Lecture Note 14 (last)

BESSEL FUNCTIONS

IF WE SEPARATE VARIABLES IN

$$u_t = D\left(u_{rr} + \frac{1}{r} u_r\right) \quad 0 \le r \le a,\ t > 0$$

$$u(r,0) = f(r), \quad u(a,t) = 0$$

WE OBTAIN $u(r,t) = R(r)\,T(t) \longrightarrow \dfrac{T'}{DT} = \dfrac{R'' + 1/r\, R'}{R} = -\lambda.$

THIS LEADS TO THE SINGULAR STURM-LIOUVILLE PROBLEM

$$(*)\quad \begin{cases} \phi'' + \frac{1}{r}\phi' + \lambda\phi = 0, & 0 \le r \le a, \\ \phi(a) = 0, \quad \phi(0) \text{ finite} \end{cases}$$

THIS CAN BE WRITTEN AS $(r\phi')' + \lambda r \phi = 0$ SO THAT IN STURM-LIOUVILLE FORM $p(r) = r$ AND $w(r) = r$ IS THE WEIGHT. IT IS A SINGULAR STURM-LIOUVILLE PROBLEM SINCE $p(0) = 0$.

THE SOLUTIONS TO $(*)$ ARE DENOTED BY $J_0(\sqrt{\lambda}\, r)$ AND $Y_0(\sqrt{\lambda}\, r)$ (BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER ZERO) AND SO

$$(+)\quad \phi = A J_0(\sqrt{\lambda}\, r) + B Y_0(\sqrt{\lambda}\, r)$$

WHERE $J_0(x)$ AND $Y_0(x)$ ARE THE TWO LINEARLY INDEPENDENT SOLUTIONS OF $x^2 y'' + x y' + x^2 y = 0$ IN $x \ge 0$. NOTICE THAT $x = 0$ IS A REGULAR SINGULAR POINT. IF WE PUT $y = x^r \to r(r-1) + r = 0$ SO $r = 0$ IS double ROOT.

local behavior

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots$$

$$Y_0(x) = \frac{2}{\pi}\left[\log(x/2) + \gamma\right] J_0(x) + \frac{2}{\pi}\left(x^2/2^2 + \dots\right)$$

NOTICE THAT $J_0(x) \sim 1$ AS $x \to 0^+$, $Y_0(x) \sim \frac{2}{\pi}\log x$ AS $x \to 0^+$.

REMARK IF WE LET $x = \sqrt{\lambda}\, r$

THEN $y(x) = \phi(x/\sqrt{\lambda})$ TRANSFORMS $\phi_{rr} + \frac{1}{r}\phi_r + \lambda\phi = 0$

INTO $y'' + \frac{1}{x} y' + y = 0$.

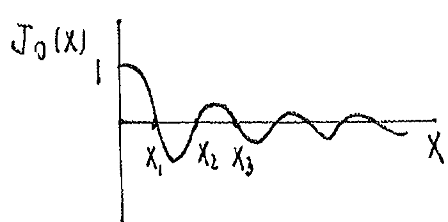
PROOF $\phi_r = y'\sqrt{\lambda}$, $\phi_{rr} = \lambda y''$

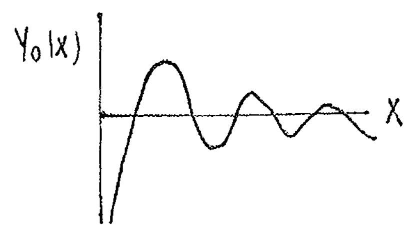
SO $\phi_{rr} + \frac{1}{r}\phi_r + \lambda\phi = \lambda y'' + \frac{\sqrt{\lambda}}{(x/\sqrt{\lambda})} y' + \lambda y = \lambda\left(y'' + \frac{1}{x}y' + y\right) = 0$

THUS $y'' + \frac{1}{x} y' + y = 0$ $\square$

NOW RETURNING TO (+) WE SET $\phi(0)$ FINITE TO GET $B = 0$.

THEN $\phi(a) = 0$ IMPLIES $J_0(\sqrt{\lambda}\, a) = 0$.





$x_1 = 2.405\ldots$

THUS THE EIGENVALUES ARE $\sqrt{\lambda_k}\, a = x_k$ OR $\lambda_k = x_k^2/a^2$, $k = 1, 2, \ldots$

WHERE x_k FOR $k = 1, 2, \ldots$ ARE THE ROOTS OF $J_0(x) = 0$.

THUS $\phi_k(r) = A\, J_0(\sqrt{\lambda_k}\, r)$.

SINCE THE WEIGHT FUNCTION IS $w(r) = r$ WE HAVE

THE ORTHOGONALITY PROPERTY

$$\int_0^a r\, J_0(\sqrt{\lambda_k}\, r)\, J_0(\sqrt{\lambda_n}\, r)\, dr = 0 \quad k \neq n.$$

IN ADDITION, SOME FURTHER WORK (NOT GIVEN) SHOWS THAT

$$\int_0^a r\left(J_0(\sqrt{\lambda_k}\, r)\right)^2 dr = \frac{a^2}{2}\left[J_0'(\sqrt{\lambda_k}\, a)\right]^2.$$

THIS DERIVATION IS GIVEN IN APPENDIX A PAGE 16 BELOW.

