})}$$

$$= \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}$$

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi$$

Poisson's Formula.

Geometric Meaning



$$|x-x'|^2 = r^2+a^2-2ar\cos(\theta-\phi)$$

$$u(x) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(x')}{|x-x'|^2}\,d\phi$$

$$dx' = ds = a\,d\phi$$

$$u(x) = \frac{a^2-|x|^2}{2\pi a}\int_{|x'|=a}\frac{u(x')}{|x-x'|^2}\,ds'$$

Theorem: Let $h(\phi)=u(x')$ be a continuous function on ∂D, then $\exists!$ sol'n u s.t.

$$\begin{cases} u(x) = h(\phi) & \text{on } \partial D. \\ \Delta u = 0 & \text{in } D \end{cases}$$

Consequences of Poisson's Formula.

Consequence 1: Mean-value Property:

$$u(0) = \frac{a}{2\pi}\int_{|x'|=a}\frac{u(x')}{|x'|^2}\,ds' = \frac{1}{2\pi a}\int_{|x'|=a}u(x')\,ds$$

$$= \frac{1}{|\partial B_a(0)|}\int_{\partial B_a(0)}u(x')$$

$u(0) =$ average of u on the circle $|x'|=a$

Maximum Principle: (strong version)

If $\Delta u = 0$ in D

weak: $\max_D u = \max_{\partial D} u$

strong: $\max_D u = \max_{\partial D} u$ & If $\max_D u = u(x)$, $x \in D$

then $u \equiv$ constant

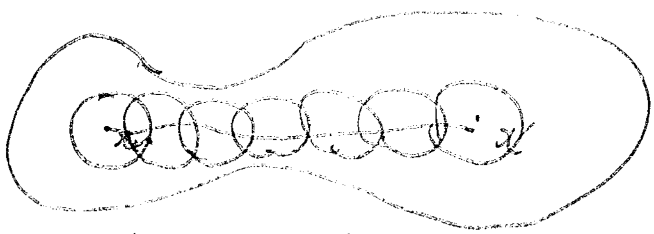
Proof: $u(x_M) = \max_D u$, $x \in D$

$u(x_M) \geq u(x)$, $\forall x \in D$

$u(x_M) = \frac{1}{|\partial B_a(x_M)|} \int_{\partial B_a(x_M)} u \leq u(x_M)$

$\Rightarrow u \equiv u(x_M) \quad \forall x \in B_a(x_M)$.

$\forall x' \in D$.



$\exists$ a finite ball such that.

$u(x') = M$.

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Differentiability:

F Let u be a harmonic fc in D, then $D^\alpha u(x,y)$ exists in D.

How do we use Poisson's Formula.

Ex.

$$\begin{cases} \Delta u = 0 & \text{in} \quad r<a \\ u = 1+3\sin\theta & \text{on } r=a \end{cases}$$

$$u(r,\theta) = \frac{a^2-r^2}{2\pi} \int_0^{\phi} \frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\, d\phi$$

$$= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi)\cos n\phi \, d\phi \quad = 0$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi)\sin n\phi \, d\phi$$

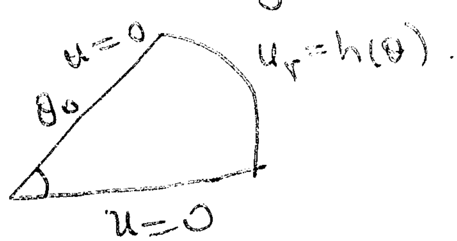
6.4 Circles, wedges & Annuli.

A wedge: $\{0<\theta<\theta_0,\ 0<r<a\}$

An annulus: $\{0<a<r<b\}$

exterior of a circle: $\{a<r<\infty\}$

Ex. 1: The wedge:



$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(\beta_0) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{\beta}\right)^2, \quad n=1,2,3,\ldots$$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r}R = 0$$

$$\alpha^2 - \lambda = 0 \;,\; \alpha = \pm\sqrt{\lambda} = \pm\frac{n\pi}{\beta}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\frac{n\pi\theta}{\beta}$$

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{-1+\frac{n\pi}{\beta}} \sin\frac{n\pi\theta}{\beta} .$$

$$A_n = a^{1-\frac{n\pi}{\beta}} \frac{2}{n\pi} \int_0^{\beta} h(\theta) \sin\frac{n\pi\theta}{\beta}\, d\theta .$$

Ex. 5:

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u = g(\theta) & \text{for } x^2 + y^2 = a^2 \\ u = h(\theta) & \text{for } x^2 + y^2 = b^2 \end{cases}$$

