STABLE SPIKES FOR A REACTION-DIFFUSION SYSTEM MODELING COLOR PATTERN FORMATION

WEIWEI AO, JUNCHENG WEI, AND MATTHIAS WINTER

ABSTRACT. We consider a reaction-diffusion system for color pattern formation with two activators and one inhibitor. Each of the activators models one of the colors being switched on, for example the first activator could represent the color blue and the second activator the color yellow. If both colors are present the pattern will have green color since the color green is achieved by a mixture of the colors blue and yellow. We prove rigorous results on the existence and stability of spikes for which one of the colors or both of them are switched on. We classify the different types of solutions which can exist depending on the choice of interaction parameters between the components and we show which of them are stable or unstable. In particular, solutions with spikes for both activators in the same position can be stable when cross-activation dominates over self-activation. On the other hand, solutions with a spike for only one activator and zero concentration for the other activator can be stable when self-activation dominates over cross-activation. The rigorous approach is based on analytical methods such as Green's function, Liapunov-Schmidt reduction and nonlocal eigenvalue problems. The analytical results are confirmed by numerical simulations.
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1. INTRODUCTION

We study a reaction-diffusion system with two activators and one inhibitor modeling color pattern formation. Many mechanisms play a role for color pattern formation. For a recent survey we refer to [13]. Color patterns in reptiles, amphibians, and fish are determined by chromatophores. They result from the spatial variation in chromatophore types, properties, and spatial arrangements.

In zebrafish the chromatophores are the pigment cells melanophores, iridophores, and xanthophores. Recent molecular genetic studies have shown that interactions between the pigment cells play major roles in pattern formation [11, 22]. The color for chameleons can be changed through active tuning of a lattice of guanine nanocrystals inside a layer of dermal iridophores. This has been confirmed by using osmotic pressure experiments and theoretical optical modeling [23].

Stripes in zebrafish have been modeled using an agent-based approach [25]. It has been shown that iridophores can act as a stabilizer of zebrafish stripes [26]. Topological analysis of zebrafish patterns has been performed in [21].

In this paper, we will prove the existence and study the stability of spike solutions for the system with two activators which display self-interaction and cross-interaction, coupled with one global inhibitor.

Spike solutions for other large reaction-diffusion with more than two components have been studied before, including the hypercycle of Eigen and Schuster [32, 34, 35] or mutual exclusion of spikes [37].

The reaction-diffusion model for color pattern formation is similar to the hypercycle but with different interaction between activators. For the hypercycle, there is only activator interaction with nearest neighbors, the interaction is not symmetric and there is no self-interaction.

The activator interaction in our system is a special case of the interaction for Schrödinger systems but the Schrödinger systems have no inhibitor. For Schrödinger systems, the existence and Morse index of spike solutions have been studied extensively by Wei and Lin, see [14, 15, 16, 17, 18, 19] and references therein. The uniqueness of positive ground states has been shown in [40] and the nondegeneracy of ground states has been proved in [2], while the Morse indices of ground state solutions for Schrödinger systems with an arbitrary number of components have been considered in [42]. For Schrödinger systems the type of solutions considered here are unstable with Morse index 1 or 2. Due to the presence of the inhibitor, it is possible to stabilize some of these solutions.

The activator-inhibitor system under investigation can be stated as follows:

$$\begin{cases} u_{1,t} = \epsilon^2 u_{1,xx} - u_1 + \dfrac{\mu_1 u_1^3 + \beta u_1 u_2^2}{v}, \quad u_{2,t} = \epsilon^2 u_{2,xx} - u_2 + \dfrac{\mu_2 u_2^3 + \beta u_1^2 u_2}{v}, \\ \tau v_t = D v_{xx} - v + \mu_1 u_1^3 + \beta u_1 u_2^2 + \mu_2 u_2^3 + \beta u_1^2 u_2. \end{cases} \tag{1.1}$$

Here $0 < \epsilon \ll 1$ and $D > 0$ are diffusion constants, μ_1, μ_2, $\beta > 0$ are positive constants for the self- and cross-activation of the activators, respectively, and τ is a nonnegative time-relaxation constant.

The positive constants μ_1, μ_2 model the strength of self-activation of each activator and β represents the strength of cross-activation.

The x-indices indicate spatial derivatives. We will derive results for the system (1.2) on a bounded interval $\Omega = (-L, L)$ for $L > 0$ with Neumann boundary conditions.

This system represents a simple biological cellular or genetic signaling network for color pattern formation in which all activator components interact with themselves and each other, and the interaction parameters can be tuned arbitrarily.

By rescaling the amplitudes of u_1 and u_2 we can always achieve that the mixed terms have the same coefficient β, so we can make this assumption without loss of generality.

We rescale the unknown functions as follows to achieve amplitudes of order $O(1)$:

$$\hat{u}_1(x) = \epsilon u_1(x), \quad \hat{u}_2(x) = \epsilon u_2(x), \quad \hat{v}(x) = \epsilon^2 v(x).$$

In terms of the rescaled functions, the system can be restated as follows:

$$\begin{cases} \hat{u}_{1,t} = \epsilon^2 \hat{u}_{1,xx} - \hat{u}_1 + \dfrac{\mu_1 \hat{u}_1^3 + \beta \hat{u}_1 \hat{u}_2^2}{\hat{v}}, \quad \hat{u}_{2,t} = \epsilon^2 \hat{u}_{2,xx} - \hat{u}_2 + \dfrac{\mu_2 \hat{u}_2^3 + \beta \hat{u}_1^2 \hat{u}_2}{\hat{v}}, \\ \tau \hat{v}_t = D \hat{v}_{xx} - \hat{v} + \left(\mu_1 \hat{u}_1^3 + \beta \hat{u}_1 \hat{u}_2^2 + \mu_2 \hat{u}_2^3 + \beta \hat{u}_1^2 \hat{u}_2 \right) \epsilon^{-1}. \end{cases} \tag{1.2}$$

It is known that these patterns are unstable without inhibitor (see for example [2, 3, 42]). In fact, without inhibitor, the Morse index of a single spike solution will be 1 or 2, depending on the β-μ condition. Here we will show that with global inhibition it is possible to get stable spiky patterns. Depending on the β-μ condition, at a certain location either both activators can have a spike forming a local pattern, or only one of the activators has a spike and the other activator zero values. Both of these types of solutions can be stable or unstable, depending on the β-μ condition.

The system (1.2) is similar to a hypercycle with cubic terms and two activators, and Gierer-Meinhardt kinetics instead of Gray-Scott kinetics, see [34]. For the study of spiky solutions of the hypercycle with quadratic interaction terms we refer the readers to [32, 35].

The feedback mechanism in (1.2) is a generalization of the well-known Gierer-Meinhardt system [6, 20] which has one local activator coupled to an inhibitor. We recall that the classical Gierer-Meinhardt system as well as the three-component system considered here are both Turing systems [24] as they allow spatial patterns to arise out of a homogeneous steady state by the so-called Turing instability. Some analytical results for the existence and stability of spiky Turing pattern for the Gierer-Meinhardt system have been obtained for example in [4], [5], [10], [27], [31], [33], [36], [38]. The results have been reviewed in [39].

Next we are going to state the rigorous results on the existence and stability of stationary spiky patterns for the system (1.2).

We prove the **existence** of three types of spiky pattern solutions:

A solution of Type 1 which has a spike for u_1 and spike for u_2, both located at zero.

A solution of Type 2 which has a spike for u_1 located at zero and $u_2 = 0$.

A solutions of Type 3 which has a spike for u_2 located at zero and $u_1 = 0$.

The spikes for u_1 and u_2 in a solution of Type 1 have the same profile except for possibly their amplitudes. To determine the amplitudes of the activator spikes we have to solve a system which depends on the coefficients of the reaction terms of the self-interaction and cross-interaction of activators. We will see that these amplitudes depend on the size of inhibitor and for larger inhibitor we need larger activator amplitudes to balance the interaction (as in the classical Gierer-Meinhardt system).

Let us also mention that the combinations of Type 2 and Type 3 solutions in a different system with reaction kinetics of Klausmeier type have been studied in [8].

Let $w(y)$ be the unique positive and even homoclinic solution of the equation

$$w_{yy} - w + w^3 = 0 \tag{1.3}$$

on the real line decaying to zero at $\pm\infty$. Let $H^2_{N,ev}(-L, L)$ be the space of functions in $H^2(-L, L)$ which satisfy Neumann boundary conditions and are even.

The main results are as follows:

Theorem 1. *Assume that $\epsilon > 0$ is small enough and*

$$\beta > \max(\mu_1, \mu_2) \text{ or } \beta < \min(\mu_1, \mu_2).$$

Then there exist spiky steady states to (1.2) in $H^2_{N,ev}(-L, L)$ such that

$$u_1^\epsilon(x) = t_1\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)\chi(x)(1 + O(\epsilon)), \quad u_2^\epsilon(x) = t_2\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)\chi(x)(1 + O(\epsilon)) \tag{1.4}$$

where $t_i > 0$ is a constant which satisfies (2.10), $v_\epsilon(0)$ is given by (2.11) and $\chi(x)$ is a cutoff function defined in (3.1).

Similarly, we can show the existence of Type 2 solutions with a spike for u_1 and $u_2 = 0$.

Theorem 2. *Assume that $\epsilon > 0$ is small enough and $\beta \neq \mu_1$. Then there exist spiky steady states to (1.2) in $H^2_{N,ev}(-L, L)$ such that*

$$u_1^\epsilon(x) = t_1\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)\chi(x)(1 + O(\epsilon)), \quad u_2^\epsilon(x) = 0, \tag{1.5}$$

where $t_1 = \frac{1}{\sqrt{\mu_1}}$, $v_\epsilon(0)$ is given by (2.15) and $\chi(x)$ is a cutoff function defined in (3.1).

Remark 1. *By symmetry, Theorem 2 is still valid if u_1 is swapped with u_2 and μ_1 is swapped with μ_2, which implies the existence of Type 3 spike solutions when $\beta \neq \mu_2$.*

The stability properties of the spiky solutions are as follows:

Theorem 3. *The steady states to (1.2) given in Theorem 1 are linearly stable if* $\beta > \max(\mu_1, \mu_2)$. *They are linearly unstable if* $\beta < \min(\mu_1, \mu_2)$.

Theorem 4. *The steady states to (1.2) given in Theorem 2 are linearly stable if* $\beta < \mu_1$. *They are linearly unstable if* $\beta > \mu_1$.

Remark 2. *For* $\mu_1 < \beta < \mu_2$ *the solution with* $u_1 \neq 0$ *and* $u_2 \neq 0$ *does not exist, in the same way as for Schrödinger systems without v component, see* [40].

