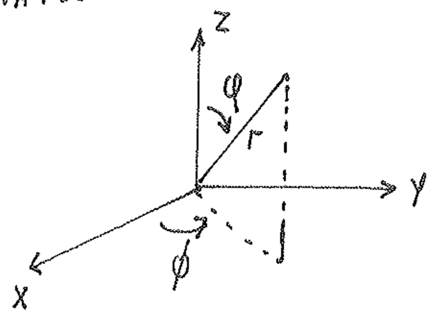
EXTENSIONS: LAPLACIAN IN SPHERICAL COORDINATES

INTRODUCE SPHERICAL COORDINATES

$$x = r\sin\varphi\cos\phi$$
$$y = r\sin\varphi\sin\phi$$
$$z = r\cos\varphi$$



$\varphi = 0, \pi$ poles

$\varphi = \pi/2$ equator

$$\varphi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$
$$\phi = \tan^{-1}(y/x)$$

IN TERMS OF THESE COORDINATES $\Delta u = 0$ BECOMES

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\, u_\varphi\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\phi^2}u = 0$$

FOR SIMPLICITY WE WILL CONSIDER THE AXI-SYMMETRIC CASE WHERE THERE IS AZIMUTHAL SYMMETRY SO THAT $u_\phi = 0$.

THEN
$$\frac{1}{r^2}\partial_r\left(r^2 u_r\right) + \frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\, u_\varphi\right) = 0, \quad 0<\varphi<\pi, \quad 0<r<a$$

WE IMPOSE THAT u HAS NO SINGULARITIES AT THE POLES $\varphi=0$ AND $\varphi=\pi$.

WE SEPARATE VARIABLES $u(r,\varphi) = R(r)\,\Phi(\varphi)$.

THEN
$$-\frac{r^2}{r^2}\frac{(r^2R')'}{R} = \frac{1}{\sin\varphi}\frac{1}{\Phi}\left(\sin\varphi\,\Phi'\right)' = \text{CONSTANT}.$$

WE WRITE THE CONSTANT AS $-\nu(\nu+1)$.

THEN
$$(1)\qquad \left(\sin\varphi\,\Phi'\right)' + \nu(\nu+1)\sin\varphi\,\Phi = 0, \quad 0<\varphi<\pi$$

$$(2)\qquad \left(r^2R'\right)' - \nu(\nu+1)R = 0$$

NOW EXAMINE THE EQUATION FOR Φ

WE LET $x = \cos\varphi$ AND REPLACE $y \mapsto \Phi$.

THEN $$\sin\varphi\, \Phi'' + \cos\varphi\, \Phi' + \nu(\nu+1)\sin\varphi\, \Phi = 0.$$

NOW $$\frac{d\Phi}{d\varphi} = \frac{dy}{dx}\frac{dx}{d\varphi} = -y'\sin\varphi$$

$$\frac{d^2\Phi}{d\varphi^2} = -y'\cos\varphi + y''\sin^2\varphi$$

SO $$\Phi'' + \frac{\cos\varphi}{\sin\varphi}\Phi' + \nu(\nu+1)\Phi = 0 \quad \text{BECOMES}$$

$$y''\sin^2\varphi - y'\cos\varphi + \frac{\cos\varphi}{\sin\varphi}(-y'\sin\varphi) + \nu(\nu+1)y = 0.$$

THIS YIELDS $$y''\sin^2\varphi - 2y'\cos\varphi + \nu(\nu+1)y = 0$$

BUT $\sin^2\varphi = 1 - x^2$. HENCE

$$(3) \quad \begin{cases} (1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0 & -1 < x < 1 \\ y \text{ BOUNDED AT } x = \pm 1 \end{cases}$$

REMARKS

(i) EQUATION (3) IS LEGENDRE'S EQUATION

(ii) $x = \pm 1$ ARE REGULAR SINGULAR POINTS.

(iii) WE WANT TO FIND THE EIGENVALUE PARAMETER ν SO THAT (3) HAS NON-TRIVIAL SOLUTIONS THAT ARE BOUNDED AT THE POLES $x = \pm 1$. NOTE THAT $x = \pm 1$ ARE $\varphi = 0, \pi$.

TO LOOK FOR SOLUTIONS OF (3) WE PUT

$$y = \sum_{=0}^{\infty} a_j x^j$$

AND DERIVE A RECURSION RELATION FOR a_j. THE UPSHOT OF THIS CALCULATION IS THE FOLLOWING:

(i) IF $\nu = n$ $n = 0, 1, 2, \ldots$ THEN (3) HAS SOLUTIONS OF THE FORM

$$y(x) = A P_n(x) + B Q_n(x)$$

WHERE $P_n(x)$ IS A POLYNOMIAL OF DEGREE n IN x

$Q_n(x)$ HAS SINGULARITIES AT $x = \pm 1$

(ii) THEREFORE THE EIGENVALUES AND EIGENFUNCTIONS OF (3) ARE

$$y_n = P_n(x), \quad \nu = n \qquad n = 0, 1, 2, \ldots$$

(iii) $P_n(x)$ ARE CALLED LEGENDRE POLYNOMIALS AND ARE DEFINED BY

$$P_n(x) = \sum_{r=0}^{m} (-1)^r \frac{(2n-2r)!\, x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!} \qquad m = \text{integer part of } \frac{n}{2}$$

IN ADDITION, $P_n(x)$ IS NORMALIZED BY

$P_n(1) = 1$. THERE IS ANOTHER WAY OF DEFINING THESE POLYNOMIALS. SEE THE APPENDIX A PAGE (A2) BELOW.