OSCILLATIONS: LARGE X BEHAVIOR OF $J_0(x)$, $Y_0(x)$.

$$x^2 y'' + x y' + x^2 y = 0$$

WE LET $y = p\psi$. THEN

$$x^2(p\psi'' + 2p'\psi' + p''\psi) + x(p\psi' + p'\psi) + x^2 p\psi = 0$$

$$\psi'' + \left(\frac{2p'}{p} + \frac{1}{x}\right)\psi' + \left(\frac{p''}{p} + \frac{p'}{xp} + 1\right)\psi = 0.$$

CHOOSE p TO ELIMINATE THE MIDDLE TERM:

$$\frac{p'}{p} = \frac{-1}{2x} \quad \text{so} \quad \ln p = -\frac{1}{2}\ln x + C \rightarrow p = x^{-1/2}.$$

THEN $p' = -\frac{1}{2}x^{-3/2}$, $p'' = \frac{3}{4}x^{-5/2}$

$$\text{so} \quad \frac{p''}{p} = \frac{3}{4x^2}, \quad \frac{p'}{xp} = \frac{(-1/2\, x^{-3/2})}{x^{1/2}} = -\frac{1}{2x^2}.$$

THIS YIELDS THAT $\psi'' + \left(1 + 1/4x^2\right)\psi = 0$.

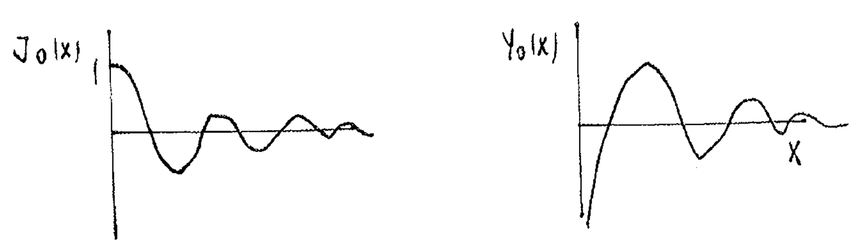
FOR $x \gg 1$ WE HAVE $\psi'' + \psi \approx 0$ SO $\psi = \begin{cases} \cos x \\ \sin x \end{cases}$

IT TURNS OUT THAT

$$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \pi/4) \qquad \text{FOR } x \gg 1$$

$$Y_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \sin(x - \pi/4)$$

DECAYING OSCILLATIONS FOR LARGE X.



FINALLY, RETURNING TO $u_t = D\left(u_{rr} + \frac{1}{r} u_r\right)$

WE OBTAIN

$$u(r,t) = \sum_{k=1}^{\infty} e^{-\lambda_k D t} J_0\left(\sqrt{\lambda_k}\, r\right) c_k$$

THEN $$u(r,0) = f(r) = \sum_{k=1}^{\infty} c_k J_0\left(\sqrt{\lambda_k}\, r\right).$$

BY ORTHOGONALITY WE OBTAIN

$$c_k = \frac{\int_0^a f(r)\, r\, J_0\left(\sqrt{\lambda_k}\, r\right) dr}{\int_0^a r\left(J_0\left(\sqrt{\lambda_k}\, r\right)\right)^2 dr} .$$

<u>EXAMPLE</u> FIND AN EIGENFUNCTION EXPANSION SOLUTION FOR

$$u_t = D\left(u_{rr} + \frac{1}{r} u_r\right), \quad 0 \le r \le a, \quad t > 0$$

$$u(r,0) = f(r), \quad u(a,t) = e^{-t}, \quad u \text{ BOUNDED AS } r \to 0.$$

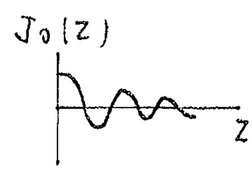
WE LET $u(r,t) = e^{-t} + v(r,t)$ TO OBTAIN HOMOGENEOUS BOUNDARY CONDITIONS SO THAT

$$\left\{ \begin{aligned} & v_t = D\left(v_{rr} + \frac{1}{r} v_r\right) + e^{-t} \\ & v(a,t) = 0, \quad v \text{ BOUNDED AS } r \to 0 \\ & v(r,0) = f(r) - 1 \end{aligned} \right.$$

WE SEPARATE VARIABLES TO GET $\dfrac{T'}{DT} = \dfrac{\phi'' + \frac{1}{r}\phi'}{\phi} = -\lambda$

FOR THE HOMOGENEOUS PROBLEM.