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

$$u(a,\theta) = \frac{1}{2}(C_0 + D_0 \log a) + \sum_{n=1}^{\infty} (C_n a^n + D_n a^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$
$$= g(\theta)$$

$$\Rightarrow \begin{cases} C_0 + D_0 \log a = \frac{1}{\pi}\int_0^{2\pi} g(\phi)\, d\phi \\ A_n(C_n a^n + D_n a^{-n}) = \frac{1}{\pi}\int_0^{2\pi} g(\phi) \cos n\phi\, d\phi \\ B_n(C_n a^n + D_n a^{-n}) = \frac{1}{\pi}\int_0^{2\pi} g(\phi) \sin n\phi\, d\phi \end{cases}$$

$$\begin{cases} C_0 + D_0 \log b = \frac{1}{\pi}\int_0^{2\pi} h(\phi)\, d\phi \\ A_n(C_n b^n + D_n b^{-n}) = \frac{1}{\pi}\int_0^{2\pi} g(\phi) \cos n\phi\, d\phi \\ B_n(C_n b^n + D_n b^{-n}) = \frac{1}{\pi}\int_0^{2\pi} g(\phi) \sin n\phi\, \partial\phi \end{cases}$$

Ex. 6.

$$\begin{cases} u_{xx}+u_{yy}=0 & \text{for } x^2+y^2>a^2 \\ u=h(\theta) & \text{for } x^2+y^2=a^2 \\ u \ \text{bdd} & \text{for } x^2+y^2\to\infty \end{cases}$$

sol'n:

$$u(r,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta+B_n\sin\theta)$$

$$h(\theta)=\frac{1}{2}A_0+\sum a^{-n}(A_n\cos n\theta+B_n\sin n\theta)$$

$$A_n=\frac{2a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta\, d\theta$$

$$B_n=\frac{2a^n}{\pi}\int_{0}^{2\pi} h(\theta)\sin n\theta\, d\theta$$

$$u(r,\theta)=(r^2-a^2)\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi}$$

for $r>a$.

~~Chapter 7~~

In summary:

$$\begin{cases} \Delta u=0 & \text{in } D \\ \text{B.C.s} & \text{on } \partial D \end{cases}$$

$$D=\begin{cases} \text{rect. + cubes} & X(x)Y(y). \\ \left.\begin{matrix}\text{disks,}\\ \text{wedges,}\\ \text{annuli}\end{matrix}\right\rangle & R(r)\Theta(\theta). \end{cases}$$

Poisson's Formula + Strong Max. Principle

EXAMPLE SOLVE THE FOLLOWING:

$$u_{xx} + u_{yy} = 0 \quad \text{IN } 0 \leq x < \infty, \; 0 \leq y \leq 1$$

$$u(x,0) = u(x,1) = 0, \qquad u(0,y) = 1-y, \qquad u \to 0 \text{ AS } x \to \infty$$

WE SEPARATE VARIABLES TO OBTAIN $u = XY$ SO $-\dfrac{X''}{X} = \dfrac{Y''}{Y} = -\lambda$

THEN $\begin{array}{l} Y'' + \lambda Y = 0, \; 0 < y < 1 \\ Y(0) = Y(1) = 0 \end{array} \rightarrow Y = \sin(n\pi y), \; \lambda = n^2\pi^2.$

THEN $X_n'' - n^2\pi^2 X_n = 0$ SO $X_n = e^{-n\pi x}$ BOUNDED AS $x \to +\infty$.

THIS YIELDS THAT $u(x,y) = \displaystyle\sum_{n=1}^{\infty} b_n e^{-n\pi x} \sin(n\pi y).$

NOW $u(0,y) = 1-y = \displaystyle\sum_{n=1}^{\infty} b_n \sin(n\pi y) \qquad b_n = 2\int_0^1 (1-y)\sin(n\pi y)\,dy.$

IF WE CALCULATE $\rightarrow b_n = \dfrac{2}{n\pi}, \quad n = 1, 2, 3, \ldots$

THIS YIELDS $u(x,y) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n\pi x} \sin(n\pi y).$

WE LET $z = e^{-\pi x + i\pi y}$ SO $u(x,y) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \mathrm{IM}\left(\frac{z^n}{n}\right) = \frac{2}{\pi}\,\mathrm{IM}\left(\sum_{n=1}^{\infty}\frac{z^n}{n}\right).$

NOW $z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \ldots = \displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n} = -\log(1-z).$

THUS $u(x,y) = -\dfrac{2}{\pi}\,\mathrm{IM}\left[\log(1-z)\right] = -\mathrm{IM}\left[\dfrac{2}{\pi}\left(\ln|1-z| + i\varphi\right)\right]$

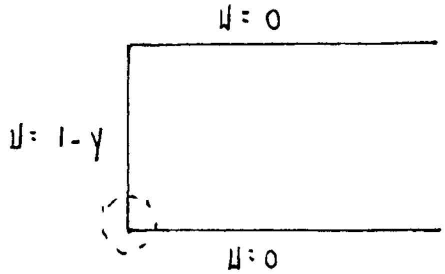
THUS $u(x,y) = -\dfrac{2}{\pi}\varphi \qquad \varphi = \tan^{-1}\left(\dfrac{\mathrm{IM}(1-z)}{\mathrm{RE}(1-z)}\right).$

$$1 - z = 1 - e^{-\pi x}\cos(\pi y) - i e^{-\pi x}\sin(\pi y)$$

$$\mathrm{RE}(1-z) = 1 - e^{-\pi x}\cos(\pi y), \qquad \mathrm{IM}(1-z) = -e^{-\pi x}\sin(\pi y).$$