(iv) THE FIRST FEW SUCH POLYNOMIALS ARE

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}.$$

LEGENDRE POLYNOMIALS : SERIES EXPANSION

CONSIDER $(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$

WE PUT IN $y = \sum_{m=0}^{\infty} a_m x^m$ TO GET

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + \nu(\nu+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

THIS GIVES $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \nu(\nu+1)\sum_{m=0}^{\infty} a_m x^m = 0.$

$$\left.\begin{array}{ll} x^0 \text{ TERM:} & 2a_2 + \nu(\nu+1)a_0 = 0 \\ x^1 \text{ TERM} & 6a_3 - 2a_1 + \nu(\nu+1)a_1 = 0. \end{array}\right\} (+)$$

COEFFICIENT OF x^m FOR $m \geq 2$ $\quad (m+2)(m+1)a_{m+2} - m(m-1)a_m - 2m\,a_m$
$+ \nu(\nu+1)a_m = 0$

THUS $\quad (*) \quad a_{m+2} = \dfrac{[m^2+m-\nu(\nu+1)]}{(m+2)(m+1)} a_m. \quad m \geq 2.$

BUT NOTICE THAT $(*)$ ALSO WORKS IF $m=0,1$ IN COMPARING WITH $(+)$.

TO GENERATE TWO LINEARLY INDEPENDENT SOLUTIONS:

(i) LET $a_0 = 1,\ a_1 = 0$. THEN $a_3 = a_5 = \ldots = 0$. ALSO NOTICE THAT IF $\nu = n$ WITH n = EVEN THEN WE GET A POLYNOMIAL OF DEGREE n, i.e. $a_{n+2} = a_{n+4} = \ldots = 0$

eg: SET $n=2$, THEN $a_0 = 1$, $a_2 = -\dfrac{2(3)}{2(1)} a_0 = -3x^2 a_0$

THUS $y_1 = a_0(1-3x^2)$. IF $y_1(1) = 1$ IS IMPOSED $y_1 = \dfrac{(3x^2-1)}{2}$.

eg: SET $n=4$, THEN $a_0 = 1$, $a_2 = -\dfrac{20}{2} a_0$, $a_2 = -10a_0$,

$a_4 = \dfrac{6-20}{30} a_2 = -\dfrac{7}{15}(-10a_0) = \dfrac{70}{15} a_0.$

SO $y_1 = a_0 - 10a_0 x^2 + {}^{70}/_{15}\, x^4 a_0.$

(ii) IF $a_0 = 0$, $a_1 = 1$ THEN $a_2 = a_4 = a_6 = \dots = 0$.

IF $\nu = n$ AND n = ODD THEN WE GET A POLYNOMIAL OF DEGREE n

SINCE $a_{n+1} = a_{n+3} = \dots = 0$.

SET $n = 1$, THEN $a_1 = 1$ AND $a_3 = 0 \longrightarrow Y_2 = X$

$n = 3$, THEN $a_1 = 1$ AND $a_3 = \dfrac{2 - 12}{6} a_1 = \dfrac{-5 a_1}{3}$

SO $Y_2 = a_1 X - 5 a_1 X^3/3$.

THE $Q_n(x)$ ARE

$$Q_0(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), \quad Q_1(x) = \frac{1}{2}x\log\left(\frac{1+x}{1-x}\right) - 1, \ \ldots$$

NOW RETURNING TO LAPLACE'S EQUATION ON PAGE 1 WE WRITE THE EQUATION FOR R AS

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

WE PUT $R = r^{\beta}$ IN EULER'S EQUATION $\longrightarrow \beta(\beta-1) + 2\beta - n(n+1) = 0.$

$$\beta^2 + \beta - n(n+1) = (\beta + (n+1))(\beta - n) = 0$$

SO $\beta = n, \ -(n+1).$

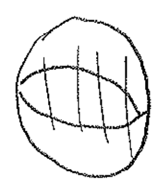
$$R_n(r) = A_n r^n + B_n r^{-(n+1)}.$$

THUS THE GENERAL SOLUTION TO LAPLACE'S EQUATION IN TERMS OF SPHERICAL COORDINATES, AND ASSUMING AZIMUTHAL SYMMETRY, IS

$$u(r,\varphi) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\varphi)$$

- IF THE DOMAIN IS THE SPHERE $0 < r < a$ WITH BOUNDARY CONDITION $u(a,\varphi) = f(\varphi)$, THEN $B_n = 0, \ \forall n$ AND

$$u(r,\varphi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\varphi)$$



WITH
$$f(\varphi) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\varphi)$$

IT REMAINS TO FIND THE A_n

- IF THE DOMAIN IS CONCENTRIC SPHERES $0 < a < r < b$ THEN

$$u(r,\varphi) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\varphi)$$

- IF THE DOMAIN IS <u>OUTSIDE</u> THE SPHERE $r \geq a$ WITH $u \to 0$ AS $r \to \infty$ THEN

$$u(r,\varphi) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\varphi).$$

PROPERTIES OF LEGENDRE POLYNOMIALS

WE CONSIDER

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad -1<x<1$$

$$y \text{ BOUNDED AT } x = \pm 1$$

THE SOLUTION IS $y_n = P_n(x)$. WE CAN WRITE THIS IN STURM-LIOUVILLE FORM AS

$$\left[(1-x^2)y' \right]' = -n(n+1)y \ , \quad -1<x<1$$

HENCE THE WEIGHT FUNCTION IS $w(x) = 1$ AND $p(x) = 1-x^2$.