Remark 3. *In case* $\mu_1 < \mu_2$ *there is a transcritical bifurcation at* $\beta = \mu_1$ *and* $\beta = \mu_2$*, respectively. For* $\beta = \mu_1$ *there is a bifurcation of the Type 1 solution from the Type 2 solution, and for* $\beta = \mu_2$ *there is a bifurcation of the Type 1 solution from the Type 3 solution.*

What do these mathematical results tell us about biological applications? The two activators u_1 and u_2 represent different colors in the living organism. For example, u_1 can represent the color blue and u_2 the color yellow. Then a spike with only u_1 positive indicates a blue spot and a spike with only u_2 positive indicates a yellow spot. Finally, a spike with both u_1 and u_2 positive represents a green spot since the color green is achieved by a mixture of the colors blue and yellow.

To summarize, suppose $\beta < \mu_1 < \mu_2$. Then Type 1 solutions are unstable, Type 2 and 3 solutions are stable. Stable patterns can have blue spots or yellow spots. Green spots are possible but they are unstable.

For $\mu_1 < \beta < \mu_2$, Type 1 solutions do not exist. Type 3 solutions are stable and Type 2 solutions are unstable. Stable patterns can have yellow spots. Blue spots are possible but they are unstable. Green spots are impossible.

For $\mu_1 < \mu_2 < \beta$, Type 1 solutions are stable, whereas both Type 2 and 3 solutions are unstable. Stable patterns can have green spots. Blue spots or yellow spots are possible but they are unstable.

Remark 4. *By choosing the parameters suitably it is possible to achieve* **stable Type 1 solutions with any proportion of mixing** *between blue and yellow color. We can see this as follows: We first compute*

$$\frac{t_2}{t_1} = \sqrt{\frac{\beta - \mu_1}{\beta - \mu_2}}.$$

Therefore, varying parameters in the range $\mu_1 < \mu_2 < \beta$ *we can get any ratio for* t_2/t_1 *in the interval*

$$1 < \frac{t_2}{t_1} < \infty$$

and varying parameters in the range $\mu_2 < \mu_1 < \beta$ *we can get any ratio for* t_2/t_1 *in the interval*

$$0 < \frac{t_2}{t_1} < 1.$$

(Note that we can keep μ_1 *and* μ_2 *fixed and vary* β *accordingly.)*

Choosing $\mu_1 = \mu_2$ *we get* $t_1 = t_2$ *and so* $\frac{t_2}{t_1} = 1$. *(If we keep* μ *fixed and vary* β *then we get* $\frac{t_2}{t_1} = 1$ *for any* β.*)*

In summary, we can achieve any value for $\frac{t_2}{t_1}$ *in the range*

$$0 < \frac{t_2}{t_1} < \infty.$$

The paper is organized as follows: In Section 2, we compute the amplitudes of spikes. In Section 3, we show the existence of solutions. In Section 4, we first derive the eigenvalue problem. Then we compute the large (i.e. $O(1)$) eigenvalues and we derive sufficient conditions for the stability of solutions with respect to these. In Section 5, we consider the small (i.e. $o(1)$) eigenvalues. We outline how to rigorously compute them to leading order and state the main criterion on the stability of solutions with respect to small eigenvalues. Sufficient conditions for this stability are derived. In Section 6, our results are confirmed by numerical simulations. In Section 7 we discuss our results.

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2. Computing the Amplitudes

In this section, we will consider spiky steady states of (1.2) of the following three types:
Type 1: spike for u_1 and spike for u_2, both located at zero,
Type 2: spike for u_1 located at zero and $u_2 = 0$.
Type 3: spike for u_2 located at zero and $u_1 = 0$.

Since Type 3 is symmetric to Type 2 by inter-changing u_1 and u_2, and μ_1 and μ_2, we will only consider Type 1 and Type 2 solutions.

We first construct steady states of the form

$$u_1(x) = t_1\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)(1+O(\epsilon)), \quad u_2(x) = t_2\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)(1+O(\epsilon)), \tag{2.1}$$

where $w(y)$ is the unique positive and even homoclinic solution of the equation

$$w_{yy} - w + w^3 = 0 \tag{2.2}$$

on the real line decaying to zero at $\pm\infty$.

From the first two equations to (1.2), we will choose t_1 and t_2 such that

$$\mu_1 t_1^2 + \beta t_2^2 = 1,\ \mu_2 t_2^2 + \beta t_1^2 = 1. \tag{2.3}$$

All functions used throughout the paper belong to the Hilbert space $H^2(-L, L)$ and the error terms are taken in the norm $H^2(-L, L)$ unless otherwise stated. After integrating (1.3) over $\mathbb{R}$, we get the relation

$$\int_{\mathbb{R}} w(y)\,dy = \int_{\mathbb{R}} w^3(y)\,dy \tag{2.4}$$

which will be used frequently, often without explicitly stating it.

Note that u_1 and u_2 are spatially small-scale variables, as $\epsilon \ll 1$, and v_ϵ is a spatially large-scale variable. For steady states, using Green's functions, the slow variable v, to leading order, can be expressed by an integral representation.

To get this representation, by (2.1) the nonlinear terms in the last equation of (1.2) can be expanded as

$$\mu_1 u_1^3(x) + \beta u_1(x)u_2^2(x) = t_1\,(v_\epsilon(0))^{3/2}\,\epsilon\left(\int_{\mathbb{R}} w^3\right)\delta_0(x) + O(\epsilon^2),$$

$$\mu_2 u_2^3(x) + \beta u_1^2(x)u_2(x) = t_2\,(v_\epsilon(0))^{3/2}\,\epsilon\left(\int_{\mathbb{R}} w^3\right)\delta_0(x) + O(\epsilon^2),$$

where $\delta_0(x)$ is the Dirac delta distribution centered at 0.

Using the Green's function $G_D(x,y)$ which is defined as the unique solution of the equation

$$D\Delta G_D(x,y) - G_D(x,y) + \delta_y(x) = 0, \quad -L < x < L, \quad G_{D,x}(-L,y) = G_{D,x}(L,y) = 0, \tag{2.5}$$

we can represent $v_\epsilon(x)$ by using the third equation of (1.2) as

$$v_\epsilon(x) = (t_1 + t_2)(v_\epsilon(0))^{3/2} \left(\int_{\mathbb{R}} w^3(y)dy \right) G_D(x,0) + O(\epsilon). \tag{2.6}$$

An elementary calculation gives

$$G_D(x,y) = \begin{cases} \frac{\theta}{\sinh(2\theta L)} \cosh\theta(L+x)\cosh\theta(L-y), & -L < x < y < L, \\ \frac{\theta}{\sinh(2\theta L)} \cosh\theta(L-x)\cosh\theta(L+y), & -L < y < x < L \end{cases} \tag{2.7}$$

with $\theta = 1/\sqrt{D}$. Note that

$$G_D(x,y) = \frac{1}{2\sqrt{D}} e^{-|x-y|/\sqrt{D}} - H_D(x,y), \tag{2.8}$$

where H_D is the regular part of the Green's function G_D. In particular, for $L = \infty$, we have

$$G_D(x,y) = \frac{1}{2\sqrt{D}} e^{-|x-y|/\sqrt{D}} =: K_D(x,y). \tag{2.9}$$

We first compute t_1 and t_2 from (2.3). Considering (2.3) as a linear system with the unknowns t_1^2 and t_2^2, we get

$$t_1^2 = \frac{\beta - \mu_2}{\beta^2 - \mu_1\mu_2}, \quad t_2^2 = \frac{\beta - \mu_1}{\beta^2 - \mu_1\mu_2}. \tag{2.10}$$

It remains to derive $v_\epsilon(0)$. From (2.6), for $x = 0$ we get

$$v_\epsilon(0) = (v_\epsilon(0))^{3/2}(t_1 + t_2)G_D(0,0) \left(\int_{\mathbb{R}} w^3(y)dy \right) + O(\epsilon).$$

This implies

$$v_\epsilon(0) = \frac{1}{G_D(0,0)^2(t_1+t_2)^2 \left(\int_{\mathbb{R}} w^3\right)^2} + O(\epsilon). \tag{2.11}$$

In the following, we state the first main result of this section on the amplitudes of Type 1 solutions:

Lemma 1. *Assume that $\epsilon > 0$ is small enough and*

$$\beta > \max(\mu_1, \mu_2) \text{ or } \beta < \min(\mu_1, \mu_2).$$

Then for spike-solutions of (1.2) of the type

$$u_1(x) = t_1\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)(1 + O(\epsilon)), \quad u_2(x) = t_2\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)(1 + O(\epsilon)),$$

where $w(y)$ is the unique solution of (2.2), the amplitudes t_1 and t_2 are given by (2.10), and $v_\epsilon(0)$ satisfies (2.11), where the Green's function G_D is defined in (2.5).

Next, we consider the Type 2 spike solutions. We will construct solutions of the form

$$u_1(x) = t_1\sqrt{v_\epsilon(0)}w\left(\frac{x}{\epsilon}\right)(1 + O(\epsilon)), \quad u_2(x) = 0. \tag{2.12}$$

From the first equation of (1.2), we get in leading order

$$t_1 = \mu_1 t_1^3$$

which implies

$$t_1 = \frac{1}{\sqrt{\mu_1}}. \tag{2.13}$$

From the third equation of (1.2), we get

$$v_\epsilon(x) = t_1 (v_\epsilon(0))^{3/2} \left(\int_{\mathbb{R}} w^3 \right) G_D(x,0) + O(\epsilon). \tag{2.14}$$

From (2.14), for $x = 0$ we get

$$v_\epsilon(0) = (v_\epsilon(0))^{3/2} t_1 G_D(0,0) \left(\int_{\mathbb{R}} w^3 \right) + O(\epsilon).$$

Together with (2.13) this implies

$$v_\epsilon(0) = \frac{1}{G_D(0,0)^2 t_1^2 \left(\int_{\mathbb{R}} w^3 \right)^2} + O(\epsilon). \tag{2.15}$$

With these in hand, we finally state the second main result of this section on the amplitude of Type 2 spike solutions:

Lemma 2. *Assume that $\epsilon > 0$ is small enough. Then for spike-solutions of (1.2) of the type*

$$u_1(x) = t_1 \sqrt{v_\epsilon(0)}\, w\left(\frac{x}{\epsilon}\right)(1 + O(\epsilon)), \quad u_2(x) = 0,$$

where $w(y)$ is the unique solution of (2.2), the amplitude t_1 is given by (2.13), and $v_\epsilon(0)$ satisfies (2.15), where the Green's function G_D is defined in (2.5).