THIS GIVES $\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0 \quad 0 \le r \le a$



$$\phi(a) = 0, \quad \phi(0) \text{ FINITE}$$

SO $\phi_k = J_0\left(\sqrt{\lambda_k}\, r\right)$ WHERE $\lambda_k = z_k^2/a^2$ $J_0(z_k) = 0$ $k = 1, 2, \ldots$

THEN WE WRITE

$$v(r,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(r) \qquad \phi_n(r) = J_0(\sqrt{\lambda_n}\, r)$$

SUBSTITUTING INTO THE PDE WE OBTAIN

$$\sum_{n=1}^{\infty} b_n'(t)\,\phi_n(r) = \sum_{n=1}^{\infty} D\, b_n(t)\left(\phi_n'' + \frac{1}{r}\phi_n'\right) + e^{-t}$$

THEN USE $\phi_n'' + \frac{1}{r}\phi_n' = -\lambda_n \phi_n$ AND EXPAND $1 = \sum_{n=1}^{\infty} \gamma_n J_0(\sqrt{\lambda_n}\, r)$.

THIS YIELDS THAT $\gamma_n = \int_0^a r J_0(\sqrt{\lambda_n}\, r)\, dr \Big/ \int_0^a r J_0^2(\sqrt{\lambda_n}\, r)\, dr$.

WE THEREFORE OBTAIN

$$\sum_{n=1}^{\infty} b_n'\,\phi_n = -\sum_{n=1}^{\infty} D\lambda_n b_n \phi_n + \sum_{n=1}^{\infty} e^{-t}\gamma_n \phi_n$$

THIS YIELDS $\displaystyle\sum_{n=1}^{\infty} \left(b_n' + D\lambda_n b_n - e^{-t}\gamma_n\right)\phi_n = 0$.

BY ORTHOGONALITY OF EIGENFUNCTIONS WE OBTAIN

$$\begin{cases} b_n' = -D\lambda_n b_n + e^{-t}\gamma_n \\ b_n(0) \text{ given} \end{cases}$$

NOTICE THAT $v(r,0) = f(r) - 1 = \sum_{n=1}^{\infty} b_n(0)\,\phi_n(r)$.

THIS IMPLIES THAT $b_n(0) = \int_0^a (f(r)-1)\, r\,\phi_n(r)\, dr \Big/ \int_0^a r(\phi_n(r))^2\, dr$.

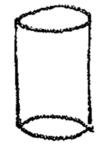
WE SOLVE FOR b_n: $\quad b_n' + D\lambda_n b_n = e^{-t}\gamma_n$.

HENCE $\quad (b_n e^{D\lambda_n t})' = e^{-t} e^{D\lambda_n t}\gamma_n \to b_n e^{D\lambda_n t} = b_n(0) + \int_0^t \gamma_n e^{-(1+D\lambda_n)\tau}\, d\tau$

THIS GIVES $\quad b_n(t) = b_n(0) e^{-D\lambda_n t} + e^{-D\lambda_n t}\int_0^t \gamma_n e^{-(1+D\lambda_n)\tau}\, d\tau$

WITH $\quad u(r,t) = e^{-t} + \sum_{n=1}^{\infty} b_n(t)\, J_0(\sqrt{\lambda_n}\, r)$.

<u>EXAMPLE</u> FIND THE SOLUTION TO



$$u_t = D\left(u_{rr} + \frac{1}{r} u_r + u_{zz} \right) \qquad \text{IN} \quad 0 < z < H, \quad 0 < r < a$$

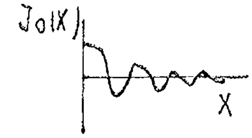
WITH $u(r,0,t) = u(r,H,t) = 0$

$u(a,z,t) = 0$, u FINITE AS $r \to 0$

$u(r,z,0) = f(r,z)$.

WE SEPARATE VARIABLES TO OBTAIN $u(r,z,t) = R(r)T(t)Z(z)$.

WE CALCULATE $$\frac{T'}{DT} = \frac{R'' + 1/r\, R'}{R} + \frac{Z''}{Z} = -\lambda .$$



WE SET $R'' + \frac{1}{r} R' = -\mu R$ SO THAT

$$\left.\begin{aligned} &R'' + \frac{1}{r} R' + \mu R = 0 \\ &R(a) = 0, \quad R(0) \text{ FINITE} \end{aligned}\right\} \rightarrow R_n(r) = J_0(\sqrt{\mu_n}\, r)$$

WHERE $\sqrt{\mu_n}\, a = X_n$, $J_0(X_n) = 0$

THUS $\mu_n = X_n^2 / a^2$ WHERE $J_0(X_n) = a$.

THEN $$-\mu_n + \frac{Z''}{Z} = -\lambda$$

SO THAT $$Z'' + (\lambda - \mu_n) Z = 0$$
$$Z(0) = Z(H) = 0$$

THIS YIELDS THAT $Z = \sin\left(\sqrt{\lambda - \mu_n}\, z\right)$ $\quad \sqrt{\lambda - \mu_n}\, H = m\pi \quad m = 1, 2, \ldots$

THIS GIVES $$\lambda_{mn} = \frac{m^2 \pi^2}{H^2} + \mu_n \qquad \mu_n = X_n^2 / a^2$$

AND $J_0(X_n) = 0$.