CONSEQUENTLY, WE HAVE THE ORTHOGONALITY RELATION

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad \text{FOR } n \neq m.$$

IF WE LET $x = \cos\varphi$ THEN THIS BECOMES

$$\int_{0}^{\pi} P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi = 0 \quad \text{FOR } n \neq m.$$

BELOW WE WILL SHOW THAT $\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}$

(A) THUS WE HAVE THE ORTHOGONALITY RELATION

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\, \delta_{mn} \qquad \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

OR EQUIVALENTLY $\displaystyle\int_0^{\pi} P_n(\cos\varphi)\, P_m(\cos\varphi) \sin\varphi\, d\varphi = \frac{2}{2n+1}\, \delta_{mn}.$

(B) NEXT, AS FOR ALL STURM-LIOUVILLE PROBLEMS, WE HAVE A COMPLETENESS PROPERTY THAT SAYS THAT FOR ANY CONTINOUS FUNCTION $f(x)$ WITH $-1 < x < 1$ THAT

$$f(x) = \sum_{n=0}^{\infty} a_n\, P_n(x) \qquad \text{WITH MEAN SQUARE CONVERGENCE}$$

TO FIND THE COEFFICIENTS WE CALCULATE

$$\int_{-1}^{1} f(x)\, P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\, dx = a_m \frac{2}{2m+1}$$

SO $$a_m = \frac{(2m+1)}{2} \int_{-1}^{1} f(x)\, P_m(x)\, dx$$

THIS GIVES THE <u>FOURIER-LEGENDRE</u> SERIES

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{(2n+1)}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx \right) P_n(x)\, dx$$

(C) WE HAVE THAT THE FIRST FEW LEGENDRE POLYNOMIALS ARE

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x), \quad P_4(x) = \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + 3/8$$

$$P_0(\cos\varphi) = 1, \quad P_1(\cos\varphi) = \cos\varphi, \quad P_2(\cos\varphi) = \tfrac{1}{2}(3\cos^2\varphi - 1)$$

$$P_3(\cos\varphi) = \tfrac{1}{2}(5\cos^3\varphi - 3\cos\varphi).$$

$$P_4(\cos\varphi) = 35/8 \cos^4\varphi - 15/4 \cos^2\varphi + 3/8$$

(D) GENERATING FUNCTION

ONE OF THE MOST CONVENIENT WAYS TO DERIVE PROPERTIES OF LEGENDRE POLYNOMIALS IS TO INTRODUCE A GENERATING FUNCTION $G(x,t)$. THE FUNCTION IS DEFINED IN SUCH A WAY THAT THE COEFFICIENTS OF $G(x,t)$ IN THE TAYLOR SERIES ABOUT $t=0$ ARE $P_n(x)$.

$$\text{i.e.} \qquad G(x,t) = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

WE WILL NOW SHOW THAT

$$(*) \quad \left\} \; \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \qquad \text{FOR } 0 < t < 1. \right.$$

TO SHOW THIS WE WRITE

$$(1) \qquad (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} Z_n(x)\, t^n$$

USING THE BINOMIAL SERIES ON THE LHS FOR t SMALL IT FOLLOWS THAT $Z_n(x)$ MUST BE A POLYNOMIAL OF DEGREE n.

TO GET SOME INTUITION FOR THIS, WE LET t BE SMALL AND CALCULATE USING $(1+h)^{-1/2} \sim 1 - \frac{h}{2} + \frac{3}{8}h^2 + \dots$

WITH $h = t^2 - 2xt$

$$\begin{aligned}
[1+(t^2-2xt)]^{-1/2} &\sim 1 - \frac{1}{2}(t^2-2xt) + \frac{3}{8}(t^2-2xt)^2 + \dots \\
&\sim 1 + xt + t^2\left(\frac{3x^2}{2} - \frac{1}{2}\right) + \dots \\
&\sim P_0(x) + t\,P_1(x) + t^2\,P_2(x) + \dots
\end{aligned}$$

SO AT LEAST UP TO $n = 0, 1, 2$, $(*)$ IS CORRECT. CAN WE SHOW $(*)$ FOR ALL n?

TO DO SO WE SHOW FROM (1) THAT $Z_n(X)$ SATISFIES LEGENDRE'S DIFFERENTIAL EQUATION. THEN SINCE $Z_n(X)$ IS A POLYNOMIAL OF DEGREE n AND $Z_n(1) = 1$ FOR $n = 0, \ldots$ (NOTE: $(1-2t+t^2)^{-1/2} = (1-t)^{-1} = \sum_{n=0}^{\infty} Z_n(1)\, t^n \to Z_n(1)=1$) IT FOLLOWS THAT $Z_n(X) = P_n(X)$.

