EXERCISES

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = \log(1 + x^2)$, $u_t(x, 0) = 4 + x$.
3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?
4. Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x + ct) + g(x - ct)$.
5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\{\text{length of } (x - ct, x + ct) \cap (-a, a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2, x + a/2) \cap (-a, a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]
6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.
7. If both ϕ and ψ are odd functions of x, show that the solution $u(x, t)$ of the wave equation is also odd in x for all t.
8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right) \quad \text{(“spherical wave equation”).}$$

   (a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
   (b) Solve for v using (3) and thereby solve the spherical wave equation.
   (c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.
9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x, 0) = x^2$, $u_t(x, 0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

is the cornerstone of the theory of relativity. It means that a signal located at the position x_0 at the instant t_0 cannot move faster than the speed of light. The domain of influence of (x_0, t_0) consists of all the points that can be reached by a signal of speed c starting from the point x_0 at the time t_0. It turns out that the solutions of the *three*-dimensional wave equation always travel at speeds exactly equal to c and never slower. Therefore, the causality principle is sharper in three dimensions than in one. This sharp form is called *Huygens's principle* (see Chapter 9).

Flatland is an imaginary two-dimensional world. You can think of yourself as a waterbug confined to the surface of a pond. You wouldn't want to live there because Huygens's principle is not valid in two dimensions (see Section 9.2). Each sound you make would automatically mix with the "echoes" of your previous sounds. And each view would be mixed fuzzily with the previous views. Three is the best of all possible dimensions.

EXERCISES

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)
2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.
3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.
4. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity
$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$
   for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.
5. For the *damped* string, equation (1.3.3), show that the energy decreases.
6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE
$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right),$$
   where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r)f(t - \beta(r))$, where $\alpha(r)$ is called the

attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

2.3 THE DIFFUSION EQUATION

In this section we begin a study of the one-dimensional diffusion equation

$$u_t = ku_{xx}. \tag{1}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations. Because (1) is harder to solve than the wave equation, we begin this section with a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we'll deduce the uniqueness of an initial-boundary problem. We postpone until the next section the derivation of the solution formula for (1) on the whole real line.

Maximum Principle. If $u(x, t)$ satisfies the diffusion equation in a rectangle (say, $0 \leq x \leq l$, $0 \leq t \leq T$) in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$) (see Figure 1).

In fact, there is a *stronger version* of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but *only on the bottom or the lateral sides* (unless u is a constant). The corners are allowed.

The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle, just apply the maximum principle to $[-u(x, t)]$.

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source, the hottest spot and the

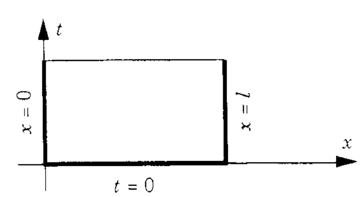

Figure 1

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x, t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x, 0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod. □

EXERCISES

1. Solve the diffusion equation with the initial condition

   $$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

   Write your answer in terms of $\mathscr{E}\text{rf}(x)$.
2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.
3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation (1).
6. Compute $\int_0^\infty e^{-x^2} dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^\infty e^{-p^2} dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

   $$\int_{-\infty}^{\infty} S(x, t)\, dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

   $$\max_{\delta \le |x| < \infty} S(x, t) \to 0 \qquad \text{as } t \to 0.$$

This means that the tail of $S(x, t)$ is "uniformly small".]

Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

(a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x, t)$ as a certain integral. Substitute $p = (x - y)/\sqrt{4kt}$ in this integral.

(b) Since the solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$

(a) Consider the diffusion equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)

(b) Show that the same is true if "odd" is replaced by "even."

(c) Show that the analogous statements are true for the wave equation.

The purpose of this exercise is to calculate $Q(x, t)$ approximately for large t. Recall that $Q(x, t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.

(a) Express $Q(x, t)$ in terms of $\mathscr{E}\text{rf}$.

(b) Find the Taylor series of $\mathscr{E}\text{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)

(c) Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x, t)$.

