EXERCISES

(a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.
(b) Explain why the note rises when the string is tightened.

Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature = 1. Suddenly both ends are plunged into a bath of temperature = 0. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.

Consider waves in a resistant medium that satisfy the problem

$$u_{tt} = c^2 u_{xx} - r u_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x),$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

Do the same for $2\pi c/l < r < 4\pi c/l$.

Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

4.2 THE NEUMANN CONDITION

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0, t) = u_x(l, t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$\boxed{-X'' = \lambda X, \quad X'(0) = X'(l) = 0,} \qquad (1)$$

other than the trivial solution $X(x) \equiv 0$.

As before, let's first search for the positive eigenvalues $\lambda = \beta^2 > 0$. As in (4.1.6), $X(x) = C\cos\beta x + D\sin\beta x$, so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$

so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l}.$$

The initial condition requires the cosine expansion (6).

EXERCISES

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.
2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).
   (a) Show that the eigenfunctions are $\cos[(n+\frac{1}{2})\pi x/l]$.
   (b) Write the series expansion for a solution $u(x,t)$.
3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.
4. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

   These are called *periodic boundary conditions*.
   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.
   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}.$$

4.3 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$X' - a_0 X = 0 \quad \text{at } x = 0 \tag{1}$$

$$X' + a_l X = 0 \quad \text{at } x = l. \tag{2}$$

The two constants a_0 and a_l should be considered as given.

contain the term

$$(A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}) X_0(x).$$

This term comes from the usual equation $-T'' = \lambda c^2 T = -(\gamma_0 c)^2 T$ for the temporal part of a separated solution (see Exercise 10).

EXERCISES

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.
(b) Find the eigenfunctions corresponding to the zero eigenvalue. (*Hint:* First solve the ODE for $X(x)$. The solutions are not sines or cosines.)

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends." (*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

5. In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).

6. If $a_0 = a_l = a$ in the Robin problem, show that:
(a) There are *no* negative eigenvalues if $a \geq 0$, there is *one* if $-2/l < a < 0$, and there are *two* if $a < -2/l$.
(b) Zero is an eigenvalue if and only if $a = 0$ or $a = -2/l$.

7. If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a \to \infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0,$$

where $\lambda_n(a) = [\beta_n(a)]^2$ is the $(n+1)$st eigenvalue.

Consider again Robin BCs at both ends for arbitrary a_0 and a_l.

(a) In the a_0a_l plane sketch the hyperbola $a_0 + a_l = -a_0a_ll$. Indicate the asymptotes. For (a_0, a_l) on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).

(b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.

(c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.

(d) Where in the plane do the Dirichlet BCs belong?

On the interval $0 \le x \le 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$

$$X'(0) + X(0) = 0 \qquad \text{and} \qquad X(1) = 0$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(d) Is there a negative eigenvalue?

Solve the wave equation with Robin boundary conditions under the assumption that (18) holds.

(a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l \left(c^{-2}u_t^2 + u_x^2\right) dx.$$

(Compare this definition with Section 2.2.)

(b) Do the same for the Neumann BCs.

(c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l \left(c^{-2}u_t^2 + u_x^2\right) dx + \tfrac{1}{2}a_l[u(l,t)]^2 + \tfrac{1}{2}a_0[u(0,t)]^2$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.

Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

(a) Show that $\lambda = 0$ is a double eigenvalue.

(b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2} l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.

(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x, 0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x, t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.

13. Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.
    (a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$

$$u(0, t) = 0 \qquad u_{tt}(l, t) = -ku_x(l, t)$$

    for some constant k.
    (b) What is the eigenvalue problem in this case?
    (c) Find the equation for the positive eigenvalues and find the eigenfunctions.

14. Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$, with $u(1) = u(e) = 0$. Assume that $\lambda > 1$. (*Hint:* Look for solutions of the form $u = x^m$ for complex m.)

15. Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \qquad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

16. Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

17. Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$. (*Hint:* First solve the fourth-order ODE.)

18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2u_{xxxx} = 0$. It has initial conditions as for the wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies

$(0, t) = u_x(0, t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l, t) = u_{xxx}(l, t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

Show that in Case 1 (radiation at both ends)

$$\lim_{n\to\infty}\left[\lambda_n - \frac{n^2\pi^2}{l^2}\right] = \frac{2}{l}(a_0 + a_l).$$

conditions still hold for both Y and Z because the eight constants in (4) are real numbers. So the *real* eigenvalue λ has the *real* eigenfunctions Y and Z. We could therefore say that X and $\overline{X}$ are replaceable by the Y and Z. The linear combinations $aX + b\overline{X}$ are the same as the linear combinations $cY + dZ$, where a and b are somehow related to c and d. This completes the proof of Theorem 2. ❑

NEGATIVE EIGENVALUES

We have seen that most of the eigenvalues turn out to be positive. An important question is whether *all* of them are positive. Here is a sufficient condition.

Theorem 3. Assume the same conditions as in Theorem 1. If

$$f(x)f'(x)\Big|_{x=a}^{x=b} \le 0 \tag{10}$$

for all (real-valued) functions $f(x)$ satisfying the BCs, then there is *no negative eigenvalue*.

This theorem is proved in Exercise 13. It is easy to verify that (10) is valid for Dirichlet, Neumann, and periodic boundary conditions, so that in these cases there are no negative eigenvalues (see Exercise 11). However, as we have already seen in Section 4.3, it could not be valid for certain Robin boundary conditions.

We have already noticed the close analogy of our analysis with linear algebra. Not only are functions acting as if they were vectors, but the operator $-d^2/dx^2$ is acting like a matrix; in fact, it *is* a linear transformation. Theorems 1 and 2 are like the corresponding theorems about real symmetric matrices. For instance, if A is a real symmetric matrix and f and g are vectors, then $(Af, g) = (f, Ag)$. In our present case, A is a differential operator with symmetric BCs and f and g are functions. The same identity $(Af, g) = (f, Ag)$ holds in our case [see (3)]. The two main differences from matrix theory are, first, that our vector space is infinite dimensional, and second, that the boundary conditions must comprise part of the definition of our linear transformation.

EXERCISES

1. (a) Find the real vectors that are orthogonal to the given vectors [1, 1, 1] and [1, −1, 0].
   (b) Choosing an answer to (a), expand the vector [2, −3, 5] as a linear combination of these three mutually orthogonal vectors.
2. (a) On the interval [−1, 1], show that the function x is orthogonal to the constant functions.
   (b) Find a quadratic polynomial that is orthogonal to both 1 and x.
   (c) Find a cubic polynomial that is orthogonal to all quadratics. (These are the first few *Legendre polynomials*.)

Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ and the initial conditions $u(x, 0) = x$, $u_t(x, 0) = 0$. Find the solution explicitly in series form.

Consider the problem $u_t = k u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = U$, $u_x(l, t) = 0$, and the initial condition $u(x, 0) = 0$, where U is a constant.

(a) Find the solution in series form. (*Hint:* Consider $u(x, t) - U$.)
(b) Using a direct argument, show that the series converges for $t > 0$.
(c) If ϵ is a given margin of error, estimate how long a time is required for the value $u(l, t)$ at the endpoint to be approximated by the constant U within the error ϵ. (*Hint:* It is an alternating series with first term U, so that the error is less than the next term.)

(a) Show that the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ lead to the eigenfunctions $(\sin(\pi x/2l), \ \sin(3\pi x/2l), \ \sin(5\pi x/2l), \ldots)$.
(b) If $\phi(x)$ is any function on $(0, l)$, derive the expansion

$$\phi(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right\} \qquad (0 < x < l)$$

by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l - x)$ for $l \le x \le 2l$. (This means that you are extending it *evenly across* $x = l$.) Write the Fourier sine series for $\tilde{\phi}(x)$ on the interval $(0, 2l)$ and write the formula for the coefficients.
(c) Show that every second coefficient vanishes.
(d) Rewrite the formula for C_n as an integral of the original function $\phi(x)$ on the interval $(0, l)$.