TO SHOW THAT $Z_n(X)$ SATISFIES LEGENDRE'S EQUATION WE DIFFERENTIATE (1) WRT X TO OBTAIN

$$(2) \qquad t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} Z_n'(x)\, t^n$$

DIFFERENTIATING AGAIN WE GET

$$(3) \qquad 3t^2(1-2xt+t^2)^{-5/2} = \sum_{n=0}^{\infty} Z_n''(x)\, t^n$$

NOW DIFFERENTIATE (1) WRT t TO GET

$$(4) \qquad (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} Z_n(x)\, t^{n-1} n$$

NOW MULTIPLY (4) BY t^2 AND DIFFERENTIATE WRT t

$$\frac{\partial}{\partial t}\left[t^2(x-t)(1-2xt+t^2)^{-3/2}\right] = \sum_{n=0}^{\infty} n(n+1)\, Z_n(x)\, t^n.$$

$$t^2\left[(x-t)\,3(x-t)(1-2xt+t^2)^{-5/2} - (1-2xt+t^2)^{-3/2}\right] + 2t(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n(n+1)\, Z_n(x)\, t^n.$$

THIS IS SIMPLIFIED TO

$$(5) \qquad (1-2xt+t^2)^{-3/2}\left[3t^2(x-t)^2(1-2xt+t^2)^{-1} - 1 + 2t(x-t)\right] = \sum_{n=0}^{\infty} n(n+1)\, Z_n(x)\, t^n$$

NOW COMBINE (2), (3) AND (5)

$$(1-x^2)\sum_{n=0}^{\infty} Z_n''(x)\,t^n - 2x\sum_{n=0}^{\infty} Z_n'(x)\,t^n + \sum_{n=0}^{\infty} n(n+1)\,Z_n(x)\,t^n$$

$$= 3t^2(1-x^2)(1-2xt+t^2)^{-5/2} - 2xt\,(1-2xt+t^2)^{-3/2}$$

$$+ (1-2xt+t^2)^{-3/2}\left[3t^2(x-t)^2(1-2xt+t^2)^{-1} - 1 + 2t(x-t)\right]$$

THE RHS IS IDENTICALLY ZERO AFTER SOME ALGEBRA, AND SO

$$\sum_{n=0}^{\infty}\left[(1-x^2)Z_n'' - 2xZ_n' + n(n+1)Z_n\right]t^n = 0$$

HENCE

$$\left.\begin{aligned}(1-x^2)Z_n'' - 2xZ_n' + n(n+1)Z_n &= 0\\ \text{WITH}\quad Z_n(1) &= 1\end{aligned}\right\}\longrightarrow Z_n(x) = P_n(x).$$

REMARK: IN THE APPENDIX A WE GIVE A DIFFERENT, EASIER, PROOF OF (*).

NOW THE GENERATING FUNCTION CAN BE USED FOR MANY RESULTS RELATING TO $P_n(x)$.

PROBLEM 1 SHOW FROM $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n$

THAT

$$(n+1)P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0 \qquad (*)$$

WITH $P_0(x) = 1$, $P_1(x) = x$. THIS RECURSION RELATION CAN BE READILY USED TO CALCULATE ALL THE $P_n(x)$.

i.e. FOR $P_2(x)$:

$$2P_2(x) = 3x\,P_1(x) - P_0(x)$$

$$\longrightarrow P_2(x) = \frac{3x^2}{2} - \frac{1}{2}.$$

TO SHOW (*) DIFFERENTIATE THE GEN. FUNCTION WRT t:

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n(x)\,n\,t^{n-1}$$

MULTIPLY BY $(1-2xt+t^2)$ TO GET

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n(x)\, n\, t^{n-1}$$

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(x)\, n\, t^{n-1} - 2x\sum_{n=0}^{\infty} P_n(x)\, n\, t^n + \sum_{n=0}^{\infty} P_n(x)\, n\, t^{n+1}$$

SHIFTING INDICES ETC, THIS CAN BE SHOWN TO YIELD (*)

__PROBLEM 2__ USE THE GENERATING FUNCTION TO PROVE THAT

$$\int_{-1}^{1} (P_n(x))^2\, dx = \frac{2}{2n+1}$$

__PROOF__ WE OUTLINE THE PROOF. THE DETAILS ARE IN HW.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$ NOW SQUARE BOTH SIDES AND COMBINE INTO A DOUBLE SUM.

SO

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x) P_m(x)\, t^n t^m$$

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty} \left(\int_{-1}^{1} (P_n(x))^2\, dx \right) t^{2n}$$ USING $\int_{-1}^{1} P_n P_m\, dx = 0 \quad n \neq m$

NOW SIMPLY integrate the LHS AND EXPAND THE RESULT IN A TAYLOR SERIES IN t. THIS YIELDS $\int_{-1}^{1} (P_n(x))^2\, dx = \frac{2}{2n+1}$.

PROBLEM 1 DETERMINE THE FOURIER-LEGENDRE SERIES OF $f(x) = x^2$ ON $-1<x<1$.

WE WRITE $$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

SO $$a_n = \frac{(2n+1)}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \qquad (*)$$

SINCE $f(x) = x^2$ THEN $a_n = 0$ FOR $n = 3, \ldots$ THIS IS BECAUSE ANY POLYNOMIAL OF DEGREE 2 CAN BE WRITTEN AS $x^2 = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$ FOR SOME c_0, c_1, c_2. HENCE $\int_{-1}^{1} (c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)) P_n(x)\, dx = 0$ $\forall n \geq 3$ BY ORTHOGONALITY $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$ FOR $m < n$.

THUS WE NEED ONLY CALCULATE a_0, a_1, a_2 IN $(*)$.

METHOD 1 $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

HENCE $2P_2 = 3x^2 - 1$ $\qquad x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$.

WE CLAIM $a_2 = 2/3$, $a_1 = 0$, $a_0 = 1/3$.