(d) *Why* is this formula a good approximation for x fixed and t large?

Prove from first principles that $Q(x, t)$ *must* have the form (4), as follows.

(a) Assuming uniqueness show that $Q(x, t) = Q(\sqrt{a}\, x,\ at)$. This identity is valid for all $a > 0$, all $t > 0$, and all x.

(b) Choose $a = 1/(4kt)$.

Let $\phi(x)$ be a continuous function such that $|\phi(x)| \leq Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x, t) \quad \text{for } 0 < x < l, t > 0 \quad u(x, 0) = \phi(x)$$
$$u_x(0, t) = g(t) \quad u_x(l, t) = h(t)$$

by the energy method.

16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x,t) = e^{-bt}v(x,t)$.)

17. Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2u = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* The solutions of the ODE $w_t + bt^2w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x,t) = e^{-bt^3/3}v(x,t)$ and derive an equation for v.)

18. Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where V is a constant. (*Hint:* Go to a moving frame of reference by substituting $y = x - Vt$.)

19. (a) Show that $S_2(x,y,t) = S(x,t)S(y,t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.
(b) Deduce that $S_2(x,y,t)$ is the source function for two-dimensional diffusions.

2.5 COMPARISON OF WAVES AND DIFFUSIONS

We have seen that the basic property of waves is that information gets transported in both directions at a finite speed. The basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually disappears. The fundamental properties of these two equations can be summarized in the following table.

Property	Waves	Diffusions
(i) Speed of propagation?	Finite ($\le c$)	Infinite
(ii) Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii) Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv) Well-posed for $t < 0$?	Yes	No
(v) Maximum principle	No	Yes
(vi) Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii) Information	Transported	Lost gradually

By the same reasoning as we used above, we end up with an explicit formula for $w(x, t)$. It is

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\, \phi(y)\, dy. \qquad (9)$$

This is carried out in Exercise 3. Notice that the only difference between (6) and (9) is a single minus sign!

Example 2.

Solve (7) with $\phi(x) = 1$. This is the same as Example 1 except for the single sign. So we can copy from that example:

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] + \left[\frac{1}{2} - \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] = 1.$$

(That was stupid: We could have guessed it!) □

EXERCISES

1. Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.
2. Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.
3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty$, $0 < t < \infty$; $w_x(0,t) = 0$; $w(x,0) = \phi(x)$.
4. Consider the following problem with a Robin boundary condition:

$$\begin{array}{lll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \\ & & \text{(and } 0 < t < \infty) \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC:} & u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0. \end{array} \qquad (*)$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?
(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?
(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).
(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(c) Deduce that $v(x, t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

5. (a) Use the method of Exercise 4 to solve the Robin problem:

DE:	$u_t = ku_{xx}$	on the half-line $0 < x < \infty$ (and $0 < t < \infty$)
IC:	$u(x,0) = x$	for $t = 0$ and $0 < x < \infty$
BC:	$u_x(0,t) - hu(0,t) = 0$	for $x = 0$,

where h is a constant.

(b) Generalize the method to the case of general initial data $\phi(x)$.

3.2 REFLECTIONS OF WAVES

Now we try the same kind of problem for the wave equation as we did in Section 3.1 for the diffusion equation. We again begin with the *Dirichlet problem* on the half-line $(0, \infty)$. Thus the problem is

DE:	$v_{tt} - c^2 v_{xx} = 0$	for $0 < x < \infty$ and $-\infty < t < \infty$	
IC:	$v(x,0) = \phi(x)$, $\quad v_t(x,0) = \psi(x)$	for $t = 0$ and $0 < x < \infty$	(1)
BC:	$v(0,t) = 0$	for $x = 0$ and $-\infty < t < \infty$.	