3. Existence of Spike Solutions

In this section, we use the contraction mapping principle to rigorously prove the existence of spiky solutions.

The issue to handle here is that the linear operator obtained by the linearization of system (1.2) around (2.1) has a nontrivial approximate kernel. This comes from the fact that taking a derivative of the equation (2.2) with respect to y implies

$$(w_y)_{yy} - w_y + 3w^2 w_y = 0.$$

Thus, w_y belongs to the kernel of the linearization of (2.2) around w. Note that the function w_y represents the translation mode of w. To eliminate the approximate kernel from the function space we will construct solutions in spaces of even functions. Since the approximate kernel consists of odd functions, we will be able to show in this section first that the linear operator restricted to even functions is uniformly invertible for ϵ small enough. Using this result, we can then complete the proof.

Recall that for given $u_1, u_2 \in H_N^2(\Omega_\epsilon)$, where $\Omega_\epsilon = (-L/\epsilon, L/\epsilon)$ and $H_N^2(\Omega_\epsilon)$ denotes the space of all functions in $H^2(\Omega_\epsilon)$ satisfying Neumann boundary conditions, since the third equation of (1.2) is linear in v, the inhibitor v is uniquely determined for given u_1, u_2. Therefore, the steady state problem can be reduced to solving the first two equations.

We are looking for solutions which satisfy

$$u_1(x) = t_1 \sqrt{v_\epsilon(0)}\, w\left(\frac{x}{\epsilon}\right) \chi(x)(1 + O(\epsilon)), \quad u_2(x) = t_1 \sqrt{v_\epsilon(0)}\, w\left(\frac{x}{\epsilon}\right) \chi(x)(1 + O(\epsilon))$$

which are even functions, i.e. $u_i(x) = u_i(-x)$, $i = 1, 2$. To this end, we assume that χ is a smooth and even cut-off function such that

$$\chi(x) = 1 \text{ for } |x| \le L/3$$

and

$$\chi(x) = 0 \text{ for } |x| \geq 2L/3. \tag{3.1}$$

To construct a solution which consists of even functions we will be working in Sobolev spaces of even functions.

We are now going to derive a solution by using the contraction mapping principle. Denoting the r.h.s. of the first two equation of (1.2) by

$$S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi + a_1, t_2\sqrt{v_\epsilon(0)}w\chi + a_2],$$

our problem can be re-written as follows: Find a_1 and a_2 such that

$$S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi + a_1, t_2\sqrt{v_\epsilon(0)}w\chi + a_2] = 0,$$

where

$$S_\epsilon : (H^2_{N,ev}(\Omega_\epsilon))^2 \to (L^2_{ev}(\Omega_\epsilon))^2.$$

Here the index "ev" stands for the restriction of the function spaces to even functions, i.e.

$$L^2_{ev}(\Omega_\epsilon) = \{u \in L^2(\Omega_\epsilon),\, u(y) = u(-y) \text{ for all } y \in \Omega_\epsilon\},$$
$$H^2_{N,ev}(\Omega_\epsilon) = \{u \in H^2_N(\Omega_\epsilon),\, u(y) = u(-y) \text{ for all } y \in \Omega_\epsilon\}.$$

To this end, we need to study the linearized operator

$$L_\epsilon : (H^2_{N,ev}(\Omega_\epsilon))^2 \to (L^2(\Omega_\epsilon))^2$$

defined by

$$L_\epsilon\phi := DS_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi, t_2\sqrt{v_\epsilon(0)}w\chi]\phi,$$

where $DS_\epsilon[\cdot]$ denotes the Fréchet derivative of the operator S_ϵ at $(t_1\sqrt{v_\epsilon(0)}w\chi, t_2\sqrt{v_\epsilon(0)}w\chi)^T$.

Then we have the following key result about the uniform invertibility of the linearized operator L_ϵ.

Proposition 1. *There exist positive constants $\bar{\epsilon}$, c such that we have for all $\epsilon \in (0, \bar{\epsilon})$,*

$$\|L_\epsilon\phi\|_{L^2(\Omega_\epsilon)} \geq c\,\|\phi\|_{H^2(\Omega_\epsilon)} \quad \text{for all} \quad \phi = (\phi_1, \phi_2)^T \in (H^2_{N,ev}(\Omega_\epsilon))^2. \tag{3.2}$$

Further, the linear mapping L_ϵ is surjective.

Proof. We prove by contradiction. Suppose that (3.2) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\phi^k\}$ with $\epsilon_k \to 0$, $\phi^k = \phi_{\epsilon_k} \in (H^2_{N,ev}(\Omega_\epsilon))^2$, $k = 1, 2, \ldots$ such that

$$\|L_{\epsilon_k}\phi^k\|_{(L^2(\Omega_{\epsilon_k}))^2} \to 0, \quad \text{as } k \to \infty, \quad \|\phi^k\|_{(H^2(\Omega_{\epsilon_k}))^2} = 1, \quad k = 1, 2, \ldots. \tag{3.3}$$

At first (after rescaling) ϕ_{ϵ_k} is only defined on Ω_{ϵ_k}. However, by a standard result (compare [7]) it can be extended to $\mathbb{R}$ such that its norm in $H^2(\mathbb{R})$ is still bounded by a constant independent of ϵ_k for ϵ_k small enough. It is then a standard procedure to show that this extension converges strongly in $H^2(\Omega_\epsilon)$ to some limit ϕ_∞ with $\|\phi_\infty\|_{(H^2(\mathbb{R}))^2} = 1$. For the functional-analytic details of the argument, we refer to [9].

Then $\phi_\infty = (\phi_1, \phi_2)^T$ solves the system

$$\begin{aligned}
&\Delta\phi_1 - \phi_1 + [(2\mu_1 t_1^2 + 1)\phi_1 + 2\beta t_1 t_2 \phi_2]w^2 \\
&-\frac{t_1}{t_1 + t_2}\left[(2\mu_1 t_1^2 + 1 + 2\beta t_1 t_2)\int_{\mathbb{R}} w^2\phi_1\,dy + (2\mu_2 t_2^2 + 1 + 2\beta t_1 t_2)\int_{\mathbb{R}} w^2\phi_2\,dy\right]\frac{w^3}{\int_{\mathbb{R}} w^3\,dy} = 0, \\
&\Delta\phi_2 - \phi_2 + [(2\mu_2 t_2^2 + 1)\phi_2 + 2\beta t_1 t_2 \phi_1]w^2
\end{aligned} \tag{3.4}$$

$$-\frac{t_2}{t_1+t_2}\left[(2\mu_2 t_2^2+1+2\beta t_1 t_2)\int_{\mathbb{R}} w^2\phi_2\,dy+(2\mu_1 t_1^2+1+2\beta t_1 t_2)\int_{\mathbb{R}} w^2\phi_1\,dy\right]\frac{w^3}{\int_{\mathbb{R}} w^3\,dy}=0. \tag{3.5}$$

This system is the special case with $\lambda = 0$ of (4.5) and (4.6) derived in Section 4. To avoid repetition of the derivations we refer to Section 4, where the derivation will be made in a more general case.

Next we prove $\phi_1 = \phi_2 = 0$. This can be done using similar arguments as in Section 4, by first showing that $-t_2\phi_1 + t_1\phi_2 = 0$, and then that $t_1\phi_1 + t_2\phi_2 = 0$. Thus, $\phi_1 = \phi_2 = 0$. To avoid repetition, here we refer to the detailed calculations given below in the proof of Proposition 2.

This contradicts the assumption $\|\phi\|_{H^2(\Omega_\epsilon)} = 1$. Therefore, (3.2) must be true.

In order to show its surjectivity, we need to show that the kernel of the adjoint operator is trivial, namely that the following system has only the zero solution $\phi = (\phi_1, \phi_2)^T$:

$$\Delta\phi_1-\phi_1+[(2\mu_1 t_1^2+1)\phi_1+2\beta t_1 t_2\phi_2]w^2$$
$$-\frac{2\mu_1 t_1^2+1+2\beta t_1 t_2}{t_1+t_2}\left[t_1\int_{\mathbb{R}} w^3\phi_1\,dy+t_2\int_{\mathbb{R}} w^3\phi_2\,dy\right]\frac{w^2}{\int_{\mathbb{R}} w^3\,dy}=0, \tag{3.6}$$
$$\Delta\phi_2-\phi_2+[(2\mu_2 t_2^2+1)\phi_2+2\beta t_1 t_2\phi_1]w^2$$
$$-\frac{2\mu_2 t_2^2+1+2\beta t_1 t_2}{t_1+t_2}\left[t_2\int_{\mathbb{R}} w^3\phi_2\,dy+t_1\int_{\mathbb{R}} w^3\phi_1\,dy\right]\frac{w^2}{\int_{\mathbb{R}} w^3\,dy}=0. \tag{3.7}$$

Combining (3.6), (3.7) and using (2.3) implies that $\hat{\phi}_1 = t_1\phi_1 + t_2\phi_2$ satisfies

$$\Delta\hat{\phi}_1-\hat{\phi}_1+3\hat{\phi}_1 w^2-3\int w^3\hat{\phi}_1\,dy\,\frac{w^2}{\int w^3\,dy}=0. \tag{3.8}$$

Multiplying (3.8) by w, integrating and using (1.3), we get

$$\int w^3\hat{\phi}_1\,dy=0.$$

Thus the nonlocal term in (3.8) vanishes and we have

$$\Delta\hat{\phi}_1-\hat{\phi}_1+3\hat{\phi}_1 w^2=0.$$

This implies $\hat{\phi}_1 = 0$ by Lemma 4.1 in [28] since it is an even function.

Thus, the nonlocal terms in (3.6), (3.7) vanish. Then for $\hat{\phi}_2 = t_2\phi_1 - t_1\phi_2$, we get

$$\Delta\hat{\phi}_2-\hat{\phi}_2+(3-2\beta(t_1^2+t_2^2))\hat{\phi}_2 w^2=0$$

which implies $\hat{\phi}_2 = 0$ by Lemma 4.1 of [28]. Going back to the original eigenfunctions, we have $\phi_1 = \phi_2 = 0$.

By the Closed Range Theorem it follows that the map L_ϵ is surjective. (The details are given for example in [9].)