WE OBTAIN $Z_m(z) = \sin\left(\frac{m\pi z}{H}\right)$, $m = 1, 2, \ldots$

$R_n(r) = J_0\left(X_n r / a\right)$ $\quad n = 1, 2, \ldots$

THEN $$\frac{T'}{DT} = -\lambda_{mn} \longrightarrow T = e^{-D(\mu_n + m^2\pi^2/H^2)t}$$

THIS YIELDS THE EIGENFUNCTION EXPANSION

$$(*) \quad u(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-D(\mu_n + m^2\pi^2/H^2)t} J_0(\sqrt{\mu_n}\, r)\sin\left(\frac{m\pi z}{H}\right)$$

FINALLY TO SATISFY THE INITIAL CONDITION WE OBTAIN AN EQUATION FOR A_{mn}:

$$f(r,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, J_0(\sqrt{\mu_n}\, r)\sin\left(\frac{m\pi z}{H}\right)$$

BY ORTHOGONALITY,

$$\int_0^a \int_0^H f(r,z)\, r\, J_0(\sqrt{\mu_n}\, r)\sin\left(\frac{m\pi z}{H}\right) dr\, dz$$

$$= A_{mn}\int_0^a \int_0^H r\, J_0^2(\sqrt{\mu_n}\, r)\, dr\, \sin^2\left(\frac{m\pi z}{H}\right) dr\, dz$$

$$= A_{mN}\frac{H}{2}\int_0^a r\, J_0^2(\sqrt{\mu_n}\, r)\, dr = A_{mn}\frac{H}{2}\left(\frac{a^2}{2}\left(J_0'(\sqrt{\mu_n}\, a)\right)^2\right)$$

$$= \frac{Ha^2}{4} A_{mn}\left(J_0'(\sqrt{\mu_n}\, a)\right)^2.$$

THIS YIELDS $$A_{mn} = \frac{4}{Ha^2[J_0'(\sqrt{\mu_n}\, a)]^2}\int_0^a \int_0^H r\, f(r,z)\, J_0(\sqrt{\mu_n}\, r)\sin\left(\frac{m\pi z}{H}\right) dr\, dz$$

WHICH YIELDS THE COEFFICIENTS IN $(*)$

EXAMPLE SOLVE THE WAVE EQUATION

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0<r<a, \quad t>0$$

$$u(r,0)=0, \quad u_t(r,0)=0, \quad u(a,t)=1, \quad u \text{ BOUNDED AS } r\to 0.$$

THIS CORRESPONDS TO DEFLECTING A MEMBRANE ALONG THE RIM TO GENERATE WAVES.

WE LET $u(r,t) = 1 + v(r,t)$ SO THAT

$$v_{tt} = c^2\left(v_{rr} + \frac{1}{r}v_r\right)$$

$$v(r,0)=-1, \quad v_t(r,0)=0$$

$$v(a,t)=0, \quad v \text{ BOUNDED AS } r\to 0.$$

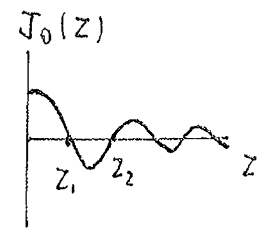
WE SEPARATE VARIABLES $v = R(r)T(t)$ TO OBTAIN

$$\frac{T''}{c^2 T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda \qquad T'' + c^2\lambda T = 0$$

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0$$

$$\phi(a)=0, \quad \phi(0) \text{ FINITE}$$



THE SOLUTION IS $\phi_k(r) = J_0(\sqrt{\lambda_k}\, r)$

WHERE $\sqrt{\lambda_k}\, a = z_k$ AND $J_0(z_k)=0$ FOR $k=1,2,3,\ldots$

HENCE, $\lambda_k = z_k^2/a^2$.

THIS YIELDS THAT $T_k'' + \omega_k^2 T_k = 0 \qquad \omega_k = c\sqrt{\lambda_k} = c z_k/a$.

THE SOLUTION IS $T_k = A_k\cos(\omega_k t) + B_k \sin(\omega_k t)$

THIS YIELDS THAT $v(r,t) = \sum_{k=1}^{\infty} \left(A_k\cos(\omega_k t) + B_k\sin(\omega_k t)\right) J_0(\sqrt{\lambda_k}\, r)$

NOW WE SATISFY $V(r,0) = -1$, $V_t(r,0) = 0$ TO OBTAIN

$$-1 = \sum_{k=1}^{\infty} A_k J_0(\sqrt{\lambda_k}\, r) \qquad A_k = -\frac{\int_0^a r J_0(\sqrt{\lambda_k}\, r)\, dr}{\int_0^a r J_0^2(\sqrt{\lambda_k}\, r\, dr)}$$

$$0 = \sum_{k=1}^{\infty} B_k w_k J_0(\sqrt{\lambda_k}\, r) \longrightarrow B_k = 0$$

THEN $$u(r,t) = 1 + \sum_{k=1}^{\infty} A_k \cos(w_k t) J_0(\sqrt{\lambda_k}\, r)$$

WHERE A_k IS AS GIVEN ABOVE.

EXAMPLE FIND AN EIGENFUNCTION REPRESENTATION FOR THE SOLUTION TO

$$u_t = D\left(u_{rr} + \frac{1}{r} u_r\right) + F \qquad 0 \le r \le a,\ t > 0$$

$$-D u_r = h(u - T_1) \text{ ON } r = a; \quad u \text{ FINITE AS } r \to 0$$

$$u(r,0) = T_2.$$

HERE D, h, T_1, T_2 ARE CONSTANTS.