The reflection method is carried out in the same way as in Section 3.1. Consider the odd extensions of both of the initial functions to the whole line, $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$. Let $u(x, t)$ be the solution of the initial-value problem on $(-\infty, \infty)$ with the initial data ϕ_{odd} and ψ_{odd}. Then $u(x, t)$ is once again an odd function of x (see Exercise 2.1.7). Therefore, $u(0, t) = 0$, so that the boundary condition is satisfied automatically. Define $v(x, t) = u(x, t)$ for $0 < x < \infty$ [the restriction of u to the half-line]. Then $v(x, t)$ is precisely the solution we are looking for. From the formula in Section 2.1, we have for $x \geq 0$,

$$v(x,\ t) = u(x,\ t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\, dy.$$

Let's "unwind" this formula, recalling the meaning of the odd extensions. First we notice that for $x > c|t|$ only positive arguments occur in the formula, The solution formula at any other point (x, t) is characterized by the number of reflections at each end ($x = 0, l$). This divides the space-time picture into diamond-shaped regions as illustrated in Figure 6. *Within each diamond the solution $v(x, t)$ is given by a different formula.* Further examples may be found in the exercises.

The formulas explain in detail how the solution looks. However, the method is impossible to generalize to two- or three-dimensional problems, nor does it work for the diffusion equation at all. Also, it is very complicated! Therefore, in Chapter 4 we shall introduce a completely different method (Fourier's) for solving problems on a finite interval.

EXERCISES

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.
2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.
3. A wave $f(x + ct)$ travels along a semi-infinite string ($0 < x < \infty$) for $t < 0$. Find the vibrations $u(x, t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.
4. Repeat Exercise 3 if the end is free.
5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0, t) = 0$, $u(x, 0) \equiv 1$, $u_t(x, 0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.
6. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $0 \le t < \infty$, $u(x, 0) = 0$, $u_t(x, 0) = V$,
$$u_t(0, t) + a u_x(0, t) = 0,$$
where V, a, and c are positive constants and $a > c$.
7. (a) Show that $\phi_{\text{odd}}(x) = (\text{sign}\, x)\phi(|x|)$.
   (b) Show that $\phi_{\text{ext}}(x) = \phi_{\text{odd}}(x - 2l[x/2l])$, where $[\cdot]$ denotes the greatest integer function.
   (c) Show that
$$\phi_{\text{ext}}(x) = \begin{cases} \phi\left(x - \left[\frac{x}{l}\right] l\right) & \text{if } \left[\frac{x}{l}\right] \text{ even} \\ -\phi\left(-x - \left[\frac{x}{l}\right] l - l\right) & \text{if } \left[\frac{x}{l}\right] \text{ odd.} \end{cases}$$
8. For the wave equation in a finite interval $(0, l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

(a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1$, $u(x, 0) = x^2(1 - x)$, $u_t(x, 0) = (1 - x)^2$, $u(0, t) = u(1, t) = 0$.

(b) Find $u(\frac{1}{4}, \frac{7}{2})$.

Solve $u_{tt} = 9u_{xx}$ in $0 < x < \pi/2$, $u(x, 0) = \cos x$, $u_t(x, 0) = 0$, $u_x(0, t) = 0$, $u(\pi/2, t) = 0$.

Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < l$, $u(x, 0) = 0$, $u_t(x, 0) = x$, $u(0, t) = u(l, t) = 0$.

3.3 DIFFUSION WITH A SOURCE

In this section we solve the *inhomogeneous* diffusion equation on the whole line,

$$\begin{aligned} &u_t - ku_{xx} = f(x, t) \qquad (-\infty < x < \infty, \quad 0 < t < \infty) \\ &u(x, 0) = \phi(x) \end{aligned} \tag{1}$$

with $f(x, t)$ and $\phi(x)$ arbitrary given functions. For instance, if $u(x, t)$ represents the temperature of a rod, then $\phi(x)$ is the initial temperature distribution and $f(x, t)$ is a source (or sink) of heat provided to the rod at later times.

We will show that the solution of (1) is

$$\begin{aligned} u(x, t) = &\int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \\ &+ \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s)\, dy\, ds. \end{aligned} \tag{2}$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function S.