□

Proof of Theorem 1:

The main existence result Theorem 1 can now be shown as follows:

We first compute $S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi, t_2\sqrt{v_\epsilon(0)}w\chi]$. From the first equation of (1.2), we get

$$\epsilon^2 u_{1,xx}-u_1+\frac{\mu_1 u_1^3+\beta u_1 u_2^2}{v_\epsilon}=t_1\sqrt{v_\epsilon(0)}(w''-w+w^3)+t_1\sqrt{v_\epsilon(0)}w^3\left(\frac{v_\epsilon(0)}{v_\epsilon(\epsilon y)}-1\right)+O(\epsilon^3)$$
$$=0+t_1\sqrt{v_\epsilon(0)}w^3\left(-\frac{v_\epsilon''(0)\epsilon^2 y^2}{2v_\epsilon(0)}+O(\epsilon^3|y|^3)\right)=O(\epsilon^2 y^2 w^3(y)).$$

Here we have used that $v_\epsilon(\epsilon y)$ is an even function and so $v_\epsilon'(0)=0$ and the $O(\epsilon)$ term vanishes.

Since

$$S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi+a_1,t_2\sqrt{v_\epsilon(0)}w\chi+a_2]=S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi,t_2\sqrt{v_\epsilon(0)}w\chi]+L_\epsilon(a_1,a_2)+G(a_1,a_2),$$

where

$$\|S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi,t_2\sqrt{v_\epsilon(0)}w\chi]\|_{(L^2(\Omega_\epsilon))^2}=O(\epsilon^2)$$

and

$$\|G(a_1,a_2)\|_{(L^2(\Omega_\epsilon))^2}=O((\|(a_1,a_2)\|_{(L^2(\Omega_\epsilon))^2})^2)$$

we can re-write

$$S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi+a_1,t_2\sqrt{v_\epsilon(0)}w\chi+a_2]=0$$

as

$$(a_1,a_2)=-L_\epsilon^{-1}S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi,t_2\sqrt{v_\epsilon(0)}w\chi]-L_\epsilon^{-1}G(a_1,a_2).$$

In other words, we need to find a fixed point $(a_1,a_2)\in H_{2,ev}(\Omega_\epsilon)$ of the mapping

$$-L_\epsilon^{-1}S_\epsilon[t_1\sqrt{v_\epsilon(0)}w\chi,t_2\sqrt{v_\epsilon(0)}w\chi]-L_\epsilon^{-1}G(a_1,a_2)\;:\;H_{2,ev}(\Omega_\epsilon)\mapsto H_{2,ev}(\Omega_\epsilon).$$

The existence of this fixed point is guaranteed by the contraction mapping principle. The details follow closely the analysis for the Gierer-Meinhardt system, see for example Section 3 in [30] or Section 5 in [38]. The existence of Type 1 solutions follows. □

For the existence of Type 2 solutions, the proof is similar and is omitted. The nonlocal eigenvalue problem is given in (4.15) and (4.16) and no transformation of eigenfunctions is required. The result about the kernel of the nonlocal eigenvalue problem is given in Proposition 3. As for Type 1 solutions it has to be shown that the kernel of the adjoint operator is trivial. To prove this result we have to consider the same NLEP as in (3.8) but now applied to ϕ_1 instead of $\hat{\phi}_1$. The proof concludes in the same way as for Type 1 solutions.

In the next two sections we consider the stability or instability of these solutions.

4. Stability I: The Eigenvalue Problem and the Large Eigenvalues

Now we study the (linearized) stability of this even steady state. To this end, we first derive the linearized operator around the steady state $(u_1^\epsilon,\,u_2^\epsilon,\,v^\epsilon)$ given in Theorem 1.

We perturb the steady state as follows:

$$u_1=u_1^\epsilon+\phi_1^\epsilon e^{\lambda t},\quad u_2=u_2^\epsilon+\phi_2^\epsilon e^{\lambda t},\quad v=v_\epsilon+\psi^\epsilon e^{\lambda t}.$$

By linearization, we obtain the following eigenvalue problem (dropping superscripts and subscripts ϵ):

$$\begin{cases}\lambda_\epsilon\phi_1=\epsilon^2\phi_{1,xx}-\phi_1+\dfrac{3\mu_1u_1^2\phi_1+\beta u_2^2\phi_1+2\beta u_1u_2\phi_2}{v}-\dfrac{\mu_1u_1^3+\beta u_1u_2^2}{v^2}\psi,\\ \lambda_\epsilon\phi_2=\epsilon^2\phi_{2,xx}-\phi_2+\dfrac{3\mu_2u_2^2\phi_2+\beta u_1^2\phi_2+2\beta u_1u_2\phi_1}{v}-\dfrac{\mu_2u_2^3+\beta u_1^2u_2}{v^2}\psi,\\ \tau\lambda_\epsilon\psi=D\psi_{xx}-\psi+(3\mu_1u_1^2\phi_1+\beta u_2^2\phi_1+2\beta u_1u_2\phi_2+3\mu_2u_2^2\phi_2+\beta u_1^2\phi_2+2\beta u_1u_2\phi_1)\epsilon^{-1},\end{cases}\tag{4.1}$$

where all components belong to the space $H_N^2(\Omega)$.

We now analyze the case $\lambda_\epsilon\to\lambda_0\neq0$ (large eigenvalues). We rescale $x=\epsilon y$, take the limit $\epsilon\to0$ in (4.1), and note that ϕ_i converges locally in $H^2(\Omega_\epsilon)$. Then we get for the first two components, using the approximations of u_1 and u_2 given in Theorem 1:

$$\begin{cases} \lambda_\epsilon \phi_1 \sim \phi_{1,yy} - \phi_1 + \dfrac{3\mu_1 u_1^2 \phi_1 + \beta u_2^2 \phi_1 + 2\beta u_1 u_2 \phi_2}{v(0)} - \dfrac{\mu_1 u_1^3 + \beta u_1 u_2^2}{v(0)^2} \psi(0), \\ \lambda_\epsilon \phi_2 \sim \phi_{2,yy} - \phi_2 + \dfrac{3\mu_2 u_2^2 \phi_2 + \beta u_1^2 \phi_2 + 2\beta u_1 u_2 \phi_1}{v(0)} - \dfrac{\mu_2 u_2^3 + \beta u_1^2 u_2}{v(0)^2} \psi(0), \end{cases} \tag{4.2}$$

where $v(0)$ is given by (2.11).

Now we calculate the term $\psi(0)$. We consider the special case $\tau = 0$ using the Green's function G_D given in (2.7).

Remark 5. *The case of small τ can be considered using a small perturbation of the case $\tau = 0$ as it can be shown that $|\lambda_\epsilon| \leq C$ for all eigenvalues such that $\lambda_\epsilon > -c_0$ with some small $c_0 > 0$, for example using a characterization of the eigenvalues by quadratic forms* [29]. *Alternatively, one could consider the case of arbitrary τ using a more general Green's function.*

From the third equation of (4.1) we get

$$\psi(0) \sim v(0) \left[[3\mu_1 t_1^2 + \beta t_2^2 + 2\beta t_1 t_2] \int_{\mathbb{R}} w^2 \phi_1 \, dy + [2\beta t_1 t_2 + 3\mu_2 t_2^2 + \beta t_1^2] \int_{\mathbb{R}} w^2 \phi_2 \, dy \right] G_D(0,0). \tag{4.3}$$

Recalling from (2.11) that

$$v(0) = \frac{1}{G_D(0,0)^2 (t_1 + t_2)^2 \left(\int_{\mathbb{R}} w^3 \right)^2} + O(\epsilon),$$

we get from (4.3)

$$\psi(0) \sim \frac{1}{G_D(0,0)(t_1+t_2)^2 (\int_{\mathbb{R}} w^3 dy)^2} \times \left[[3\mu_1 t_1^2 + \beta t_2^2 + 2\beta t_1 t_2] \int_{\mathbb{R}} w^2 \phi_1 \, dy + [2\beta t_1 t_2 + 3\mu_2 t_2^2 + \beta t_1^2] \int_{\mathbb{R}} w^2 \phi_2 \, dy \right]. \tag{4.4}$$

Then (4.2) gives the following nonlocal eigenvalue problem (NLEP)

$$\Delta \phi_1 - \phi_1 + [(2\mu_1 t_1^2 + 1)\phi_1 + 2\beta t_1 t_2 \phi_2] w^2 - \frac{t_1}{t_1 + t_2} \left[(2\mu_1 t_1^2 + 1 + 2\beta t_1 t_2) \int_{\mathbb{R}} w^2 \phi_1 \, dy + (2\mu_2 t_2^2 + 1 + 2\beta t_1 t_2) \int_{\mathbb{R}} w^2 \phi_2 \, dy \right] \frac{w^3}{\int_{\mathbb{R}} w^3 \, dy} = \lambda \phi_1, \tag{4.5}$$

$$\Delta \phi_2 - \phi_2 + [(2\mu_2 t_2^2 + 1)\phi_2 + 2\beta t_1 t_2 \phi_1] w^2 - \frac{t_2}{t_1 + t_2} \left[(2\mu_2 t_2^2 + 1 + 2\beta t_1 t_2) \int_{\mathbb{R}} w^2 \phi_2 \, dy + (2\mu_1 t_1^2 + 1 + 2\beta t_1 t_2) \int_{\mathbb{R}} w^2 \phi_1 \, dy \right] \frac{w^3}{\int_{\mathbb{R}} w^3 \, dy} = \lambda \phi_2, \tag{4.6}$$

where ϕ_1, $\phi_2 \in H^2(\mathbb{R})$.

We first diagonalize the local terms of the NLEP (4.5), (4.6). Written in vector form, the local terms are

$$\Delta \phi - \phi + \mathcal{B} \phi w^2, \tag{4.7}$$

where $\phi = (\phi_1, \phi_2)^T$ and

$$\mathcal{B} = \begin{pmatrix} 2\mu_1 t_1^2 + 1 & 2\beta t_1 t_2 \\ 2\beta t_1 t_2 & 2\mu_2 t_2^2 + 1 \end{pmatrix}. \tag{4.8}$$

Using (2.3), the eigenvalues of $\mathcal{B}$ are 3 and $3 - 2\beta(t_1^2 + t_2^2)$, with corresponding eigenvectors $(t_1, t_2)^T$ and $(-t_2, t_1)^T$, respectively.

Thus, setting

$$\hat{\phi}_1 = t_1\phi_1 + t_2\phi_2 \tag{4.9}$$

and

$$\hat{\phi}_2 = -t_2\phi_1 + t_1\phi_2, \tag{4.10}$$

the NLEP is transformed to

$$\Delta\hat{\phi}_1 - \hat{\phi}_1 + 3\hat{\phi}_1 w^2 - \left[3\int_{\mathbb{R}} w^2\hat{\phi}_1\,dy + \frac{t_1-t_2}{t_1+t_2}\left(3-2\beta(t_1^2+t_2^2)\right)\int_{\mathbb{R}} w^2\hat{\phi}_2\,dy\right]\frac{w^3}{\int w^3\,dy} = \lambda\hat{\phi}_1, \tag{4.11}$$

and

$$\Delta\hat{\phi}_2 - \hat{\phi}_2 + (3-2\beta(t_1^2+t_2^2))\hat{\phi}_2 w^2 = \lambda\hat{\phi}_2. \tag{4.12}$$

Note that the transformed NLEP has a special structure: the second equation is decoupled from the first equation and it is a local equation. Therefore it can be considered first. Only the first equation has a nonlocal term.