METHOD 2 WE CALCULATE DIRECTLY THAT

$$a_0 = \frac{1}{2} \int_{-1}^{1} x^2 P_0(x)\, dx = \frac{1}{2} \int_{-1}^{1} x^2\, dx = \frac{1}{3}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} \underbrace{x^2\, x}_{\text{ODD}}\, dx = 0$$

$$a_2 = \frac{5}{2} \int_{-1}^{1} x^2 \left(\frac{3x^2}{2} - \frac{1}{2} \right) dx = \frac{5}{2} \left(\frac{3x^5}{10} - \frac{x^3}{6} \right) \Big|_{-1}^{1} = \frac{2}{3}.$$

PROBLEM 2 SOLVE LAPLACE'S EQUATION OUTSIDE A SPHERE,

$$\Delta u = 0, \quad r \geq a, \quad 0 \leq \varphi \leq \pi$$

$$u_r(a, \varphi) = 0$$

WITH $u \sim r\cos\varphi$ AS $r \to \infty$.

THIS REPRESENTS THE VELOCITY POTENTIAL OF A FLUID THAT IS INCOMPRESSIBLE AND INVISCID

WE WRITE THE GENERAL SOLUTION AS

$$u(r, \varphi) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-1-n}) P_n(\cos\varphi)$$

NOW $A_1 = 1$, $A_n = 0$ FOR $n = 0, 2, \ldots$ $P_1(\cos\varphi) = \cos\varphi$.

HENCE $$u(r, \varphi) = \left(r + B_1/r^2\right)\cos\varphi$$

NOW $u_r = 0$ ON $r = a$ $\rightarrow$ $1 + \left(\frac{-2B}{a^3}\right) = 0$ $\quad B = +\frac{a^3}{2}$

THUS $$u(r, \varphi) = \left(r + a^3/2r^2\right)\cos\varphi.$$

PROBLEM 3 SOLVE LAPLACE'S EQUATION IN A SPHERE WITH AZIMUTHAL SYMMETRY

$$\Delta u = 0 \quad \text{IN} \quad 0 \leq r \leq a, \quad 0 \leq \varphi \leq \pi$$

$$u(a, \varphi) = f(\varphi) \quad \text{ON} \quad r = a.$$

WE WRITE $$u(r, \varphi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\varphi)$$

NOW $$f(\varphi) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\varphi)$$

ORTHOGONALITY GIVES $$A_n a^n \frac{2}{2n+1} = \int_0^{\pi} f(\varphi) P_n(\cos\varphi)\sin\varphi\, d\varphi$$

SO $$A_n = \frac{(2n+1)}{2} \frac{1}{a^n} \int_0^{\pi} f(\varphi) P_n(\cos\varphi)\sin\varphi\, d\varphi.$$

SUPPOSE NOW THAT $f(\varphi) = T_0 \sin^4\varphi$ WITH T_0 CONSTANT. WE WILL CALCULATE THE SOLUTION EXPLICITLY.

WE WRITE $$u(r,\varphi) = T_0 \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\varphi)$$

THEN $$\sin^4\varphi = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$$

BUT IF $X = \cos\varphi$ THEN $\sin^4\varphi = (1-\cos^2\varphi)(1-\cos^2\varphi) = (1-X^2)^2$

SO $$X^4 - 2X^2 + 1 = \sum_{n=0}^{\infty} A_n P_n(X).$$ WE MUST HAVE $A_n = 0,\ n \geq 5$

NOW RECALL $P_4(X) = \frac{35}{8} X^4 - \frac{15}{4} X^2 + \frac{3}{8}$, $P_2(X) = \frac{3X^2}{2} - \frac{1}{2}$

$$P_0(X) = 1.$$

SO $$X^4 = \frac{8}{35} P_4(X) - \frac{8}{35}\left(\frac{-15}{4} X^2 + \frac{3}{8}\right) = \frac{8}{35} P_4(X) + \frac{6X^2}{7} - \frac{3}{35}$$

$$X^4 - 2X^2 = \frac{8}{35} P_4(X) - \frac{8}{7} X^2 - \frac{3}{35}$$

$$= \frac{8}{35} P_4(X) - \frac{8}{7}\left(\frac{2}{3} P_2(X) + \frac{1}{3}\right) \frac{-3}{35}$$

$$X^4 - 2X^2 = \frac{8}{35} P_4(X) - \frac{16}{21} P_2(X) - \frac{8}{21} - \frac{3}{35}$$

$$X^4 - 2X^2 + 1 = \frac{8}{35} P_4(X) - \frac{16}{21} P_2(X) + \frac{8}{15}$$

HENCE $A_0 = \frac{8}{15}$, $A_2 = \frac{-16}{21}$, $A_4 = \frac{8}{35}$, $A_n = 0$ FOR $n \neq 0, 2, 4$

$$\longrightarrow u(r,\varphi) = T_0 \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\varphi)$$

<u>REMARK</u> THERE IS ANOTHER WAY TO FIND COEFFICIENTS FOR

$$x^4 - 2x^2 + 1 = \sum_{n=0}^{\infty} A_n P_n(x).$$

NOW THE LHS IS EVEN IN X AND IS DEGREE 4. HENCE SINCE P_1, P_3 ARE ODD WE NEED

$$x^4 - 2x^2 + 1 = A_0 P_0(x) + A_2 P_2(x) + A_4 P_4(x)$$

$P_0(x) = 1$

$P_2(x) = \frac{3x^2}{2} - \frac{1}{2}$

$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}.$

EQUATING COEFFICIENTS OF x^4, x^2 AND 1 GIVE

x^4: $\quad 1 = \frac{35}{8} A_4 \longrightarrow A_4 = 8/35$

x^2: $\quad -2 = -\frac{15}{4} A_4 + \frac{3}{2} A_2 \longrightarrow -2 = -\frac{15}{4}\left(\frac{8}{35}\right) + \frac{3}{2} A_2 \rightarrow \frac{3}{2} A_2 = -2 + \frac{6}{7} = -\frac{8}{7}$