Let's begin by explaining where (2) comes from. Later we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) Our explanation is by analogy. The simplest analogy is the ODE

$$\frac{du}{dt} + Au(t) = f(t), \qquad u(0) = \phi, \tag{3}$$

where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)A} f(s)\, ds. \tag{4}$$

A more elaborate analogy is the following. Let's suppose that ϕ is an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $n \times n$ matrix.

SOURCE ON A HALF-LINE

For inhomogeneous diffusion on the half-line we can use the method of reflection just as in Section 3.1 (see Exercise 1).

Now consider the more complicated problem of a *boundary source* $h(t)$ on the half-line; that is,

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ \boldsymbol{v(0,t)} &= \boldsymbol{h(t)} \\ v(x,0) &= \phi(x). \end{aligned} \tag{9}$$

We may use the following subtraction device to reduce (9) to a simpler problem. Let $V(x,t) = v(x,t) - h(t)$. Then $V(x,t)$ will satisfy

$$\begin{aligned} V_t - kV_{xx} &= f(x,t) - h'(t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ V(0,t) &= 0 \\ V(x,0) &= \phi(x) - h(0). \end{aligned} \tag{10}$$

To verify (10), just subtract! This new problem has a homogeneous boundary condition to which we can apply the method of reflection. Once we find V, we recover v by $v(x,t) = V(x,t) + h(t)$. This simple subtraction device is often used to reduce one linear problem to another.

The domain of independent variables (x, t) in this case is a quarter-plane with specified conditions on both of its half-lines. If they do not agree at the corner [i.e., if $\phi(0) \neq h(0)$], then the solution is discontinuous there (but continuous everywhere else). This is physically sensible. Think for instance, of suddenly at $t = 0$ sticking a hot iron bar into a cold bath.

For the inhomogeneous *Neumann* problem on the half-line,

$$\begin{aligned} w_t - kw_{xx} &= f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ \boldsymbol{w_x(0,t)} &= \boldsymbol{h(t)} \\ w(x,0) &= \phi(x), \end{aligned} \tag{11}$$

we would subtract off the function $xh(t)$. That is, $W(x,t) = w(x,t) - xh(t)$. Differentiation implies that $W_x(0,t) = 0$. Some of these problems are worked out in the exercises.

EXERCISES

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty, \quad 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

using the method of reflection.

2. Solve the completely inhomogeneous diffusion problem on the half-line

$$v_t - kv_{xx} = f(x,t) \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$v(0,t) = h(t) \qquad v(x,0) = \phi(x),$$

by carrying out the subtraction method begun in the text.

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$w_t - kw_{xx} = 0 \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$w_x(0,t) = h(t) \qquad w(x,0) = \phi(x),$$

by the subtraction method indicated in the text.

3.4 WAVES WITH A SOURCE

The purpose of this section is to solve

$$\boxed{u_{tt} - c^2 u_{xx} = f(x,t)} \tag{1}$$

on the whole line, together with the usual initial conditions

$$\boxed{\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}} \tag{2}$$

where $f(x,t)$ is a given function. For instance, $f(x,t)$ could be interpreted as an external force acting on an infinitely long vibrating string.

Because $L = \partial_t^2 - c^2\partial_x^2$ is a linear operator, the solution will be the *sum* of three *terms*, one for ϕ, one for ψ, and one for f. The first two terms are given already in Section 2.1 and we must find the third term. We'll derive the following formula.

Theorem 1. The unique solution of (1),(2) is

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint_\Delta f} \tag{3}$$

where Δ is the characteristic triangle (see Figure 1).

The double integral in (3) is equal to the iterated integral

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

We will give three different derivations of this formula! But first, let's note what the formula says. It says that the effect of a force f on $u(x,t)$ is obtained

EXERCISES

1. Solve $u_{tt} = c^2 u_{xx} + xt$, $\quad u(x, 0) = 0$, $\quad u_t(x, 0) = 0$.
2. Solve $u_{tt} = c^2 u_{xx} + e^{ax}$, $\quad u(x, 0) = 0$, $\quad u_t(x, 0) = 0$.
3. Solve $u_{tt} = c^2 u_{xx} + \cos x$, $\quad u(x, 0) = \sin x$, $\quad u_t(x, 0) = 1 + x$.
4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.