WE FIRST CALCULATE THE STEADY-STATE SOLUTION WHICH SATISFIES

$$D\left(u_s'' + \frac{1}{r} u_s'\right) + F = 0$$

$$-D u_s' = h(u_s - T_1) \quad \text{ON } r = a$$

WE LET $u_s = \frac{F}{4D} r^2 + A_0$, WHICH SOLVES $u_s'' + \frac{1}{r} u_s' = -\frac{F}{D}$.

THEN TO FIND A_0 WE HAVE

$$-\left(\frac{F}{4D}\right) 2aD = h\left(\frac{F}{4D} a^2 + A_0 - T_1\right)$$

THIS YIELDS $$A_0 = T_1 - \frac{Fa}{2h} - \frac{Fa^2}{4D}.$$

THIS YIELDS THE STEADY-STATE SOLUTION:

$$u_s(r) = \frac{F}{4D}(r^2 - a^2) + T_1 - \frac{Fa}{2h}.$$

WE THEN WRITE $u(r,t) = v(r,t) + u_s(r)$.

THIS LEADS TO

$$v_t = D\left(v_{rr} + \frac{1}{r} v_r\right)$$

$$-Dv_r = hv \text{ ON } r = a, \quad v \text{ finite as } r \to 0$$

$$v(r,0) = T_2 - u_s(r).$$

SEPARATING VARIABLES WE OBTAIN $\dfrac{T'}{DT} = \dfrac{R'' + 1/r\, R'}{R} = -\lambda.$

THIS GIVES THE EIGENVALUE PROBLEM

$$\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0, \quad 0 < r < a$$

$$-D\phi'(a) = h\phi(a), \quad \phi(0) \text{ FINITE.}$$

WE OBTAIN $\phi_k(r) = J_0(\sqrt{\lambda_k}\, r)$ WHERE $\sqrt{\lambda_k}$ IS FOUND FROM THE ROOTS OF THE TRANSCENDENTAL RELATION:

$$-D\sqrt{\lambda_k}\, J_0'(\sqrt{\lambda_k}\, a) = h J_0(\sqrt{\lambda_k}\, a)$$

THE SOLUTION IS THEN $v(r,t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}\, r) e^{-\lambda_n D t}$

SATISFYING $v(r,0) = T_2 - u_s(r) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}\, r)$

WE OBTAIN THAT $$A_n = \frac{\int_0^a r\,[T_2 - u_s(r)]\, J_0(\sqrt{\lambda_n}\, r)\, dr}{\int_0^a r\,(J_0(\sqrt{\lambda_n}\, r))^2\, dr}.$$

HIGHER BESSEL FUNCTIONS

WE BEGIN WITH

$$u_t = D\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} \right) \quad \text{IN} \quad 0 \le \varphi \le 2\pi,\ 0 \le r \le a$$

$$u(a, \varphi, t) = 0, \quad u \text{ finite as } r \to 0$$

$$u(r, \varphi, 0) = f(r, \varphi)$$

WE SEPARATE VARIABLES $u(r,\varphi,t) = T(t)\, R(r)\, \Phi(\varphi)$.

THEN $$T' R \Phi = D T \left(\left(R'' + \frac{1}{r} R'\right)\Phi + \frac{1}{r^2} R \Phi'' \right)$$

THEN $$\frac{T'}{DT} = \frac{R'' + {}^1/_r R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = \text{CONSTANT.} = -\lambda$$

WE HAVE $$\frac{r^2\left(R'' + \frac{1}{r} R'\right)}{R} + \frac{\Phi''}{\Phi} = r^2 \,(\text{constant})$$

HENCE $\Phi''/\Phi = -\mu$ WITH Φ 2π PERIODIC.

WE HAVE $$\Phi'' + \mu \Phi = 0, \quad 0 < \varphi < 2\pi \qquad \longrightarrow \qquad \mu_n = n^2$$

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi) \qquad\qquad \Phi(\varphi) = A \cos n\varphi + B \sin n\varphi .$$

THEN WE GET $$\frac{R'' + \frac{1}{r} R'}{R} - \frac{\mu}{r^2} = -\lambda$$

THIS YIELDS THAT $$R'' + \frac{1}{r} R' + \left(\lambda - \mu/r^2\right) R = 0 .$$

THEN WE OBTAIN

$$r^2 R'' + r R' + (\lambda r^2 - \mu) R = 0 \qquad 0 \le r \le a$$

$$R(a) = 0, \quad R \text{ FINITE AS } r \to 0.$$

WITH $\mu = n^2$ THIS LEADS TO THE EIGENVALUE PROBLEM

$$r^2 \phi'' + r\phi' + (\lambda r^2 - n^2)\phi = 0$$

$$\phi(a) = 0, \quad \phi \text{ finite as } r \to 0.$$

IF WE LET $x = \sqrt{\lambda}\, r$ AND REPLACE $\phi(r) = y(\sqrt{\lambda}\, r)$ TO OBTAIN

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

$$y(a) = 0, \quad y(0) \text{ FINITE}$$

IF WE SUBSTITUTE $y = x^{\alpha}$ WE GET $\alpha(\alpha - 1) + \alpha - n^2 = 0$

AND SO $\alpha = \pm n$. THIS IMPLIES $y_1 \sim c_1 x^n$ AND $y_2 \sim c_2 x^{-n}$ AS $x \to 0$.