1: $\quad 1 = A_0 - A_2/2 + 3A_4/8$

so $A_2 = -16/21$

THEN $\quad 1 = A_0 + \frac{8}{21} + \frac{3}{35} = A_0 + \frac{1}{7}\left(\frac{8}{3} + \frac{3}{5}\right) = A_0 + \frac{1}{7}\left(\frac{49}{15}\right) = A_0 + \frac{7}{15}.$

SO $A_0 = 1 - 7/15 = 8/15.$

WE CONCLUDE THAT $x^4 - 2x^2 + 1 = \frac{8}{15} P_0(x) - \frac{16}{21} P_2(x) + \frac{8}{35} P_4(x).$

<u>PROBLEM</u> PROVE THAT $\int_{-1}^{1} x^n P_m(x)\,dx = 0$ IF $m > n$. $\quad m, n$ integers > 0.

<u>PROOF</u> WE CAN WRITE $x^n = \sum_{j=0}^{n} A_j P_j(x)$ SINCE x^n IS A POLYNOMIAL OF DEGREE n. THEN COEFFICIENTS A_j CAN BE FOUND. SUBSTITUTE AND USE ORTHOGONALITY

$$\int_{-1}^{1} x^n P_m(x)\,dx = \int_{-1}^{1} \sum_{j=0}^{n} A_j P_j(x)\,P_m(x)\,dx = \sum_{j=0}^{n} A_j \int_{-1}^{1} P_j P_m\,dx = 0.$$

SINCE $j \neq m$.

THUS FOR ANY POLYNOMIAL $Q(x)$ OF DEGREE n WE HAVE

$$\int_{-1}^{1} Q(x) P_m(x)\,dx = 0 \quad \text{IF } m > n. \text{ (SIMILAR PROOF).}$$

REMARK IF WE USED

$$A_n = \frac{(2n+1)}{2} \int_{-1}^{1} (x^4 - 2x^2 + 1)\, P_n(x)\, dx$$

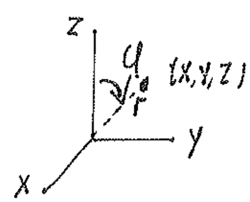
WE WOULD OBSERVE THAT $A_1 = A_3 = 0$ SINCE P_1, P_3 ARE ODD FUNCTIONS WHILE $(x^4 - 2x^2 + 1)$ IS EVEN. BUT CALCULATING THESE INTEGRALS FOR A_0, A_2, A_4 IS TEDIOUS BY HAND. THE OTHER METHOD IS BETTER.

PROBLEM 4 SHOW THAT THE ELECTROSTATIC POTENTIAL FOR A POINT CHARGE AT $\underline{x}_0 = (0, 0, a)$ ON Z-AXIS CAN BE EXPANDED IN TERMS OF LEGENDRE POLYNOMIALS AS

$$\frac{1}{|\underline{x} - \underline{x}_0|} = \frac{1}{a} \sum_{n=0}^{\infty} P_n(\cos\varphi) \left(\frac{r}{a}\right)^n \quad \text{WITH } r = |\underline{x}|, \text{ WHEN } r < a.$$

DERIVATION WE WRITE $|\underline{x} - \underline{x}_0|^2 = x^2 + y^2 + (z-a)^2 = x^2 + y^2 + z^2 - 2az + a^2 = r^2 + a^2 - 2az$

THEN PUT $z = r\cos\varphi$ WITH φ = LATITUDE



HENCE $|\underline{x} - \underline{x}_0|^2 = r^2 + a^2 - 2ar\cos\varphi$

SO
$$\frac{1}{|\underline{x} - \underline{x}_0|} = \frac{1}{a} \frac{1}{\sqrt{r^2/a^2 - 2\frac{r}{a}\cos\varphi + 1}}$$

NOW RECALL THE GENERATING FUNCTION $\displaystyle \frac{1}{\sqrt{t^2 - 2xt + 1}} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$

HENCE, WE LET $t = r/a$, $x = \cos\varphi$.

THIS GIVES
$$\frac{1}{|\underline{x} - \underline{x}_0|} = \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n P_n(\cos\varphi)$$

WHICH CONVERGES WHEN $0 < r < a$.

DIFFUSION ON A SPHERE

NEXT WE CONSIDER DIFFUSION ON THE SURFACE OF A SPHERE MODELED BY

$$u_t = \frac{1}{a^2 \sin\varphi} \partial_\varphi \left(\sin\varphi \, \partial_\varphi u \right), \quad 0 < \varphi < \pi, \quad t > 0$$

$$u(\varphi, 0) = f(\varphi)$$

HERE a IS THE RADIUS OF THE SPHERE AND WE HAVE ASSUMED AZIMUTHAL SYMMETRY.