THEN $$\varphi = \tan^{-1}\left[\frac{-e^{-\pi x}\sin(\pi y)}{1-e^{-\pi x}\cos(\pi y)}\right] = -\tan^{-1}\left(\frac{\sin(\pi y)}{e^{\pi x}-\cos(\pi y)}\right).$$

THIS YIELDS THAT $$u(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{\sin(\pi y)}{e^{\pi x}-\cos(\pi y)}\right).$$



NOTICE THAT THERE IS A DISCONTINUITY IN THE BOUNDARY DATA NEAR $x = 0,\ y = 0$.

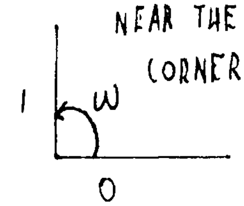
THEN $\sin(\pi y) \approx \pi y$

$$e^{\pi x} - \cos(\pi y) \approx 1 + \pi x - \left[1 - \frac{\pi^2 y^2}{2}\right] \approx \pi x.$$

THIS YIELDS NEAR $x \approx 0$ AND $y \approx 0$ THAT

$$u(x,y) \approx \frac{2}{\pi}\tan^{-1}\left(y/x\right) = \frac{2}{\pi}\,\omega$$

NEAR THE CORNER.



WHEN $\omega = 0 \longrightarrow u = 0$

$\omega = \pi/2 \longrightarrow u = 1$

NOW NEAR THE CORNER

$$u_x \sim \frac{2}{\pi}\,\frac{-y/x^2}{1+y^2/x^2} \approx -\frac{2}{\pi}\,\frac{y}{x^2+y^2}$$

which looks like $u_x \approx -\frac{2}{\pi}\left(\frac{\sin\varphi}{r}\right)$

SIMILARLY, $u_y \approx \frac{2}{\pi}\left(\frac{\cos\varphi}{r}\right)$

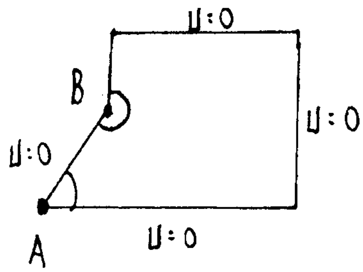
AND SO THE DERIVATIVES BLOW UP AS $(x,y) \to (0,0)$.

CORNER SINGULARITIES

SUPPOSE THAT WE WANT TO SOLVE

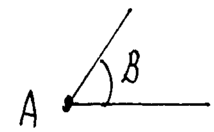
$$\Delta U = F$$

IN THE DOMAIN AS SHOWN BELOW WITH $U=0$ ON ALL SIDES.



WE WANT TO SEE WHAT IS THE BEHAVIOR OF THE SOLUTION NEAR THE CORNERS AT POINTS A AND B.

WE ZOOM IN A NEIGHBORHOOD NEAR A



AND INTRODUCE A LOCAL COORDINATE SYSTEM NEAR A.

NEAR A WE INTRODUCE POLAR COORDINATES r AND φ TO GET

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} = F(r,\varphi). \qquad r=0 \longrightarrow \text{point } A$$

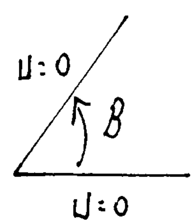
WE THEN LET $r = \varepsilon \rho$ WHERE ε IS SMALL TO LOCALIZE THE REGION NEAR A. THEN $U_r = U_\rho \, 1/\varepsilon$ $\quad U_{rr} = \frac{1}{\varepsilon^2} U_{\rho\rho}$

AND SO

$$\frac{1}{\varepsilon^2}\left(U_{\rho\rho} + \frac{1}{\rho} U_\rho + \frac{1}{\rho^2} U_{\varphi\varphi}\right) = F(\varepsilon\rho, \varphi).$$

SO IF F IS BOUNDED AS $r \to 0$ THEN IN A SMALL NEIGHBOURHOOD OF POINT A WE MUST SOLVE

$$\left\{\begin{array}{l} U_{\rho\rho} + \frac{1}{\rho} U_\rho + \frac{1}{\rho^2} U_{\varphi\varphi} = 0 \\ U = 0 \text{ ON } \varphi = 0 \\ U = 0 \text{ ON } \varphi = B \end{array}\right.$$



WE SEPARATE VARIABLES $U = P(\rho)\,\Phi(\varphi)$

$$\frac{\rho^2\left(P'' + \frac{1}{\rho} P'\right)}{P} = \frac{-\Phi''}{\Phi} = \lambda$$

THIS YIELDS THAT

$$\left.\begin{array}{l}\Phi'' + \lambda\Phi = 0 \\ \Phi(0) = \Phi(\beta) = 0\end{array}\right\} \longrightarrow \Phi = \sin(\sqrt{\lambda}\,\varphi) \quad \text{so} \quad \sqrt{\lambda}\,\beta = \pi$$

$$\text{or} \quad \lambda = \frac{\pi^2}{\beta^2} \quad \text{is } \underline{\underline{\text{smallest}}}\ \lambda.$$

THEN $\rho^2 P'' + \rho P' - \dfrac{\pi^2}{\beta^2} P = 0$ SO $P = \rho^{\pi/\beta}$ IS BOUNDED SOLUTION.