Find the complex eigenvalues of the first-derivative operator d/dx subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

Show *by direct integration* that the eigenfunctions associated with the Robin BCs, namely,

$$\phi_n(x) = \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x \quad \text{where } \lambda_n = \beta_n^2,$$

are mutually orthogonal on $0 \le x \le l$, where β_n are the positive roots of (4.3.8).

Show directly that $(-X_1' X_2 + X_1 X_2')|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.

Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

(*The Gram–Schmidt orthogonalization procedure*) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define

$$Y_2 = X_2 - (X_2, Z_1)Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on.

(a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

11. (a) Show that the condition $f(x)f'(x)\big|_a^b \le 0$ is valid for any function $f(x)$ that satisfies Dirichlet, Neumann, or periodic boundary conditions.

(b) Show that it is also valid for Robin BCs provided that the constants a_0 and a_l are positive.

12. Prove *Green's first identity:* For every pair of functions $f(x)$, $g(x)$ on (a, b),

$$\int_a^b f''(x)g(x)\,dx = -\int_a^b f'(x)g'(x)\,dx + f'g\bigg|_a^b.$$

13. Use Green's first identity to prove Theorem 3. (*Hint:* Substitute $f(x) = X(x) = g(x)$, a real eigenfunction.)

14. What do the terms in the series

$$\frac{\pi}{4} = \sin 1 + \frac{1}{3}\sin 3 + \frac{1}{5}\sin 5 + \cdots$$

look like? Make a graph of $\sin n$ for $n = 1, 2, 3, 4, \ldots, 20$ without drawing the intervening curve; that is, just plot the 20 points. Use a calculator; remember that we are using radians. In some sense the numbers $\sin n$ are *randomly* located in the interval $(-1, 1)$. There is a great deal of "random cancellation" in the series.

15. Use the same idea as in Exercises 12 and 13 to show that none of the eigenvalues of the fourth-order operator $+d^4/dx^4$ with the boundary conditions $X(0) = X(l) = X''(0) = X''(l) = 0$ are negative. □

5.4 COMPLETENESS

In this section we state the basic theorems about the convergence of Fourier series. We discuss three senses of convergence of functions. The basic theorems obviously satisfies the BCs. If we let

$$v(x,t) = u(x,t) - \mathcal{U}(x,t),$$

then $v(x,t)$ satisfies the same problem but with zero boundary data, with initial data $\phi(x) - \mathcal{U}(x,0)$ and $\psi(x) - \mathcal{U}_t(x,0)$, and with right-hand side f replaced by $f - \mathcal{U}_{tt}$.

The boundary condition and the differential equation can simultaneously be made homogeneous by subtracting any known function that satisfies them. One case when this can surely be accomplished is the case of "stationary data" when h, k, and $f(x)$ all are independent of time. Then it is easy to find a solution of

$$-c^2\mathcal{U}_{xx} = f(x) \qquad \mathcal{U}(0) = h \qquad \mathcal{U}(l) = k.$$

Then $v(x,t) = u(x,t) - \mathcal{U}(x)$ solves the problem with zero boundary data, zero right-hand side, and initial data $\phi(x) - \mathcal{U}(x)$ and $\psi(x)$.