WE SEPARATE VARIABLES TO OBTAIN $u = T(t)\,\Phi(\varphi)$

$$a^2 \frac{T'}{T} = \frac{1}{\sin\varphi} \frac{\left(\sin\varphi \; \Phi' \right)'}{\Phi} = -\nu(\nu+1)$$

HENCE

$$(\sin\varphi \; \Phi')' + \nu(\nu+1)\sin\varphi \; \Phi = 0, \quad 0 < \varphi < \pi$$

Φ BOUNDED AT $\varphi = 0, \pi$

$$T' = -\frac{\nu(\nu+1)}{a^2} T$$

THEN

$$\Phi_n(\varphi) = P_n(\cos\varphi) \qquad n = 0, 1, 2, \ldots$$

$$T = \exp\left(-\frac{n(n+1)}{a^2} t \right)$$

HENCE WE OBTAIN THE SEPARATION OF VARIABLES SOLUTION

$$u(\varphi, t) = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi) \exp\left[-\frac{n(n+1)}{a^2} t \right]$$

IF

$$u(\varphi, 0) = f(\varphi) = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$$

THEN

$$A_n = \frac{(2n+1)}{2} \int_0^{\pi} f(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi, \quad n = 0, 1, \ldots$$

NOTICE THAT $A_0 = \frac{1}{2}\int_0^{\pi} f(\varphi)\sin\varphi \, d\varphi = \lim_{t\to\infty} u(\varphi, t)$.

NOW SUPPOSE $f(\varphi) = 2\cos^2\varphi - 1$.

WE WRITE $$2x^2 - 1 = \sum_{n=0}^{\infty} A_n P_n(x)$$

HENCE $A_n = 0 \;\; \forall\, n \geq 3$. WE HAVE $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

HENCE $$2x^2 = \frac{4}{3}\left(P_2(x) + \frac{1}{2}\right) = \frac{4P_2(x)}{3} + \frac{2}{3}$$

SO $$2x^2 - 1 = \frac{4}{3} P_2(x) - \frac{1}{3} = \frac{4}{3} P_2(x) - \frac{1}{3} P_0(x).$$

THIS YIELDS THAT $u(\varphi, t)$ IS GIVEN EXPLICITLY BY
$$u(\varphi, t) = -\frac{1}{3} + \frac{4}{3} P_2(\cos\varphi) \exp\left(-\tfrac{6}{a^2} t\right)$$

<u>REMARK</u> IF WE CHANGED THIS PROBLEM TO
$$u_t = \frac{1}{a^2 \sin\varphi}\left(\sin\varphi \, u_\varphi\right)_\varphi \qquad \varphi_c < \varphi < \pi \qquad \varphi_c > 0$$

THEN $$\frac{1}{\sin\varphi}\left(\sin\varphi \, \Phi'\right)' + \nu(\nu+1)\Phi = 0, \qquad \varphi_c < \varphi < \pi.$$

THE SOLUTION THAT HAS NO SINGULARITY AT $\varphi = \pi$ IS THE LEGENDRE FUNCTION $P_\nu(\cos\varphi)$. (ν NOT necessarily AN integer)

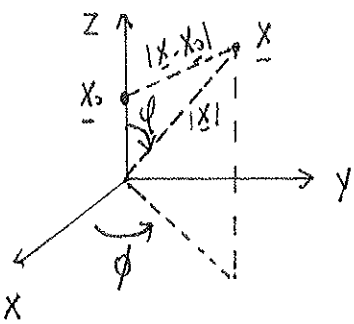
THE CONDITION $\Phi(\varphi_c) = 0$ WOULD GIVE THE EIGENVALUE RELATION $P_\nu(\cos\varphi_c) = 0$ WHICH IS AN IMPLICIT EQUATION FOR $\nu = \nu_n$, $n = 1, 2, \ldots$

APPENDIX A

DERIVATION OF GENERATING FUNCTION

LET $\underset{\sim}{X}_0$ BE SOME FIXED POINT IN $\mathbb{R}^3$. THEN $1/|\underset{\sim}{X}-\underset{\sim}{X}_0|$ IS A HARMONIC FUNCTION IN $\mathbb{R}^3$ SO THAT $\Delta\left(\frac{1}{|\underset{\sim}{X}-\underset{\sim}{X}_0|}\right)=0$. THIS FOLLOWS SINCE IF WE LET $\rho=|\underset{\sim}{X}-\underset{\sim}{X}_0|$ THEN $\Delta(1/\rho)\equiv(1/\rho)''+\frac{2}{\rho}(1/\rho)'=0$.

NOW WITHOUT LOSS OF GENERALITY LET $\underset{\sim}{X}_0$ BE ALIGNED WITH POSITIVE Z-AXIS. WE TAKE $\underset{\sim}{X}_0$ TO BE UNIT VECTOR. WE GET THE PICTURE SHOWN



$$\text{NOW } |\underset{\sim}{X}-\underset{\sim}{X}_0|^2 = (\underset{\sim}{X}-\underset{\sim}{X}_0)^T\cdot(\underset{\sim}{X}-\underset{\sim}{X}_0)$$
$$= \underset{\sim}{X}^T\underset{\sim}{X} - 2\underset{\sim}{X}^T\underset{\sim}{X}_0 + \underset{\sim}{X}_0^T\underset{\sim}{X}_0$$
$$= r^2 - 2r\cos\varphi + 1$$

(NOTE: $\underset{\sim}{X}^T\underset{\sim}{X}=|\underset{\sim}{X}|^2=r^2$, $\underset{\sim}{X}^T\underset{\sim}{X}_0=\underset{\sim}{X}\cdot\underset{\sim}{X}_0=|\underset{\sim}{X}||\underset{\sim}{X}_0|\cos\varphi = r\cos\varphi$, $\underset{\sim}{X}_0^T\underset{\sim}{X}_0=|\underset{\sim}{X}_0|^2=1$

THUS $\frac{1}{|\underset{\sim}{X}-\underset{\sim}{X}_0|}=\frac{1}{(r^2-2r\cos\varphi+1)^{1/2}}$ IS A SOLUTION TO LAPLACE'S EQUATION IN 3-D WITH AZIMUTHAL SYMMETRY. (INDEPENDENT OF ANGLE ϕ).