By Lemma 3.2 of [12] we have exact information about the eigenvalues of the NLEP

$$\Delta\phi - \phi + 3\phi w^2 - 3\int_{\mathbb{R}} w^2\phi\,dy\frac{w^3}{\int w^3\,dy} = \lambda\phi. \tag{4.13}$$

Using the identity $L_0 w^2 = 3w^2$, where $L_0\phi = \Delta\phi - \phi + 3\phi w^2$, it has been shown in [12] that the the point spectrum for the non-selfadjoint problem (4.13) is real, and it can be determined exactly. For the principal eigenvalue we have

$$\lambda = 3\left(1 - \frac{\int_{\mathbb{R}} w^5\,dy}{\int_{\mathbb{R}} w^3\,dy}\right) = 3\left(1-\frac{3}{2}\right) = -\frac{3}{2}$$

using $w(y) = \sqrt{2}\,\mathrm{sech}\,y$. Further, the continuous spectrum of (4.13) is $\lambda < -1$. The kernel of (4.13) equals span $\{w_y\}$.

In (4.12) we rewrite

$$3 - 2\beta(t_1^2+t_2^2) = 1 - 2g(\beta),$$

where $g(\beta) = \beta(t_1^2+t_2^2) - 1$. For stability, it will be crucial to determine the sign of $g(\beta)$. Using (2.3), (2.10), we compute

$$g(\beta) = (\beta-\mu_2)t_2^2 = \frac{(\beta-\mu_1)(\beta-\mu_2)}{\beta^2-\mu_1\mu_2}.$$

Since $t_1^2t_2^2 > 0$, from (2.10) we get $(\beta-\mu_1)(\beta-\mu_2) > 0$. Therefore $g(\beta)$ has the same sign as $\beta^2-\mu_1\mu_2$. Therefore, $g(\beta) > 0$ if $\beta > \max(\mu_1,\mu_2)$ and $g(\beta) < 0$ if $\beta < \min(\mu_1,\mu_2)$.

Then for the kernel of the NLEP (4.5), (4.6) we have the following result:

Proposition 2. *Suppose that*

$$\beta > \max(\mu_1,\mu_2) \text{ or } \quad \beta < \min(\mu_1,\mu_2).$$

If $\lambda = 0$, we get for the NLEP (4.5), (4.6)

$$\phi_1 = t_1\alpha w_y, \quad \phi_2 = t_2\alpha w_y, \quad \text{(with some real number } \alpha).$$

Proof. We apply Lemma 4.1 in [28] to the second equation of the transformed NLEP (4.11), (4.12) and get $\hat{\phi}_2 = 0$ since $g(\beta) \neq 0$. Then we apply Lemma 3.2 of [12] to the first equation and get $\hat{\phi}_1 = \alpha w_y$, where α is a real number.

Transforming back, for the kernel of the **original NLEP** (4.5), (4.6) we have

$$\phi_1 = t_1\alpha w_y, \quad \phi_2 = t_2\alpha w_y, \quad \text{(with some real number } \alpha).$$

□

In the next section, we will show that all the small eigenvalues must have negative real part. Here we will show the following:

Proof of Theorem 3: We consider the eigenvalues of the transformed NLEP (4.11), (4.12). We first consider the case $\beta > \max(\mu_1, \mu_2)$. Then we have $g(\beta) > 0$. If $\text{Re}(\lambda) \geq -c$ for some $c > 0$ small enough, and $\lambda \neq 0$, then by Lemma 4.1 (4) of [34] we have

$$\hat{\phi}_2 = 0.$$

Since $\hat{\phi}_2 = 0$, the first equation becomes

$$\Delta\hat{\phi}_1 - \hat{\phi}_1 + 3\hat{\phi}_1 w^2 - 3\int w^2\hat{\phi}_1\,dy\,\frac{w^3}{\int w^3\,dy} = \lambda\hat{\phi}_1.$$

Therefore, $\hat{\phi}_1$ satisfies (4.13). By Lemma 3.2 of [12] we have $\hat{\phi}_1 = 0$. Transforming back, this implies that $\phi_1 = \phi_2 = 0$, and there is no eigenvalue with $\text{Re}(\lambda) \geq -c$ for some $c > 0$ small enough and $\lambda \neq 0$. The stability part of Theorem 3 follows.

If $\beta < \min(\mu_1, \mu_2)$, it follows that $g(\beta) < 0$ and we will show that the spike solutions are unstable.

In fact, by Lemma 4.1 (3) of [34], for (4.16) there is an eigenfunction $\hat{\phi}_2$ with $\lambda > 0$. Using this eigenfunction $\hat{\phi}_2$ we can compute $\hat{\phi}_1$ as follows:

Let

$$\mathcal{L}\hat{\phi}_1 = \Delta\hat{\phi}_1 - \hat{\phi}_1 + 3\hat{\phi}_1 w^2 - \frac{3w^3}{\int w^3\,dy}\int w^2\hat{\phi}_1\,dy. \tag{4.14}$$

Note that the operator $\mathcal{L} - \lambda I : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible by Lemma 3.2 of [12]. Then we compute

$$\hat{\phi}_1 = -(\mathcal{L} - \lambda I)^{-1}\frac{t_1 - t_2}{t_1 + t_2}\left(3 - 2\beta(t_1^2 + t_2^2)\right)\int w^2\hat{\phi}_2\,dy\frac{w^3}{\int w^3\,dy}.$$

Transforming back, we get an eigenfunction (ϕ_1, ϕ_2) with eigenvalue $\lambda > 0$.

Arguing as in the proof of Theorem 1 of [1] the eigenvalue problem (4.5), (4.6) captures all converging sequences of eigenvalues λ_ϵ of (4.1) which converge to an eigenvalue λ with $\text{Re}(\lambda) > -1$. On the other hand, for any eigenvalue λ of (4.5), (4.6) with $\text{Re}(\lambda) > -1$ there is a converging sequence of eigenvalues λ_ϵ of (4.1) with λ as its limit. Therefore the eigenvalue problem (4.1) is stable concerning eigenvalues sequences λ_ϵ converging to a limit which is not zero. The case of zero limiting eigenvalue will be studied in Section 5 below. The result is stated in (5.20). Together, it follows that the proof of Theorem 3 is complete.

□

The proof of Theorem 4 follows the same strategy. For Type 2 spike solutions, using (4.2), the NLEP becomes

$$\Delta\phi_1 - \phi_1 + 3\phi_1 w^2 - \left[3\int_\Omega w^2\phi_1\,dy + \frac{\beta}{\mu_1}\int_\Omega w^2\phi_2\,dy\right]\frac{w^3}{\int_\Omega w^3\,dy} = \lambda\phi_1, \tag{4.15}$$

$$\Delta\phi_2 - \phi_2 + \frac{\beta}{\mu_1}\phi_2 w^2 = \lambda\phi_2, \tag{4.16}$$

where ϕ_1, $\phi_2 \in H^2(\mathbb{R})$.

The NLEP has a special structure: Only the first equation is a NLEP. The second equation is a decoupled local equation. No transformation of the eigenfunctions is required.

First, we have the following result about the kernel of the NLEP (4.15), (4.16):

Proposition 3. *Suppose that* $\frac{\beta}{\mu_1} \neq 1$. *If* $\lambda = 0$, *we get for the NLEP (4.15), (4.16)* $\phi_1 = \alpha w_y$, $\phi_2 = 0$, *where* α *is a real number.*

Proof. We apply Lemma 4.1 in [28] to the second equation and get $\phi_2 = 0$. Then we apply Lemma 3.2 of [12] to the first equation and get $\phi_1 = \alpha w_y$, where α is a real number. □

Proof of Theorem 4: We consider the eigenvalues of the NLEP (4.15), (4.16). If $\mathrm{Re}(\lambda) \geq -c$ for some $c > 0$ small enough, and $\lambda \neq 0$, we get $\phi_2 = 0$, provided that $\frac{\beta}{\mu_1} < 1$, by using Lemma 4.1 (4) of [34]. Then we apply Lemma 3.2 of [12] to the first equation and get $\phi_1 = 0$. This implies **stability** of the eigenvalue problem (4.15), (4.16).

On the other hand, if $\frac{\beta}{\mu_1} > 1$ we can construct an unstable eigenfunction, first for ϕ_2, and then also for ϕ_1. In fact, by Lemma 4.1 (3) of [34], for (4.16) there is an eigenfunction ϕ_2 with $\lambda > 0$. Using this eigenfunction ϕ_2 we can compute ϕ_1 as follows:

Let $\mathcal{L}$ be the operator defined in (4.14) but now applied to ϕ_1 instead of $\hat{\phi}_1$. Recall that the operator $\mathcal{L} - \lambda I : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible by Lemma 3.2 of [12]. Then we compute

$$\phi_1 = -(\mathcal{L} - \lambda I)^{-1} \frac{\beta}{\mu_1} \int w^2 \phi_2 \, dy \frac{w^3}{\int w^3 \, dy}.$$

Thus (ϕ_1, ϕ_2) is an eigenfunction for eigenvalue $\lambda > 0$.

Again, using the argument [1], the the eigenvalue problem (4.1) is stable concerning eigenvalue sequences λ_ϵ whose limit is not zero. The case of zero limiting eigenvalue will be studied in Section 5 below. The result is stated in (5.22). Together, it follows that the proof of Theorem 4 is complete.

□

In the next section we complete the proof of Theorems 3 and 4 by considering small eigenvalues λ_ϵ which converge to zero.

5. Stability II: The Small Eigenvalues

Now we study the small eigenvalues for (4.1), namely those with $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. For simplicity, we set $\tau = 0$. Since $\tau\lambda_\epsilon \ll 1$ the results in this section are also valid for τ finite. The case of general $\tau > 0$ can be treated as in Section 7 of [33]. We will show that the small eigenvalues are of order $O(\epsilon^2)$.

For given $f \in L^2(\Omega)$, let $T[f]$ be the unique solution in $H^2_N(\Omega)$ of the problem

$$D\Delta(T[f]) - T[f] + \epsilon^{-1} f = 0. \tag{5.1}$$

We present the argument in detail for Type 1 solutions. We will explain the differences for Type 2 and Type 3 solutions in Remark 6.