THEREFORE NEAR THE CORNER WE HAVE

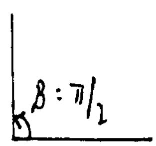
$$u(\rho,\varphi) \approx C\rho^{\pi/\beta}\sin\left(\frac{\pi\varphi}{\beta}\right) \qquad \text{for some } C.$$

WE CALCULATE THE ELECTRIC FIELD

$$\nabla u \approx \left(\partial u/\partial\rho,\ 1/\rho\ \partial u/\partial\varphi\right) = C\left(\frac{\pi}{\beta}\rho^{\pi/\beta - 1}\sin\left(\frac{\pi\varphi}{\beta}\right), \frac{\pi}{\beta}\rho^{\pi/\beta - 1}\cos\left(\frac{\pi\varphi}{\beta}\right)\right)$$

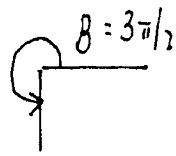
THIS YIELDS $\quad \nabla u \approx C\dfrac{\pi}{\beta}\rho^{\pi/\beta - 1}\left(\sin(\pi\varphi/\beta), \cos(\pi\varphi/\beta)\right).$

WE CONSIDER A FEW CASES:

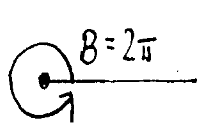


$\longrightarrow \nabla u = O(\rho) \longrightarrow$ DOES NOT BLOW UP AS $\rho \to 0$.

$\longrightarrow \nabla u = O(1) \longrightarrow$ NO BLOW-UP AS $\rho \to 0$



$\longrightarrow \nabla u = O(\rho^{-1/3})$ BLOWS UP AS $\rho \to 0$



$\longrightarrow \nabla u = O(\rho^{-1/2})$ BLOWS UP AS $\rho \to 0$

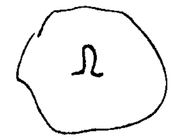
THIS LAST EXAMPLE IS A "LIGHTNING ROD" THAT CAN STORE LOTS OF CHARGE AT TIP.

UNIQUENESS AND EXISTENCE

WE WANT TO PROVE THAT THERE IS A UNIQUE SOLUTION TO

$$\Delta u = F \quad \text{IN } \Omega$$

$$\partial_n u + K(u-g) = 0 \quad \text{ON } \partial\Omega \quad \text{WITH } K>0 \text{ AND CONSTANT.}$$



HERE Ω IS AN ARBITRARY BOUNDED DOMAIN.

SUPPOSE THAT u_1, u_2 ARE TWO SOLUTIONS AND LET $v = u_1 - u_2$.

THEN v SATISFIES
$$\Delta v = 0 \quad \text{IN } \Omega$$
$$\partial_n v + Kv = 0 \quad \text{ON } \partial\Omega, \quad K>0.$$

WE WANT TO PROVE THAT $v \equiv 0$ IN Ω SO THAT $u_1 \equiv u_2$ IN Ω.

WE RECALL A VECTOR IDENTITY
$$\nabla \cdot [\underline{F}\phi] = \phi\, \nabla\cdot \underline{F} + \underline{F}\cdot\nabla\phi$$

SO THAT $\quad \nabla\cdot[v\nabla v] = \nabla v\cdot\nabla v + v\Delta v = |\nabla v|^2 + v\Delta v.$

THEN $\quad 0 = \int_\Omega v\,\Delta v\, d\underline{x} = \int_\Omega v\,\nabla\cdot[\nabla v]\, d\underline{x} = \int_\Omega \nabla\cdot[v\nabla v]\, d\underline{x} - \int_\Omega |\nabla v|^2\, d\underline{x}$

NOW USING DIVERGENCE THEOREM,
$$\int_{\partial\Omega} v\,\nabla v\cdot\hat{n}\, ds = \int_\Omega |\nabla v|^2\, d\underline{x}$$

BUT $\quad \nabla v\cdot\hat{n} = \partial_n v = -Kv \quad$ ON $\partial\Omega$. THEN,
$$-\int_{\partial\Omega} K v^2\, ds = \int_\Omega |\nabla v|^2\, d\underline{x} \longrightarrow \int_\Omega |\nabla v|^2\, d\underline{x} + \int_{\partial\Omega} K v^2\, ds = 0.$$

SINCE $K>0$ THIS IMPLIES THAT $v \equiv 0$ IN Ω SO $u_1 \equiv u_2$.