For another example, take problem (11) for a simple periodic case:

$$f(x,t) = F(x)\cos\omega t \qquad h(t) = H\cos\omega t \qquad k(t) = K\cos\omega t,$$

that is, with the same time behavior in all the data. We wish to subtract a solution of

$$\mathcal{U}_{tt} - c^2\mathcal{U}_{xx} = F(x)\cos\omega t$$
$$\mathcal{U}(0,t) = H\cos\omega t \qquad \mathcal{U}(l,t) = K\cos\omega t.$$

A good guess is that $\mathcal{U}$ should have the form $\mathcal{U}(x,\ t) = \mathcal{U}_0(x)\cos\omega t$. This will happen if $\mathcal{U}_0(x)$ satisfies

$$-\omega^2\mathcal{U}_0 - c^2\mathcal{U}_0'' = F(x) \qquad \mathcal{U}_0(0) = H \qquad \mathcal{U}_0(l) = K. \qquad \square$$

There is also the method of Laplace transforms, which can be found in Section 12.5.

EXERCISES

1. (a) Solve as a series the equation $u_t = u_{xx}$ in $(0,1)$ with $u_x(0,t) = 0$, $u(1,t) = 1$, and $u(x,0) = x^2$. Compute the first two coefficients explicitly.
   (b) What is the equilibrium state (the term that does not tend to zero)?
2. For problem (1), complete the calculation of the series in case $j(t) = 0$ and $h(t) = e^t$.
3. Repeat problem (1) for the case of Neumann BCs.
4. Solve $u_{tt} = c^2u_{xx} + k$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = V$. Here k and V are constants.
5. Solve $u_{tt} = c^2u_{xx} + e^t\sin 5x$ for $0 < x < \pi$, with $u(0,t) = u(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \sin 3x$.

Solve $u_{tt} = c^2 u_{xx} + g(x)\sin\omega t$ for $0 < x < l$, with $u = 0$ at both ends and $u = u_t = 0$ when $t = 0$. For which values of ω can resonance occur? (Resonance means growth in time.)

Repeat Exercise 6 for the damped wave equation $u_{tt} = c^2 u_{xx} - r u_t + g(x)\sin\omega t$, where r is a positive constant.

Solve $u_t = k u_{xx}$ in $(0, l)$, with $u(0, t) = 0$, $u(l, t) = At$, $u(x, 0) = 0$, where A is a constant.

Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \le x \le 1 = l$, with $u(0, t) = h$, $u(1, t) = k$, where h and k are given constants, and $u(x, 0) = 0$, $u_t(x, 0) = 0$.

Find the temperature of a metal rod that is in the shape of a solid circular cone with cross-sectional area $A(x) = b(1 - x/l)^2$ for $0 \le x \le l$, where b is a constant. Assume that the rod is made of a uniform material, is insulated on its sides, is maintained at zero temperature on its flat end ($x = 0$), and has an unspecified initial temperature distribution $\phi(x)$. Assume that the temperature is independent of y and z. [*Hint:* Derive the PDE $(1 - x/l)^2 u_t = k\{(1 - x/l)^2 u_x\}_x$. Separate variables $u = T(t)X(x)$ and then substitute $v(x) = (1 - x/l)X(x)$.]

Write out the solution of problem (11) explicitly, starting from the discussion in Section 5.6.

Carry out the solution of (11) in the case that

$$f(x, t) = F(x)\cos\omega t \qquad h(t) = H\cos\omega t \qquad k(t) = K\cos\omega t.$$

If friction is present, the wave equation takes the form

$$u_{tt} - c^2 u_{xx} = -r u_t,$$

where the resistance $r > 0$ is a constant. Consider a periodic source at one end: $u(0, t) = 0$, $u(l, t) = Ae^{i\omega t}$.

(a) Show that the PDE and the BC are satisfied by

$$\mathcal{U}(x, t) = Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}, \qquad \text{where } \beta^2 c^2 = \omega^2 - ir\omega.$$

(b) No matter what the IC, $u(x, 0)$ and $u_t(x, 0)$, are, show that $\mathcal{U}(x, t)$ is the asymptotic form of the solution $u(x, t)$ as $t \to \infty$.

(c) Show that you can get resonance as $r \to 0$ if $\omega = m\pi c/l$ for some integer m.

(d) Show that friction can prevent resonance from occurring.