THE TWO SOLUTIONS ARE

$$y = A\, J_n(x) + B\, Y_n(x)$$

J_n, Y_n BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n.

$$J_n(x) \sim c\, x^n \text{ AS } x \to 0, \quad Y_n(x) \sim c/x^n \text{ AS } x \to 0.$$

$$Y_n(0) \text{ UNBOUNDED}, \quad J_n(0) = 0 \quad n > 0.$$

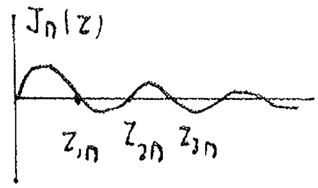
HENCE WE HAVE $\phi(r) = A\, J_n(\sqrt{\lambda}\, r)$ $\quad B = 0$ FOR BOUNDEDNESS

THEN WITH $\phi(a) = 0 \longrightarrow J_n(\sqrt{\lambda}\, a) = 0$

SO $\sqrt{\lambda_{mn}}\, a = z_{mn}$ WHERE $J_n(z_{mn}) = 0$ FOR

$m = 1, 2, 3, \ldots$ AND EACH n.

$$\longrightarrow \lambda_{mn} = z_{mn}^2 / a^2$$



THIS LEADS TO $u(r,\varphi,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_{mn} D t} J_n(\sqrt{\lambda_{mn}}\, r)\left(A_{mn} \cos n\varphi + B_{mn} \sin n\varphi\right)$

NOW WITH $u(r,\varphi,0) = f(r,\varphi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{mn}}\, r)\left[A_{mn} \cos n\varphi + B_{mn} \sin n\varphi\right]$

THEN

$$A_{mn} = \frac{\int_0^a \int_0^{2\pi} r\, f(r,\varphi) \cos(n\varphi)\, J_n(\sqrt{\lambda_{mn}}\, r)\, dr}{\int_0^a \int_0^{2\pi} \cos^2(n\varphi)\, r\, J_n^2(\sqrt{\lambda_{mn}}\, r)\, dr}$$

, SIMILAR FOR B_n

BY ORTHOGONALITY WE MUST HAVE

$$\int_0^a r\, J_n(\sqrt{\lambda_{mn}}\, r)\, J_n(\sqrt{\lambda_{jn}}\, r)\, dr = 0 \quad \text{FOR } m \neq j.$$

EXAMPLE SOLVE $u_t = D\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}\right)$ IN $0 \le r \le a$, $0 \le \varphi \le \alpha$

WITH $u(a, \varphi, t) = u(r, 0, t) = u(r, \alpha, t) = 0$, $u(r, \varphi, 0) = f(r, \varphi)$.

WE SEPARATE VARIABLES TO OBTAIN $u(r, \varphi, t) = T(t) R(r) \Phi(\varphi)$ TO GET

$$\frac{T'}{DT} = \frac{R'' + {}^1\!/_r R'}{R} + \frac{1}{r^2} \frac{\Phi''}{\Phi} = -\lambda.$$

$$\left.\begin{array}{l} \Phi'' + \mu \Phi = 0, \quad 0 < \varphi < \alpha \\ \Phi(0) = \Phi(\alpha) = 0 \end{array}\right\} \to \begin{array}{l} \Phi = \sin(m\pi\varphi/\alpha) \\ \mu_m = m^2\pi^2/a^2 \end{array}$$

THEN

$$\left.\begin{array}{l} R'' + \frac{1}{r} R' + (\lambda - \mu/r^2) R = 0 \\ R(a) = 0, \quad R(0) \text{ FINITE} \end{array}\right\} \to R(r) = J_{m\pi/a}(\sqrt{\lambda}\, r)$$

THEN $J_{m\pi/\alpha}(\sqrt{\lambda_{mn}}\, a) = 0$ SO $\sqrt{\lambda_{mn}}\, a = \sigma_{mn}$ AND $J_{m\pi/\alpha}(\sigma_{mn}) = 0$.

THEN $T(t) = e^{-D\lambda_{mn} t}$

THE SOLUTION IS

$$u(r, \varphi, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-D\lambda_{mn} t} \sin\left(\frac{m\pi\varphi}{\alpha}\right) J_{\frac{m\pi}{\alpha}}(\sqrt{\lambda_{mn}}\, r)$$

THEN WITH

$$u(r, \varphi, 0) = f(r, \varphi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi\varphi}{\alpha}\right) J_{\frac{m\pi}{\alpha}}(\sqrt{\lambda_{mn}}\, r)$$