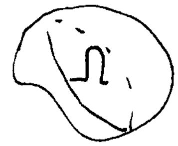
IF $K=0$ (NO FLUX) THEN v = CONSTANT IN Ω AND SO ANY TWO SOLUTIONS DIFFER BY A CONSTANT.

UNBOUNDED REGIONS - UNIQUENESS THEOREMS

CONSIDER Ω TO BE A DOMAIN IN $\mathbb{R}^3$ WITH

$$\Delta u = 0 \quad \text{outside } \Omega$$

$$u = 0 \quad \text{ON } \partial\Omega$$



AND LET'S IMPOSE u IS BOUNDED AS $r = |\underline{x}| \to \infty$.

DOES THIS MEAN THAT $u \equiv 0$ AT EACH x OUTSIDE Ω?

NO! LET Ω BE A SPHERE OF RADIUS 1 SO THAT

$$u_{rr} + \frac{2}{r} u_r = 0 \quad \text{WITH } r \geq 1$$

$$u = 0 \quad \text{ON } r = 1$$

$$u \text{ BOUNDED AS } r \to \infty$$

THEN THIS PROBLEM HAS AN INFINITE NUMBER OF SOLUTIONS GIVEN BY

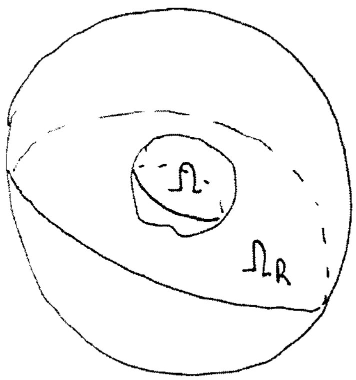
$$u = A\left(1 - 1/r\right) \quad \text{FOR ANY } A.$$

IN ORDER TO GET THE UNIQUE SOLUTION $u = 0$ WE MUST IMPOSE THE STRONGER CONDITION THAT

$$u = O\left(1/r\right) \quad \text{AS } r \to \infty.$$

i.e. $u \to 0$ AS $r \to \infty$ LIKE $u = C/r$.

TO SEE THAT THIS CONDITION IS SUFFICIENT TO GUARANTEE THAT $u \equiv 0$ WE "SOLVE" THE FOLLOWING PROBLEM IN THE DOMAIN WHERE Ω IS SURROUNDED BY A LARGE SPHERE OF RADIUS R.



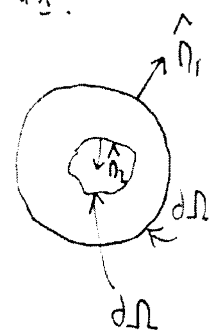
LET $\Omega_R = \{ |\underline{x}| \le R \,/\, \Omega \}$

REGION OUTSIDE Ω BUT INSIDE SPHERE.

THEN RECALL $u \Delta u = u \nabla \cdot (\nabla u)$

$$= \nabla \cdot [u \nabla u] - |\nabla u|^2 .$$

THUS $$0 = \int_{\Omega_R} u \Delta u \, d\underline{x} = \int_{\Omega_R} \nabla \cdot [u \nabla u] \, d\underline{x} - \int_{\Omega_R} |\nabla u|^2 \, d\underline{x} .$$



NOW USE DIVERGENCE THEOREM

$$\int_{\partial\Omega_R} u \nabla u \cdot \hat{n}_1 \, dS + \int_{\partial\Omega} u \nabla u \cdot \hat{n}_2 \, dS = \int_{\Omega_R} |\nabla u|^2 \, d\underline{x}$$

BUT $u = 0$ ON $\partial\Omega$ AND $\partial\Omega_R$ IS THE BOUNDARY OF A SPHERE OF RADIUS R.

$$\longrightarrow \quad \int_0^{2\pi} \int_0^{\pi} u \frac{\partial u}{\partial r}\bigg|_{r=R} R^2 \sin\phi \, d\phi \, d\varphi = \int_{\Omega_R} |\nabla u|^2 \, d\underline{x}$$

NOW IF WE CAN SHOW THAT LHS $\longrightarrow 0$ AS $R \to \infty$

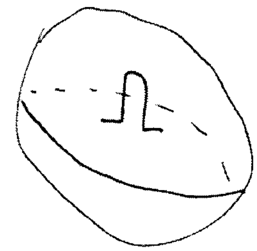
THEN $\displaystyle\int_{\text{outside } \Omega} |\nabla u|^2 \, d\underline{x} = 0 \longrightarrow u =$ CONSTANT AND SINCE $u = 0$ ON $\partial\Omega \longrightarrow u \equiv 0$.

SO WE NEED $\displaystyle u \frac{\partial u}{\partial r}\bigg|_{r=R} \hat{\approx} \frac{1}{R^{2+\delta}}$ FOR ANY $\delta > 0$.