By Theorem 1 we have for the spiky steady states

$$u_1^\epsilon = \hat{t}_1 w_\epsilon + O(\epsilon), \quad u_2^\epsilon = \hat{t}_2 w_\epsilon + O(\epsilon),$$

$$v_\epsilon = T[\mu_1 (u_1^\epsilon)^3 + \beta u_1^\epsilon (u_2^\epsilon)^2 + \mu_2 (u_2^\epsilon)^3 + \beta (u_1^\epsilon)^2 u_2^\epsilon], \tag{5.2}$$

where

$$\hat{t}_i = \sqrt{v_\epsilon(0)} t_i. \tag{5.3}$$

After rescaling $x = \epsilon y$ for the first two components, the eigenvalue problem (4.1) becomes

$$\begin{aligned}
\lambda_\epsilon \phi_1 &= \phi_{1,yy} - \phi_1 + \frac{3\mu_1 (u_1^\epsilon)^2 \phi_1 + \beta (u_2^\epsilon)^2 \phi_1 + 2\beta u_1^\epsilon u_2^\epsilon \phi_2}{v_\epsilon} - \frac{\mu_1 (u_1^\epsilon)^3 + \beta u_1^\epsilon (u_2^\epsilon)^2}{v_\epsilon^2} \psi, \\
\lambda_\epsilon \phi_2 &= \phi_{2,yy} - \phi_2 + \frac{3\mu_2 (u_2^\epsilon)^2 \phi_2 + \beta (u_1^\epsilon)^2 \phi_2 + 2\beta u_1^\epsilon u_2^\epsilon \phi_1}{v_\epsilon} - \frac{\mu_2 (u_2^\epsilon)^3 + \beta (u_1^\epsilon)^2 u_2^\epsilon}{v_\epsilon^2} \psi, \\
\tau \lambda_\epsilon \psi &= D\psi_{xx} - \psi + \left[3\mu_1 (u_1^\epsilon)^2 \phi_1 + \beta (u_2^\epsilon)^2 \phi_1 + 2\beta u_1^\epsilon u_2^\epsilon \phi_2 + 3\mu_2 (u_2^\epsilon)^2 \phi_2 + \beta (u_1^\epsilon)^2 \phi_2 + 2\beta u_1^\epsilon u_2^\epsilon \phi_1\right] \epsilon^{-1},
\end{aligned} \tag{5.4}$$

where the unknown functions ϕ_1, ϕ_2 are in $H_N^2(\Omega_\epsilon)$ and ψ is in $H_N^2(\Omega)$.

Let us define

$$\tilde{u}_{\epsilon,j}(\epsilon y) = \chi(\epsilon y) u_j^\epsilon(\epsilon y), \quad j = 1, 2, \tag{5.5}$$

where χ is the smooth, even cut-off function defined in (3.1). Then

$$\tilde{u}_{\epsilon,j}(x) = u_j^\epsilon(x) + \text{e.s.t.}, \quad j = 1, 2, \tag{5.6}$$

where e.s.t. denotes an exponentially small term in $H_N^2(\Omega_\epsilon)$. We note that $\tilde{u}_{\epsilon,j}$, $j = 1, 2$ are even functions.

Next we transform the eigenfunctions as in (4.9), (4.10):

$$\hat{\phi}_1 = t_1 \phi_1 + t_2 \phi_2,$$

$$\hat{\phi}_2 = -t_2 \phi_1 + t_1 \phi_2.$$

The transformed eigenvalue problem becomes in leading order

$$\begin{aligned}
\lambda_\epsilon \hat{\phi}_1 &= \hat{\phi}_{1,yy} - \hat{\phi}_1 + 3w^2 \chi^2 \hat{\phi}_1 \frac{v_\epsilon(0)}{v_\epsilon}(1 + O(\epsilon)) - (t_1^2 + t_2^2) w^3 \chi^3 \frac{(v_\epsilon(0))^{3/2} \hat{\psi}}{v_\epsilon^2}(1 + O(\epsilon)), \\
\lambda_\epsilon \hat{\phi}_2 &= \hat{\phi}_{2,yy} - \hat{\phi}_2 + (3 - 2\beta(t_1^2 + t_2^2)) w^2 \chi^2 \hat{\phi}_2 \frac{v_\epsilon(0)}{v_\epsilon}(1 + O(\epsilon)), \\
\tau \lambda_\epsilon \hat{\psi} &= D\hat{\psi}_{xx} - \hat{\psi} + \left[3w^2 \chi^2 \hat{\phi}_1 + \tfrac{t_1 - t_2}{t_1 + t_2}(3 - 2\beta(t_1^2 + t_2^2)) w^2 \chi^2 \hat{\phi}_2\right] v_\epsilon(0) \epsilon^{-1}(1 + O(\epsilon)),
\end{aligned} \tag{5.7}$$

where all unknown functions $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\psi}$ are in $H_N^2(\Omega)$.

Next we define the approximate kernel and co-kernel for the transformed eigenvalue problem (5.7)

$$\mathcal{K}_\epsilon := \text{span}\left\{\left(t_1 \frac{d}{dy}\tilde{u}_{\epsilon,1}(\epsilon y) + t_2 \frac{d}{dy}\tilde{u}_{\epsilon,2}(\epsilon y), 0\right)\right\} \subset (H_N^2(\Omega_\epsilon))^2,$$

$$\mathcal{C}_\epsilon := \text{span}\left\{\left(t_1 \frac{d}{dy}\tilde{u}_{\epsilon,1}(\epsilon y) + t_2 \frac{d}{dy}\tilde{u}_{\epsilon,2}(\epsilon y), 0\right)\right\} \subset (L^2(\Omega_\epsilon))^2,$$

where $\Omega_\epsilon = \left(-\frac{L}{\epsilon}, \frac{L}{\epsilon}\right)$.

Note that, by Theorem 1, $\tilde{u}_{\epsilon,j}$ satisfies

$$\Delta_y \tilde{u}_{\epsilon,j} - \tilde{u}_{\epsilon,j} + \frac{\mu_j \tilde{u}_{\epsilon,j}^3 + \beta \tilde{u}_{\epsilon,j} \tilde{u}_{\epsilon,3-j}^2}{v_\epsilon} + \text{e.s.t} = 0, \quad j = 1, 2.$$

Thus $\tilde{u}'_{\epsilon,j} := \frac{d\tilde{u}_{\epsilon,j}}{dy}$, $v'_\epsilon := \epsilon \frac{dv_\epsilon(x)}{dx}$ satisfies

$$\Delta_{yy} \tilde{u}'_{\epsilon,j} - \tilde{u}'_{\epsilon,j} + \frac{3\mu_1 \tilde{u}_{\epsilon,j}^2 + \beta \tilde{u}_{\epsilon,3-j}^2}{v_\epsilon} \tilde{u}'_{\epsilon,j} + \frac{2\beta \tilde{u}_{\epsilon,j} \tilde{u}_{\epsilon,3-j}}{v_\epsilon} \tilde{u}'_{\epsilon,3-j} - \frac{\mu_j \tilde{u}_{\epsilon,j}^3 + \beta \tilde{u}_{\epsilon,j} \tilde{u}_{\epsilon,3-j}^2}{(v_\epsilon)^2} v'_\epsilon + \text{e.s.t} = 0. \tag{5.8}$$

This implies

$$(t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'})_{yy}-(t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'})+\frac{3w^2\chi^2v_\epsilon(0)}{v_\epsilon}(t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'})$$
$$-\frac{(t_1^2+t_2^2)\chi^3w^3(v_\epsilon(0))^{3/2}}{(v_\epsilon)^2}v_\epsilon^{'}+O(\epsilon)=0. \tag{5.9}$$

Let us now decompose $\hat{\phi}_\epsilon=(\hat{\phi}_{\epsilon,1},\hat{\phi}_{\epsilon,2})$, where

$$\hat{\phi}_{\epsilon,1}=a^\epsilon(t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'})+\phi_\epsilon^\perp, \tag{5.10}$$

with complex numbers a^ϵ. Here the factor ϵ is for scaling purposes, to achieve that a^ϵ is of order $O(1)$, and

$$(\phi_\epsilon^\perp,0)\in\mathcal{K}_\epsilon^\perp,$$

where orthogonality is taken with respect to the scalar product of the product space $(L^2(\Omega_\epsilon))^2$. We will show that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)}=O(\epsilon^2),\quad \|\hat{\phi}_{\epsilon,2}\|_{H^2(\Omega_\epsilon)}=O(\epsilon^2),$$

and so $\phi_\epsilon^\perp$ and $\hat{\phi}_{\epsilon,2}$ will not play a leading role in our results.

Suppose that $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}=1$. Then $|a^\epsilon|\le C$.

Similarly, we decompose

$$\psi_\epsilon=a^\epsilon\psi_{\epsilon,1}+\psi_\epsilon^\perp, \tag{5.11}$$

where $\psi_{\epsilon,1}$ satisfies

$$\psi_{\epsilon,1}=T[3w^2\chi^2(t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'})]v_\epsilon(0)(1+O(\epsilon)) \tag{5.12}$$

and $\psi_\epsilon^\perp$ is given by

$$\psi_\epsilon^\perp=T[3w^2\chi^2\phi_\epsilon^\perp+\frac{t_1-t_2}{t_1+t_2}(3-2\beta(t_1^2+t_2^2))w^2\chi^2\hat{\phi}_2]v_\epsilon(0)(1+O(\epsilon)). \tag{5.13}$$

By the second equation of (5.7) we get

$$\hat{\phi}_{\epsilon,2,yy}-(1+\lambda_\epsilon)\hat{\phi}_{\epsilon,2}+(3-2\beta(t_1^2+t_2^2))w^2\chi^2\hat{\phi}_{\epsilon,2}\frac{v_\epsilon(0)}{v_\epsilon(\epsilon y)}(1+O(\epsilon))=0.$$

Since

$$\frac{v_\epsilon(0)}{v_\epsilon(\epsilon y)}=1-\frac{1}{2(v_\epsilon(0))}v_\epsilon''(0)\epsilon^2y^2+O(\epsilon^3|y|^3)=1+O(\epsilon^2y^2) \tag{5.14}$$

we get $\|\hat{\phi}_{\epsilon,2}\|_{H^2(\Omega_\epsilon)}=O(\epsilon^2)$. Here we have used that $v_\epsilon(\epsilon y)$ is an even function and so $v_\epsilon'(0)=0$ and the $O(\epsilon)$ term vanishes.

