One Example of fully nonlinear first order PDE

Example: Solve
$$u_x^2 - u_y^2 = 1$$
with $u(x,y) = x^2$ on $y = x$

Solution: paramotrize the initial data curve as

(ξ, ξ), $u_0(\xi) = \xi^2$

Hence
$$\begin{cases} p_0^2 - q_0^2 = 1 \\ 2\xi = p_0 + q_0 \end{cases} \Rightarrow \begin{array}{l} (p_0 - q_0)(p_0 + q_0) = 1 \\ p_0 - q_0 = \frac{1}{2\xi} \end{array}$$

$$p_0 = \xi + \frac{1}{4\xi}, \quad q_0 = \xi - \frac{1}{4\xi}$$

Method of Characteristics:

$$F = p^2 - q^2 = 1$$
$$F_p = 2p, \quad F_q = -2q, \quad F_x = 0, \quad F_y = 0, \quad F_u = 0$$

$$\begin{cases} \frac{dx}{ds} = 2p, \quad x(0) = \xi \\ \frac{dy}{ds} = -2q, \quad y(0) = \xi \\ \frac{dp}{ds} = 0, \quad p(0) = \xi + \frac{1}{4\xi} \quad \Rightarrow p = \xi + \frac{1}{4\xi} \\ \frac{dq}{ds} = 0, \quad q(0) = \xi - \frac{1}{4\xi} \quad \Rightarrow q = \xi - \frac{1}{4\xi} \\ \frac{du}{ds} = p(2p) - q(2q) = 2(p^2 - q^2) = 2, \quad u(0) = \xi^2 \end{cases}$$

$$\left.\begin{cases} x = 2(\xi + \frac{1}{4\xi})s + \xi \\ y = -2(\xi - \frac{1}{4\xi})s + \xi \\ u = 2s + \xi^2 \end{cases}\right\} \Rightarrow \begin{cases} 8\xi^3 - 4\xi^2(x+y) + x - y = 0 \\ s = -2\xi^2 + \xi(x+y) \\ u = 2s + \xi^2 \end{cases}$$

The solution is implicit.