THIS IS CLEARLY SATISFIED IF $u \hat{=} C/r$ AS $r \to \infty$

FOR THEN $\displaystyle u \frac{\partial u}{\partial r}\bigg|_{r=R} = -\frac{C^2}{R^3}$ WITH $\delta = 1$.

THEREFORE IN $\mathbb{R}^3$ THE FOLLOWING PROBLEM HAS A UNIQUE SOLUTION:



$$\begin{cases} \Delta u = 0 & \text{outside } \Omega \\ u = 1 & \text{ON } \partial\Omega \\ u \approx C/r & \text{AS } r = |\underline{x}| \longrightarrow \infty \end{cases}$$

THE CONSTANT C IS CALLED THE "CAPACITANCE" OF THE "BODY" Ω. IT MEASURES HOW MUCH CHARGE CAN BE STORED ON THE SURFACE OF Ω (HINT: USE THE DIVERGENCE THEOREM).

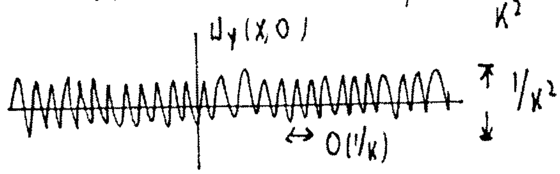
ILL-POSED PROBLEM CONSIDER THE FOLLOWING PROBLEM WHERE WE GIVE BOTH u AND THE FLUX u_y AT $y=0$ FOR

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0$$

$$u(x,0) = 0, \qquad u_y(x,0) = \frac{1}{k^2} \sin(kx)$$

IF $k \gg 1$ IS LARGE THEN $|u_y| \leq \frac{1}{k^2} \longrightarrow 0$ BUT u_y HIGHLY OSCILLATORY

SO THAT



THE EXACT SOLUTION IS $u = f(y)\sin(kx) \longrightarrow f'' - k^2 f = 0$ WITH $f(0) = 0$ AND $f'(0) = 1/k^2$. THE SOLUTION IS $f(y) = \frac{1}{k^3}\sinh(ky)$

AND SO $\quad u(x,y) = \frac{1}{k^3}\sin(kx)\sinh(ky).$

NOTICE THAT FOR EACH FIXED $y>0$ WE GET THAT $|u| \leq \frac{c}{k^3} e^{ky}$ WHICH HAS UNBOUNDED GROWTH FOR HIGH FREQUENCY $k \to \infty$.

THE SOLUTION DOES NOT HAVE CONTINUOUS DEPENDENCE ON DATA $\to$ ILL-POSED

UNIQUENESS OF SOLUTION TO HEAT EQUATION

WE CONSIDER THE FOLLOWING PDE:

$$u_t = \nabla \cdot [p(\underline{x}) \nabla u] - q u - F \quad \text{IN } \Omega$$

$$\partial_n u + K(u - g) = 0 \quad \text{ON } \partial\Omega$$

$$u(\underline{x}, 0) = u_0(\underline{x}).$$

WE ASSUME THAT $p(\underline{x}) > 0$, $q(\underline{x}) > 0$ AND $K > 0$ FOR X IN Ω.

NOW SHOW THERE IS A UNIQUE SOLUTION.

LET u_1, u_2 BE TWO SOLUTIONS AND DEFINE $V \equiv u_1 - u_2$.

WE WANT TO SHOW THAT $V \equiv 0$ IN Ω AND FOR $t \geq 0$.

BY SUBTRACTION, WE OBTAIN THAT

$$V_t = \nabla \cdot [p(\underline{x}) \nabla V] - q V \quad \text{IN } \Omega$$

$$\partial_n V + K V = 0 \quad \text{ON } \partial\Omega$$

$$V = 0 \quad \text{AT } t = 0.$$

WE WANT TO SHOW $V \equiv 0$. WE MULTIPLY BY V AND INTEGRATE OVER Ω TO OBTAIN:

$$V V_t = V \nabla \cdot [p(\underline{x}) \nabla V] - q V^2.$$

THEN, $$\frac{1}{2} \frac{d}{dt} \int_\Omega V^2 \, d\underline{x} = \int_\Omega V \nabla \cdot [p(\underline{x}) \nabla V] \, d\underline{x} - \int_\Omega q V^2 \, d\underline{x}.$$

NOW $$\nabla \cdot [p V \nabla V] = V \nabla \cdot [p \nabla V] + p \nabla V \cdot \nabla V$$

THIS GIVES $$\frac{d}{dt} \frac{1}{2} \int_\Omega V^2 \, d\underline{x} = \int_\Omega \left(\nabla \cdot [p V \nabla V] - p |\nabla V|^2 \right) d\underline{x} - \int_\Omega q V^2 \, d\underline{x}$$