Substituting the decomposition of $\hat{\phi}_{\epsilon,1}$ and ψ_ϵ as well as $\hat{\phi}_{\epsilon,2}$ into the first part of (5.7) we have

$$\epsilon\left(a^\epsilon(t_1^2+t_2^2)w^3\chi^3\frac{v_\epsilon(0)^{3/2}}{v_\epsilon^2}v_\epsilon^{'}-a^\epsilon(t_1^2+t_2^2)\frac{w^3\chi^3(v_\epsilon(0))^{3/2}}{v_\epsilon^2}\psi_{\epsilon,1}\right)$$
$$+\Delta\phi_\epsilon^\perp-\phi_\epsilon^\perp+3w^2\chi^2\frac{v_\epsilon(0)}{v_\epsilon}\phi_\epsilon^\perp-\frac{(t_1^2+t_2^2)w^3\chi^3(v_\epsilon(0))^{3/2}}{v_\epsilon^2}\psi_\epsilon^\perp-\lambda_\epsilon\phi_\epsilon^\perp+\text{e.s.t}$$
$$=\lambda_\epsilon a^\epsilon(t_1^2+t_2^2)\sqrt{v_\epsilon(0)}w'(1+o(1)), \tag{5.15}$$

since $t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'}$ satisfies (5.9) and $t_1\tilde{u}_{\epsilon,1}^{'}+t_2\tilde{u}_{\epsilon,2}^{'}\sim(t_1^2+t_2^2)\sqrt{v_\epsilon(0)}w'$.

Using $\|\psi_\epsilon^\perp\|=O(\|\phi_\epsilon\|^\perp)+O(\epsilon^2)$, we derive $\|\phi_\epsilon^\perp\|=O(\epsilon^2)$ and $\|\psi_\epsilon^\perp\|=O(\epsilon^2)$ since the operator $\mathcal{L}-\lambda I$ with $\mathcal{L}$ defined in (4.14) is invertible by Lemma 3.2 of [12].

Multiplying both sides of (5.15) by w' and integrating, the l.h.s and the r.h.s of (5.15) become

$$\text{l.h.s.} = a^\epsilon (t_1^2 + t_2^2)\sqrt{v_\epsilon(0)} \int_{\mathbb{R}} w^3 \chi^3 \frac{v_\epsilon(0)}{v_\epsilon^2}(v_\epsilon' - \psi_{\epsilon,1}) w' \, dy + O(\epsilon^3) \tag{5.16}$$

and

$$\text{r.h.s.} = \lambda_\epsilon a^\epsilon (t_1^2 + t_2^2)\sqrt{v_\epsilon(0)} \int_{\mathbb{R}} (w'(y))^2 \, dy \, (1 + o(1)), \tag{5.17}$$

respectively.

Note that the integrals resulting from the second line of (5.15) are in leading order a product of an even function of order $O(\epsilon^2)$ and an odd function. Therefore, for the integrals the terms of $O(\epsilon^2)$ vanish and they can estimated by $O(\epsilon^3)$.

Using the Green's function representation of v_ϵ and ψ_ϵ we compute using (2.7)

$$\begin{aligned}
v_\epsilon'(\epsilon y) - \psi_{\epsilon,1}(\epsilon y) &= -(t_1+t_2)(v_\epsilon(0))^{3/2}\epsilon \int_{\mathbb{R}} \left(\nabla_1 H_D(\epsilon y, \epsilon z) w^3(z) - H_D(\epsilon y, \epsilon z) \frac{d}{dz} w^3(z) \right) dz + O(\epsilon^2) \\
&= -(t_1+t_2)(v_\epsilon(0))^{3/2}\epsilon \int_{\mathbb{R}} (\nabla_1 H_D(\epsilon y, \epsilon z) + \nabla_2 H_D(\epsilon y, \epsilon z)) w^3(z) \, dz + O(\epsilon^2) \\
&= -(t_1+t_2)(v_\epsilon(0))^{3/2}\epsilon (\nabla_1 H_D(\epsilon y, 0) + \nabla_2 H_D(\epsilon y, 0)) \int_{\mathbb{R}} w^3(z) \, dz + O(\epsilon^2),
\end{aligned}$$

where $\nabla_1 H_D(P,Q) = \frac{\partial}{\partial P} H_D(P,Q)$ and $\nabla_2 H_D(P,Q) = \frac{\partial}{\partial Q} H_D(P,Q)$.

We have used that the contribution from K_D vanishes. This can be seen as follows: since $K_D(x,y) = K_D(|x-y|)$ we compute

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\partial}{\partial y} K_D(|\epsilon y - \epsilon z|) w^3(z) \, dz &= -\int_{\mathbb{R}} \left(\frac{\partial}{\partial z} K_D(|\epsilon y - \epsilon z|) \right) w^3(z) \, dz \\
&= \int_{\mathbb{R}} K_D(|\epsilon y - \epsilon z|) \frac{d}{dz} w^3(z) \, dz.
\end{aligned}$$

Compare Section 7 of [33].

For (5.16), we get

$$\begin{aligned}
&\text{l.h.s.} = \\
&= -\epsilon a^\epsilon (t_1^2 + t_2^2)\sqrt{v_\epsilon(0)} \int_{\mathbb{R}} (t_1+t_2)\sqrt{v_\epsilon(0)} w'(y) w^3(y) (\nabla_1 H_D(\epsilon y, 0) + \nabla_2 H_D(\epsilon y, 0)) \, dy \\
&\qquad \times \int_{\mathbb{R}} w^3(z) \, dz + O(\epsilon^3) \\
&= -\epsilon^2 a^\epsilon (t_1^2 + t_2^2)(t_1+t_2) v_\epsilon(0) \int_{\mathbb{R}} w'(y) w^3(y) y \, dy ((\nabla_1)^2 H_D(0,0) + \nabla_1 \nabla_2 H_D(0,0)) \\
&\qquad \times \int_{\mathbb{R}} w^3(z) dz + O(\epsilon^3) \\
&= \epsilon^2 a^\epsilon (t_1^2 + t_2^2)(t_1+t_2) v_\epsilon(0) \left(\frac{1}{4} \int_{\mathbb{R}} w^4(y) \, dy \right) \left(\int_{\mathbb{R}} w^3(z) \, dz \right) \\
&\qquad \times ((\nabla_1)^2 H_D(0,0) + \nabla_1 \nabla_2 H_D(0,0)) + O(\epsilon^3) \\
&= \epsilon^2 a^\epsilon (t_1^2 + t_2^2)(t_1+t_2) v_\epsilon(0) \left(\frac{1}{4} \int_{\mathbb{R}} w^4 \, dy \right) \left(\int_{\mathbb{R}} w^3 \, dz \right) \\
&\qquad \times \left(\nabla_P^2 H_D(P,Q)|_{P=Q=0} + \nabla_P \nabla_Q H_D(P,Q)|_{P=Q=0} \right) + O(\epsilon^3) \\
&= \epsilon^2 a^\epsilon (t_1^2 + t_2^2)(t_1+t_2) v_\epsilon(0) \left(\frac{1}{8} \int_{\mathbb{R}} w^4 \, dy \right) \left(\int_{\mathbb{R}} w^3 \, dz \right) \nabla_P^2 H_D(P,P)|_{P=0} + O(\epsilon^3).
\end{aligned} \tag{5.18}$$

Combining (5.17) and (5.18), the small eigenvalues λ_ϵ satisfy

$$\lambda_\epsilon \sim \epsilon^2(t_1+t_2)\sqrt{v_\epsilon(0)}\Big(\frac{1}{8}\int_{\mathbb{R}} w^4\,dy\Big)\Big(\int_{\mathbb{R}} w^3\,dz\Big)\frac{1}{\int_{\mathbb{R}}(w')^2\,dy}\nabla_P^2 H_D(P,P)|_{P=0}. \tag{5.19}$$

Using (2.11), we get

$$\lambda_\epsilon \sim \epsilon^2\frac{\int_{\mathbb{R}} w^4\,dy}{8\int_{\mathbb{R}}(w')^2\,dy}\,\frac{\nabla_P^2 H_D(P,P)|_{P=0}}{G_D(0,0)}. \tag{5.20}$$

Since $G_D(0,0)>0$ and $\nabla_P^2 H_D(P,P)<0$ it follows that $\lambda_\epsilon<0$. These inequalities follows from (2.7) as follows:

$$G_D(P,P) = \frac{\theta}{\sinh(2\theta L)}\cosh\theta(L+P)\cosh\theta(L-P),$$

$$G_D(0,0) = \frac{\theta}{\sinh(2\theta L)}\cosh^2(\theta L) = \frac{\theta}{2}\coth(\theta L) > 0,$$

$$\nabla_P^2 H_D(P,P)|_{P=0} = -\nabla_P^2 G_D(P,P)|_{P=0} = -\frac{2\theta^3}{\sinh(2\theta L)}\cosh^2(2\theta P)|_{P=0} = -\frac{2\theta^3}{\sinh(2\theta L)} < 0.$$

Remark 6. *For Type 2 and Type 3 spikes we can make similar computations for the small eigenvalues. We do not have to make the transformation of the eigenfunctions and we use the same Green's function G_D. We get*

$$\lambda_\epsilon \sim \epsilon^2 t_1\sqrt{v_\epsilon(0)}\Big(\frac{1}{8}\int_{\mathbb{R}} w^4\,dy\Big)\Big(\int_{\mathbb{R}} w^3(z)\,dz\Big)\frac{1}{\int_{\mathbb{R}}(w')^2\,dy}\nabla_P^2 H_D(P,P)|_{P=0}. \tag{5.21}$$

Using (2.15), we get

$$\lambda_\epsilon \sim \frac{\int_{\mathbb{R}} w^4\,dy}{8\int_{\mathbb{R}}(w')^2\,dy}\,\frac{\nabla_P^2 H_D(P,P)|_{P=0}}{G_D(0,0)}. \tag{5.22}$$

We can see that the small eigenvalues for the Type 2 and Type 3 solutions are the same as for the Type 1 solutions. Since $G_D(0,0)>0$ and $\nabla_P^2 H_D(P,P)<0$ it follows that $\lambda_\epsilon<0$.

In the next section we present some numerical computations of the solutions.

6. Numerical Simulations

For the numerical simulations we use the domain $\Omega=(-1,1)$ and Neumann boundary conditions for all components.

The pictures show the numerically obtained long-term limit of the three components u_1, u_2, v, i.e. the state at $t=3,000$. It has been observed numerically that for larger times the solution remains at this steady state and does not change anymore. This confirms the analytical result that the steady state with spikes for the two activators is stable.

The choice of constants for the numerical simulations has been motivated by the analysis. In particular, ϵ^2 has to be rather small compared to D and μ_1,μ_2,β have to be chosen so that the assumptions of the stability results Theorem 3 and Theorem 4, respectively, are satisfied.