BY ORTHOGONALITY:

$$A_{mn} = \frac{\int_0^\alpha \int_0^a r f(r, \varphi) \sin\left(\frac{m\pi\varphi}{\alpha}\right) J_{\frac{m\pi}{\alpha}}(\sqrt{\lambda_{mn}}\, r)\, dr}{\int_0^\alpha \int_0^a r \left(J_{\frac{m\pi}{\alpha}}(\sqrt{\lambda_{mn}}\, r)\right)^2 \sin^2\left(\frac{m\pi\varphi}{\alpha}\right) dr\, d\varphi}$$

EXAMPLE FIND THE SOLUTION TO THE HEAT EQUATION IN A CYLINDER. WE HAVE THAT $u(r,\varphi,t)$ SATISFIES

$$u_t = D\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} \right) \qquad 0 \le r \le a,\ \ 0 \le \varphi \le 2\pi,\ \ t>0$$

$$u(a,\varphi,t) = 0 \quad \text{ON} \quad r=a, \quad u \text{ BOUNDED AS } r\to 0.$$

$$u(r,\varphi,0) = \left(1-\frac{r}{a}\right)\cos\varphi,$$

WE LET $u(r,\varphi,t) = v(r,t)\cos\varphi$.

WE SUBSTITUTE INTO THE PDE TO OBTAIN:

$$v_t = D\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v \right) \qquad 0\le r\le a,\ \ t>0$$

$$v(a,t)=0, \qquad v(r,0) = \left(1 - r/a\right)$$

WE SEPARATE VARIABLES TO OBTAIN $\dfrac{T'}{DT} = \dfrac{R'' + \frac{1}{r}R' - \frac{1}{r^2}R}{R} = -\lambda.$

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\phi'' + \frac{1}{r}\phi' + \left(\lambda - 1/r^2\right)\phi = 0 \qquad 0\le r\le a$$

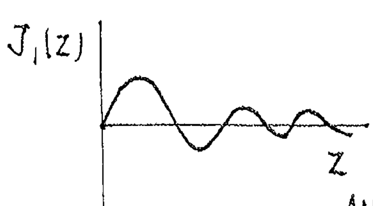
$$\phi(a) = 0, \quad \phi \text{ BOUNDED AS } r\to 0$$

WE WRITE THIS AS $(r\phi')' + (\lambda r - 1/r)\phi = 0 \rightarrow$ weight $w=r$.

THE SOLUTION is $\phi = AJ_1(\sqrt{\lambda}\,r) + BY_1(\sqrt{\lambda}\,r)$.

WE NEED $B=0$ FOR BOUNDEDNESS AND $\phi(a)=0$ YIELDS THAT $J_1(\sqrt{\lambda}\,a)=0$ SO $\sqrt{\lambda_k}\,a = z_k \rightarrow \lambda_k = z_k^2/a^2$

WHERE $J_1(z_k)=0$. WE OBTAIN THEN $T_k(t) = e^{-\lambda_k D t}$



BY SUPERPOSITION $v(r,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k D t} J_1(\sqrt{\lambda_k}\,r)$

WITH $v(r,0) = 1 - r/a$ WE GET $c_k = \dfrac{\int_0^a \left(1-\frac{r}{a}\right) r J_1(\sqrt{\lambda_k}\,r)\,dr}{\int_0^a r\left[J_0(\sqrt{\lambda_k}\,r)\right]^2 dr}$

AND $u(r,\varphi,t) = v(r,t)\cos\varphi$.

BESSEL'S EQUATION OF ORDER ν

WE WRITE $\quad x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad y = x^\alpha \rightarrow \alpha = \pm \nu$

THEN $\quad y = A J_\nu(x) + B Y_\nu(x) \qquad \nu > 0$ WLOG

$Y_\nu(x)$ SINGULAR AS $x \rightarrow 0$, $J_\nu(x)$ ANALYTIC AS $x \rightarrow 0$.

THUS $\quad x^2 \phi'' + x \phi' + (\lambda x^2 - \nu^2)\phi = 0 \qquad 0 < x < a$

$\phi(0)$ FINITE $\quad \phi(a) = 0$

THE SOLUTION IS $\quad \phi = J_\nu(\sqrt{\lambda}\, x)$

WITH $\quad J_\nu(\sqrt{\lambda}\, a) = 0$ SO $\sqrt{\lambda}\, a = z_k$, $k = 1, 2, ..$

WHICH GIVES $\quad \lambda_k = z_k^2 / a^2$, $k = 1, 2, ..$

THERE IS AN IDENTITY OF THE FORM

$$\int_0^a x \left(J_\nu(\sqrt{\lambda}\, x) \right)^2 dx = \frac{a^2}{2} \left[J_\nu'(\sqrt{\lambda}\, a) \right]^2$$

WHEN $\quad J_\nu(\sqrt{\lambda}\, a) = 0$. (SEE APPENDIX A)

APPENDIX A

LEMMA SUPPOSE THAT $\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0$ IN $0 < r < a$

$\phi(a) = 0$, ϕ, ϕ' BOUNDED AS $r \to 0$.

PROVE THAT $\int_0^a r\phi^2\,dr = \frac{a^2}{2\lambda}(\phi'(a))^2$.

PROOF $(r\phi')' + \lambda r\phi = 0$.