5. Let $f(x, t)$ be any function and let $u(x, t) = (1/2c)\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f \quad \text{and} \quad u(x, 0) \equiv u_t(x, 0) \equiv 0.$$

(*Hint:* Begin by writing the formula as the *iterated* integral

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s)\, dy\, ds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

6. Derive the formula for the inhomogeneous wave equation in yet another way.
   (a) Write it as the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

   (b) Solve the first equation for u in terms of v as

$$u(x, t) = \int_0^t v(x - ct + cs, s)\, ds.$$

   (c) Similarly, solve the second equation for v in terms of f.
   (d) Substitute part (c) into part (b) and write as an iterated integral.

7. Let A be a positive-definite $n \times n$ matrix. Let

$$S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} t^{2m+1}}{(2m + 1)!}.$$

   (a) Show that this series of matrices converges uniformly for bounded t and its sum $S(t)$ solves the problem $S''(t) + A^2 S(t) = 0$, $S(0) = 0$, $S'(0) = I$, where I is the identity matrix. Therefore, it makes sense to denote $S(t)$ as $A^{-1} \sin tA$ and to denote its derivative $S'(t)$ as $\cos(tA)$.
   (b) Show that the solution of (13) is (14).

8. Show that the source operator for the wave equation solves the problem

$$\mathscr{S}_{tt} - c^2 \mathscr{S}_{xx} = 0, \quad \mathscr{S}(0) = 0, \quad \mathscr{S}_t(0) = I,$$

where I is the identity operator.

9. Let $u(t) = \int_0^t \mathcal{S}(t-s) f(s)\, ds$. Using *only* Exercise 8, show that u solves the inhomogeneous wave equation with zero initial data.
10. Use any method to show that $u = 1/(2c) \iint_D f$ solves the inhomogeneous wave equation on the half-line with zero initial and boundary data, where D is the domain of dependence for the half-line.
11. Show by direct substitution that $u(x, t) = h(t - x/c)$ for $x < ct$ and $u(x, t) = 0$ for $x \geq ct$ solves the wave equation on the half-line $(0, \infty)$ with zero initial data and boundary condition $u(0, t) = h(t)$.
12. Derive the solution of the fully inhomogeneous wave equation on the half-line
$$v_{tt} - c^2 v_{xx} = f(x, t) \quad \text{in } 0 < x < \infty$$
$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x)$$
$$v(0, t) = h(t),$$
by means of the method using Green's theorem. (*Hint:* Integrate over the domain of dependence.)
13. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$,
$u(0, t) = t^2, \quad u(x, 0) = x, \quad u_t(x, 0) = 0.$
14. Solve the homogeneous wave equation on the half-line $(0, \infty)$ with zero initial data and with the Neumann boundary condition $u_x(0, t) = k(t)$. Use any method you wish.
15. Derive the solution of the wave equation in a finite interval with inhomogeneous boundary conditions $v(0, t) = h(t)$, $v(l, t) = k(t)$, and with $\phi = \psi = f = 0$.

3.5 DIFFUSION REVISITED

In this section we make a careful mathematical analysis of the solution of the diffusion equation that we found in Section 2.4. (On the other hand, the formula for the solution of the wave equation is so much simpler that it doesn't require a special justification.)

The solution formula for the diffusion equation is an example of a *convolution*, the convolution of ϕ with S (at a fixed t). It is

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\, \phi(y)\, dy = \int_{-\infty}^{\infty} S(z, t)\, \phi(x - z)\, dz, \tag{1}$$

where $S(z, t) = 1/\sqrt{4\pi k t}\, e^{-z^2/4kt}$. If we introduce the variable $p = z/\sqrt{kt}$, it takes the equivalent form

$$\boxed{u(x, t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp.} \tag{2}$$

Now we are prepared to state a precise theorem.