HENCE FOR $r\neq 1$, $\varphi\neq 0$, WE MUST HAVE FOR SOME A_n THAT

$$\frac{1}{(r^2-2r\cos\varphi+1)^{1/2}}=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\varphi)$$

FOR SOME CONSTANTS A_n (INDEPENDENT OF φ AND r). TO FIND A_n SET $\varphi=0$ AND TAKE $r<1$. THEN $(r^2-2r\cos\varphi+1)^{-1/2}=(r^2-2r+1)^{-1/2}=(1-r)^{-1}$.

HENCE $\frac{1}{1-r}=\sum_{n=0}^{\infty}A_n P_n(1)r^n$. BUT $P_n(1)=1 \rightarrow \frac{1}{1-r}=\sum_{n=0}^{\infty}A_n r^n \rightarrow A_n=1,\ \forall n$

THUS WE HAVE

$$\frac{1}{(r^2-2r\cos\varphi+1)^{1/2}} = \sum_{n=0}^{\infty} r^n P_n(\cos\varphi).$$

LET $x=\cos\varphi$, $t=r$, WITH $0<t<1$.

HENCE $$\frac{1}{(t^2-2tx+1)^{1/2}} = \sum_{n=0}^{\infty} t^n P_n(x). \qquad \square.$$

ANOTHER WAY TO DEFINE THE LEGENDRE POLYNOMIALS

WE CLAIM THAT $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[x^2-1\right]^n$. (RODRIGUEZ'S FORMULA).

CLEARLY $P_n(x)$ IS A POLYNOMIAL OF DEGREE n AND BY WRITING $(x^2-1)=(x-1)(x+1)$ WE HAVE $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\Big((x-1)^n(x+1)^n\Big)$

UPON DIFFERENTIATING n TIMES AND EVALUATING AT $x=1$ WE GET ONE NON-ZERO TERM GIVEN BY $\left[\dfrac{d^n}{dx^n}(x-1)^n\right](x+1)^n\Big|_{x=1} = n!\,2^n$. THUS $P_n(1)=1$.

NOW WE MUST SHOW THAT $P_n(x)$ SATISFIES

$$(*) \qquad (1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

DEFINE $h(x)=(1-x^2)^n$. WE NEED ONLY SHOW THAT $y\equiv\dfrac{d^n}{dx^n}(1-x^2)^n$ SATISFIES $(*)$ AND THEN WE ARE DONE.

WE CALCULATE $h'(x) = -2nx(1-x^2)^{n-1}$

SO $$(1-x^2)h' + 2nx(1-x^2)^n = 0 \longrightarrow (1-x^2)h' + 2nxh = 0. \quad (+)$$

NOW LET A, B BE ANY TWO FUNCTION OF X. BY REPEATED APPLICATION OF CHAIN RULE

$$(AB)'' = A''B + 2A'B' + AB''$$

$$(AB)''' = A'''B + 3A''B' + 3A'B'' + AB'''$$

$$(AB)^{(m)} = A^{(m)}B + mA^{(m-1)}B' + \binom{m}{2} A^{(m-2)}B'' + \dots + mAB^{(m-1)} + AB^{(m)}.$$

(BINOMIAL COEFFICIENTS)

NOW DIFFERENTIATE $(+)$ $n+1$ TIMES

$$[(1-x^2)h']^{(n+1)} + 2n[xh]^{(n+1)} = 0$$

SET $A = h'$, $B = (1-x^2)$, $m = n+1$ IN FIRST TERM

$A = h$, $B = x$, $m = n+1$ IN SECOND TERM.

SO

$$(1-x^2)h^{(n+2)} + (n+1)h^{(n+1)}(-2x) + \binom{n+1}{2} h^{(n)}(-2)$$
$$+ 2n\left[x h^{(n+1)} + (n+1)h^{(n)}\right] = 0.$$

THUS

$$(1-x^2)h^{(n+2)} + h^{(n+1)}\left[-2x(n+1) + 2nx\right] + h^{(n)}\left[2n(n+1) - 2\binom{n+1}{2}\right] = 0.$$

BUT $\binom{n+1}{2} = \dfrac{(n+1)!}{(n-1)!\,2!} = \dfrac{n(n+1)}{2}$.

THIS GIVES $2n(n+1) - 2\dfrac{n(n+1)}{2} = n(n+1)$. WE CONCLUDE THAT

$h^{(n)} = \dfrac{d^n}{dx^n}[(1-x^2)]$ SATISFIES $(1-x^2)(h^{(n)})'' - 2x(h^{(n)})' + n(n+1)h^{(n)} = 0.$

THUS $v = h^{(n)}$ SATISFIES $(*)$ ON PREVIOUS PAGE (LEGENDRE'S DIFFERENTIAL EQUATION).

$$\longrightarrow \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$