WE USE THE DIVERGENCE THEOREM NEXT TO OBTAIN,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} v^2\,d\underline{x} = \int_{\partial\Omega} p\,v\,\nabla v\cdot\hat{n}\,ds - \int_{\Omega}\left(p\,|\nabla v|^2 + q\,v^2\right)d\underline{x}$$

$$(*)\qquad \frac{1}{2}\frac{d}{dt}\int_{\Omega} v^2\,d\underline{x} = -\int_{\partial\Omega} p\,\kappa\,v^2\,ds - \int_{\Omega}\left(p\,|\nabla v|^2 + q\,v^2\right)d\underline{x}.$$

NOW DEFINE $\quad E(t) = \frac{1}{2}\int_{\Omega} v^2(\underline{x},t)\,d\underline{x}.$

THEN $E(t)$ IS CONTINUOUS, $E(0) = 0$ SINCE $v(\underline{x},0) = 0$ FOR $\underline{x}\in\Omega$, AND $E(t)\geq 0$ SINCE $v^2 \geq 0$ IN Ω.

BUT SINCE $p\geq 0$, $q\geq 0$ AND $\kappa > 0$, $(*)$ YIELDS $\frac{dE}{dt}\leq 0$.

THEREFORE, BY CALCULUS $E(t)\equiv 0$ FOR ALL t.

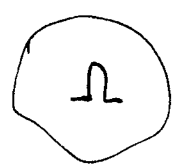
THIS IMPLIES THAT $v\equiv 0$ IN Ω AND FOR $t\geq 0$,

WHICH YIELDS $u_1 = u_2$.

REMARK

(i) THE CONDITION FOR THE EXISTENCE OF A SOLUTION

TO
$$\Delta u = F \quad \text{IN } \Omega$$
$$\partial_n u = g \quad \text{ON } \partial\Omega$$



IS THAT $\displaystyle\int_{\Omega} F\,d\underline{x} = \int_{\partial\Omega} g\,ds.$

PROOF USE DIVERGENCE THEOREM $\displaystyle\int_{\Omega}\nabla\cdot(\nabla u)\,d\underline{x} = \int_{\partial\Omega}\nabla u\cdot\hat{n}\,ds.$

HENCE $\displaystyle\int_{\Omega} F\,d\underline{x} = \int_{\partial\Omega}\partial_n u\,ds = \int_{\partial\Omega} g\,ds$ IS NEEDED.

REMARKS ON UNIQUENESS

NONLINEAR PROBLEMS CAN HAVE MORE THAN ONE SOLUTION.

FOR INSTANCE CONSIDER THE NONLINEAR PROBLEM IN A DISK:

$$\Delta u + B e^{u} = 0, \quad 0 \le r \le 1, \; 0 \le \varphi \le 2\pi$$

$$u = 0 \text{ ON } r = 1, \quad u \text{ BOUNDED AS } r \to 0.$$

WE LOOK FOR RADIALLY SYMMETRIC SOLUTIONS $u = u(r)$ SATISFYING

$$u_{rr} + \frac{1}{r} u_r + B e^{u} = 0, \quad 0 \le r \le 1$$

$$u = 0 \text{ ON } r = 1, \quad u \text{ BOUNDED AS } r \to 0$$

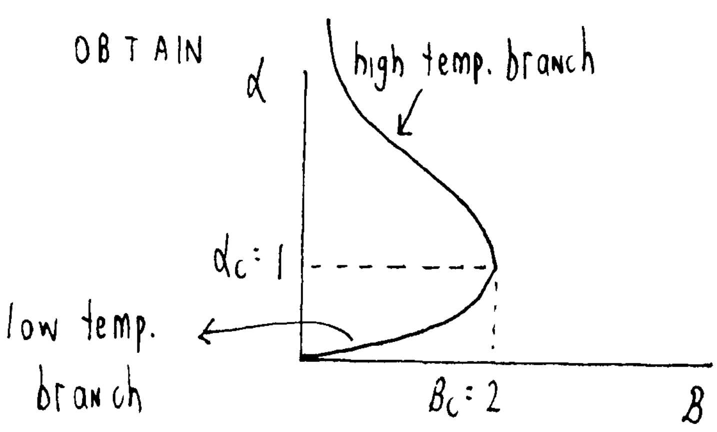
THE SOLUTIONS HAVE THE FORM

$$(*) \quad \left\{ \; u = 2 \log\left(\frac{1+\alpha}{1+\alpha r^2} \right) \quad \text{WITH} \quad B = \frac{8\alpha}{(1+\alpha)^2}, \quad \alpha > 0. \right.$$

(WE VERIFY THIS BELOW) NOTICE $u(0) = 2\log(1+\alpha)$ SO THAT α IS A MEASURE OF THE TEMPERATURE AT THE CENTER.

NOTICE THAT $u(0)$ IS THE MAXIMUM TEMPERATURE.

IF WE PLOT α VERSUS B, THEN BY CALCULUS WE OBTAIN



NOTICE $dB/d\alpha = 0$ AT $\alpha_c = 1$.