MULTIPLY BY $r\phi'$ $\quad (r\phi')(r\phi')' + \lambda r^2\phi\phi' = 0$.

INTEGRATE TO GET $\quad \frac{1}{2}\left[(r\phi')^2\right]\Big|_0^a + \lambda\int_0^a \frac{r^2}{2}(\phi^2)'\,dr = 0$.

NOW INTEGRATE BY PARTS:

$$\frac{1}{2}a^2(\phi'(a))^2 + \lambda\left[r^2\phi^2\Big|_0^a - \int_0^a r\phi^2\,dr\right] = 0.$$

BUT $\phi(a) = 0$ SO $\quad \int_0^a r\phi^2\,dr = \frac{1}{2\lambda}a^2(\phi'(a))^2$.

NOW IF $\phi = J_0(\sqrt{\lambda}\,r)$ AND $J_0(\sqrt{\lambda}\,a) = 0$ DETERMINES λ, THEN THE IDENTITY ABOVE GIVES

$$\int_0^a r\left(J_0(\sqrt{\lambda}\,r)\right)^2 dr = \frac{1}{2}a^2\left(J_0'(\sqrt{\lambda}\,a)\right)^2. \quad \blacksquare$$

REMARK THE SAME CALCULATION CAN BE DONE FOR

$$r^2\phi'' + r\phi' + (\lambda r^2 - \nu^2)\phi = 0 \qquad \phi(0) \text{ BOUNDED}, \quad \phi(a) = 0.$$

THE SOLUTION IS $\phi = J_\nu(r\sqrt{\lambda})$ WITH $J_\nu(a\sqrt{\lambda}) = 0$.

WE CLAIM THAT $\quad \int_0^a r\left(J_\nu(\sqrt{\lambda}\,r)\right)^2 dr = \frac{a^2}{2}\left(J_\nu'(\sqrt{\lambda}\,a)\right)^2$.

PROOF $\quad (r\phi')' + (\lambda r - \nu^2/r)\phi = 0$.

MULTIPLY BY $r\phi'$ AND INTEGRATE $\int_0^a$.

$$\frac{1}{2}\left[(r\phi')^2\right]\Big|_0^a + \lambda\int_0^a r^2 \frac{1}{2}\frac{d}{dr}(\phi^2)\,dr - \nu^2\frac{\phi^2}{2}\Big|_0^a = 0.$$

BUT FOR $\nu > 0$, $\phi(0) = 0$ AND FOR $\nu = 0$ THE LAST TERM VANISHES. SINCE $\phi(a) = 0$ WE GET

$$\frac{1}{2}a^2(\phi'(a))^2 + \frac{\lambda}{2}\int_0^a r^2(\phi^2)'\,dr = 0.$$

INTEGRATE BY PARTS AS BEFORE AND USE $\phi(a) = 0$ TO GET

$$\int_0^a r\phi^2\,dr = \frac{1}{2\lambda}a^2(\phi'(a))^2$$

SINCE $\phi(r) = J_\nu(r\sqrt{\lambda})$ THEN $\phi'(r) = \sqrt{\lambda}\,J_\nu'(r\sqrt{\lambda})$.

SO $$\int_0^a r\left(J_\nu(r\sqrt{\lambda})\right)^2 dr = \frac{a^2}{2}\left(J_\nu'(a\sqrt{\lambda})\right)^2.$$

Homework Assignment 6 (Due Date: April 8, 2014)

1. (30pts) Put the following two problems in Sturm-Liouville form, identify the weight function $w(x)$, and calculate the eigenvalues and eigenfunctions. Also what is the orthogonality relation for the eigenfunctions?

$$x^2\phi_{xx} + 5x\phi_x + \lambda\phi = 0, 1 \le x \le 2; \phi(1) = \phi(2) = 0$$

Hint: try $\phi(x) = x^r$

$$\phi_{xx} - 2\phi_x + \lambda\phi = 0, 0 \le x \le 1, \phi(0) = \phi(1) = 0$$

Hint: try $\phi(x) = e^{rx}$

2. (20pts) Use the method of separation variables to solve

$$\begin{cases} u_{tt} = u_{xx} + e^t \sin(3x), \quad 0 < x < \pi \\ u(x,0) = \sin(3x), u_t(x,0) = \sin(5x) \ 0 < x < \pi \\ u(0,t) = t, u(\pi,t) = 0 \end{cases} \tag{1}$$

3. (30pts) (a) (20pts) Use the method of separation variables to solve the following PDE:

$$u_{xx} + u_{yy} = 0 \quad \text{in } D = (0,\pi) \times (0,\pi)$$

$$u_y(x,0) = u(x,\pi) = 0, u(\pi,y) = 0$$

$$u(0,y) = \cos^2(y)$$

(b) (10pts) Prove that the solution obtained in (a) is unique.

4. (20pts) Use the method of separation of variables to solve the following PDE:

$$u_{xx} + u_{yy} = 1 \quad \text{in } D = \{(x,y)|x^2 + y^2 < 4\}$$

$$u(x,y) = x^2 - y^2 \quad \text{on } \partial D = \{(x,y)|x^2 + y^2 = 4\}$$

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