THEN $B(\alpha_c) = \dfrac{8}{(1+1)^2} = 2 = B_c$

- FOR $0 < B < B_c = 2$, THERE ARE TWO RADIALLY SYMMETRIC SOLUTIONS.
- NO RADIALLY SYMMETRIC SOLUTIONS FOR $B > B_c = 2$.

VERIFICATION OF (*) AS SOLUTION

WE WRITE $u = -2\log(1+\alpha r^2) + 2\log(1+\alpha)$.

THEN $u_r = -4\alpha r(1+\alpha r^2)^{-1}$

$$u_{rr} = -4\alpha(1+\alpha r^2)^{-1} + 8\alpha^2 r^2 (1+\alpha r^2)^{-2}$$

NOW $u_{rr} + \frac{1}{r}u_r = 8\alpha^2 r^2(1+\alpha r^2)^{-2} - 8\alpha(1+\alpha r^2)^{-1} = 8\alpha(1+\alpha r^2)^{-2}\left[\alpha r^2 - (1+\alpha r^2)\right]$

SO $u_{rr} + \frac{1}{r}u_r = -8\alpha(1+\alpha r^2)^{-2}$.

NOW $e^u = \dfrac{(1+\alpha)^2}{(1+\alpha r^2)^2}$ SO $u_{rr} + \frac{1}{r}u_r + Be^u$

$$= \frac{-8\alpha}{(1+\alpha r^2)^2} + \frac{B(1+\alpha)^2}{(1+\alpha r^2)^2} = 0$$

THUS $8\alpha = B(1+\alpha)^2$ OR $B = 8\alpha/(1+\alpha)^2$.

STABILITY AND STURM-LIOUVILLE THEORY

CONSIDER $u_t = u_{rr} + \frac{1}{r}u_r + Be^u$ IN $0 \le r \le 1$, $t > 0$

$u(1,t) = 0$, u BOUNDED AS $r \to 0$.

LET $u_s(r)$ BE STEADY-STATE SOLUTION GIVEN BY

$$u_s(r) = 2\log\left(\frac{1+\alpha}{1+\alpha r^2}\right) \qquad B = \frac{8\alpha}{(1+\alpha)^2}.$$

IS THIS SOLUTION STABLE? IF WE START WITH INITIAL CONDITION NEAR $u_s(r)$ DO WE REMAIN CLOSE AS $t \to \infty$?

TO STUDY THIS WE LINEARIZE THE PDE AROUND $u_s(r)$. WE WRITE $u(r,t) = u_s(r) + \delta e^{\lambda t}\phi(r)$

WITH $\delta \ll 1$ (δ SMALL).

THIS YIELDS THAT

$$\lambda \delta e^{\lambda t} \phi = u_s'' + \frac{1}{r} u_s' + \delta e^{\lambda t} \left(\phi'' + \frac{1}{r} \phi' \right) + \beta e^{u_s + \delta e^{\lambda t} \phi}$$

$$= u_s'' + \frac{1}{r} u_s' + \delta e^{\lambda t} \left(\phi'' + \frac{1}{r} \phi' \right) + \beta e^{u_s} \left(1 + \delta e^{\lambda t} \phi + \dots \right)$$

UPON USING $e^h \approx 1 + h + \dots$ AS $h \to 0$.

THEN SINCE $u_s'' + \frac{1}{r} u_s' + \beta e^{u_s} = 0$ WE OBTAIN THAT

$$\lambda \phi = \phi'' + \frac{1}{r} \phi' + \beta e^{u_s} \phi$$

NOW $$\beta e^{u_s} = \frac{8\alpha}{(1+\alpha)^2} e^{2 \log\left(\frac{1+\alpha}{1+\alpha r^2} \right)} = \frac{8\alpha}{(1+\alpha)^2} \frac{(1+\alpha)^2}{(1+\alpha r^2)^2} = \frac{8\alpha}{(1+\alpha r^2)^2}.$$

THE EIGENVALUE PROBLEM BECOMES A SL PROBLEM

$$\phi'' + \frac{1}{r} \phi' + g(r) \phi = \lambda \phi$$

$$\phi(1) = 0, \quad \phi(0) \text{ BOUNDED}$$

WITH $g(r) = \frac{8\alpha}{(1+\alpha r^2)^2}$. $\exists$ INFINITE NUMBER OF EIGENVALUES $\lambda_1, \lambda_2, \dots$

IF ONE CAN SHOW THAT ANY EIGENVALUE λ SATISFIES $\lambda_K < 0$, $K = 1, 2, \dots$ $\longrightarrow$ THEN WE HAVE STABILITY OF $u_s(r)$.

IT TURNS OUT TO BE TRUE ONLY FOR $0 < \alpha < 1$ (Lower BRANCH).

IF $\exists\, \lambda_1 > 0$, THEN $u_s(r)$ IS UNSTABLE. THIS OCCURS ON THE HIGH TEMPERATURE BRANCH WHERE $\alpha > 1$.