Different types of stable solutions have been computed.

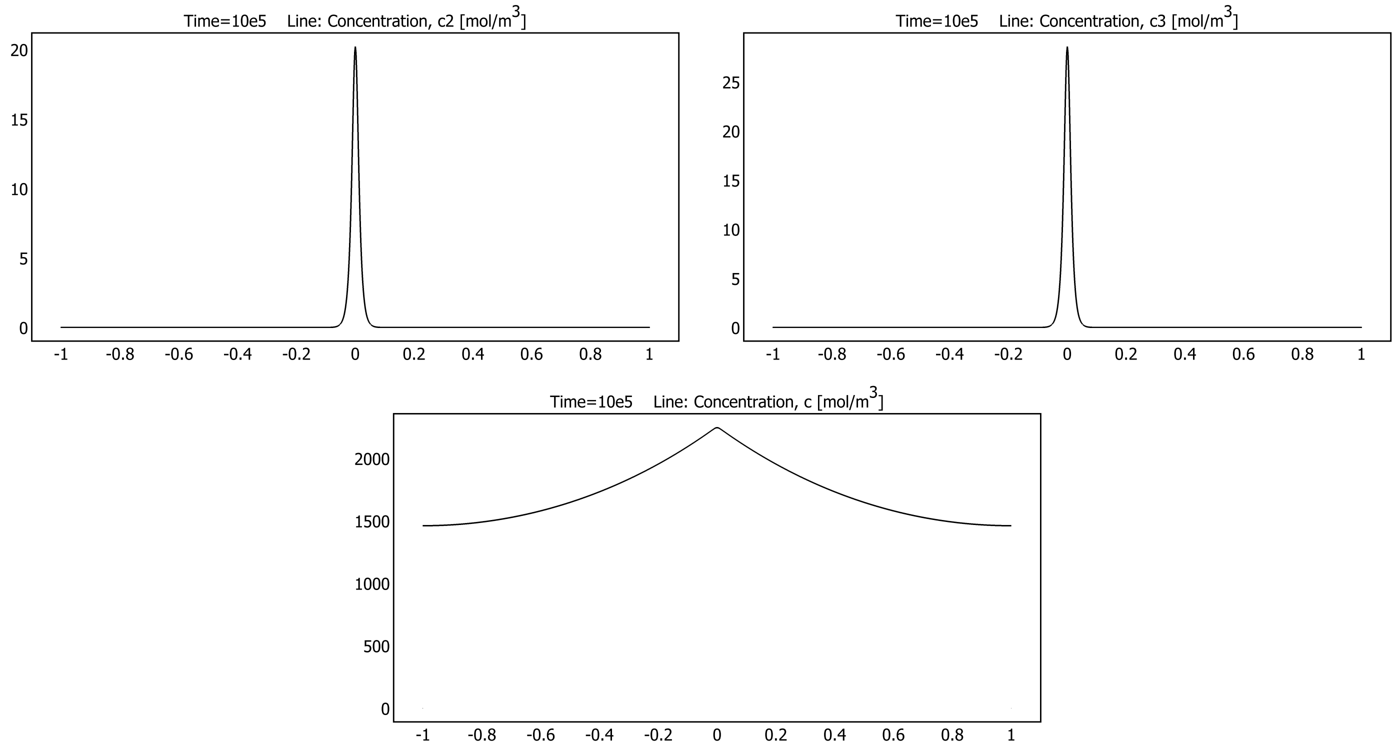

Figure 1. Type 1 Solution ($u_1 > 0$, $u_2 > 0$), Color Green, Single Spike,

$$\epsilon^2 = 0.0001,\ D = 1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 5.$$

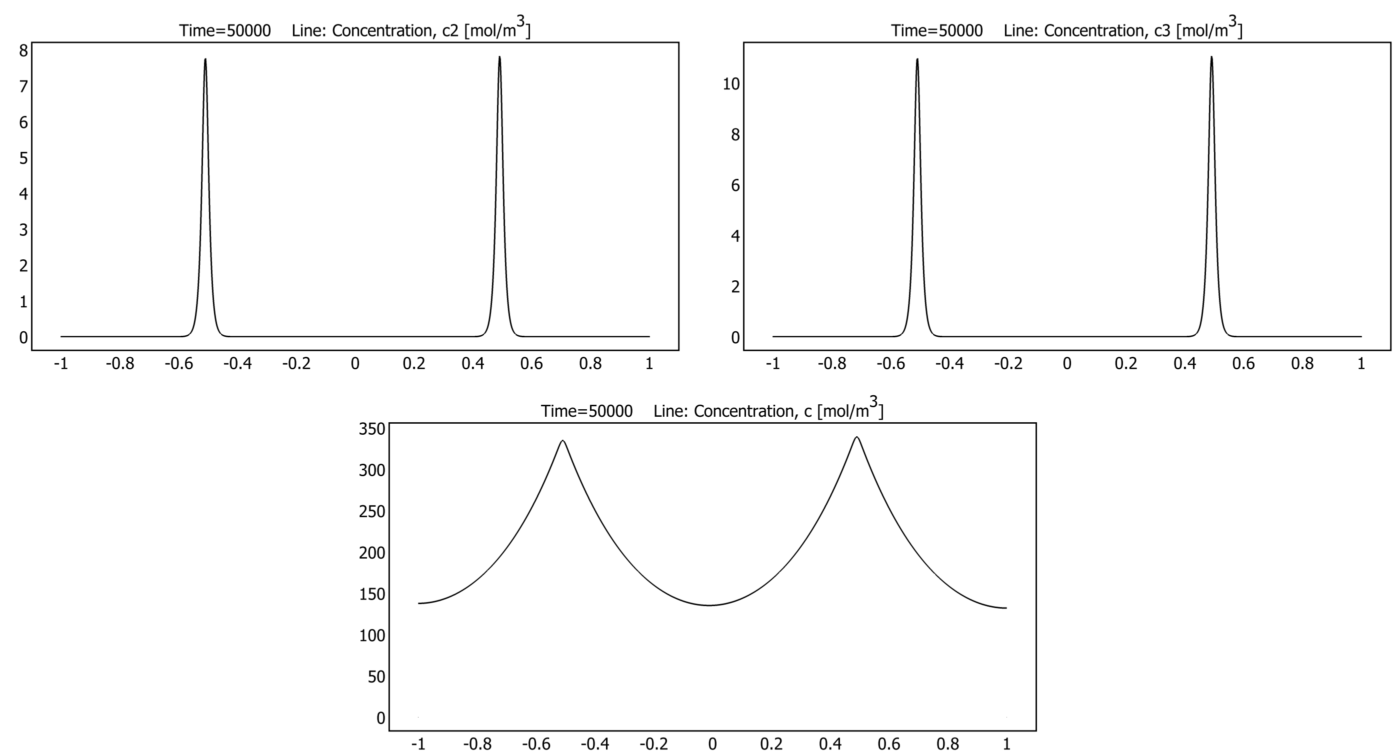

Figure 2. Type 1 Solution ($u_1 > 0$, $u_2 > 0$), Color Green, 2 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 5.$$

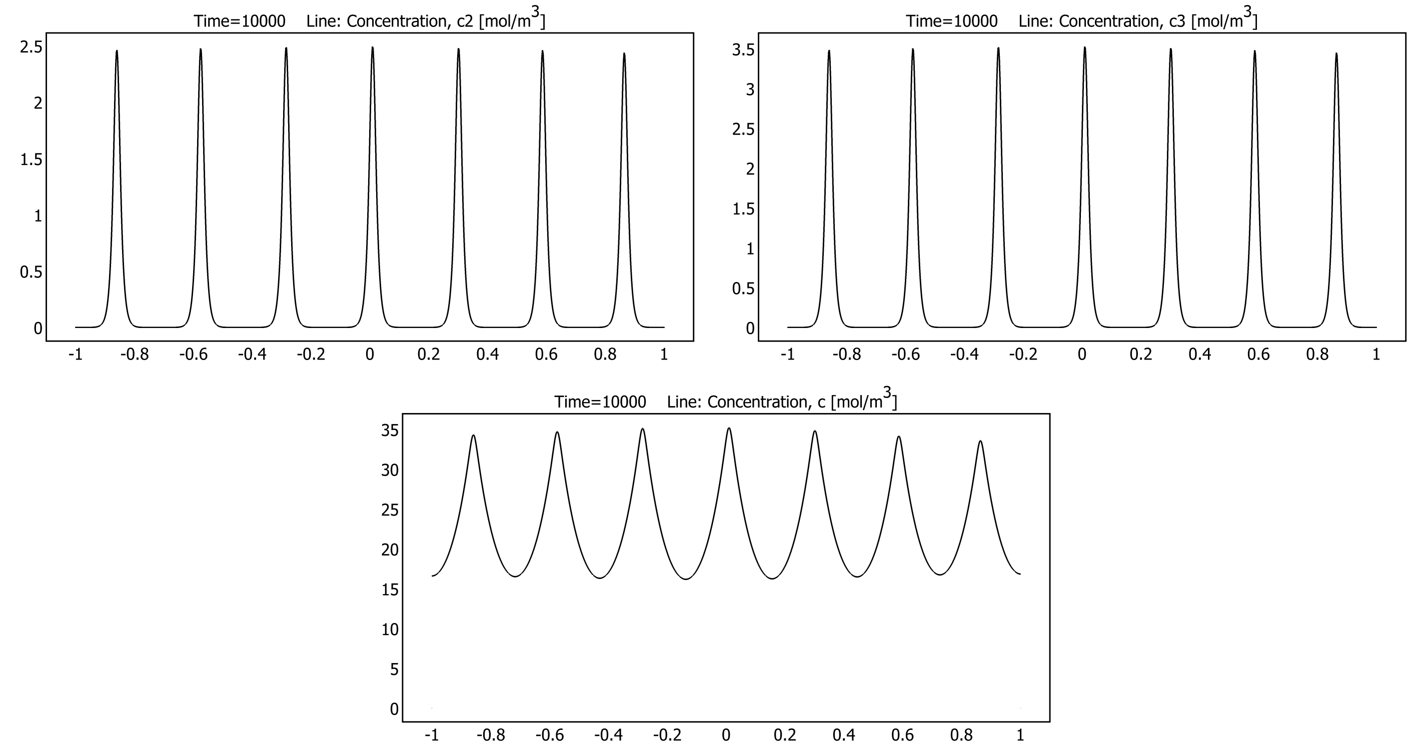

Figure 3. Type 1 Solution ($u_1 > 0$, $u_2 > 0$), Color Green, 7 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.01,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 5.$$

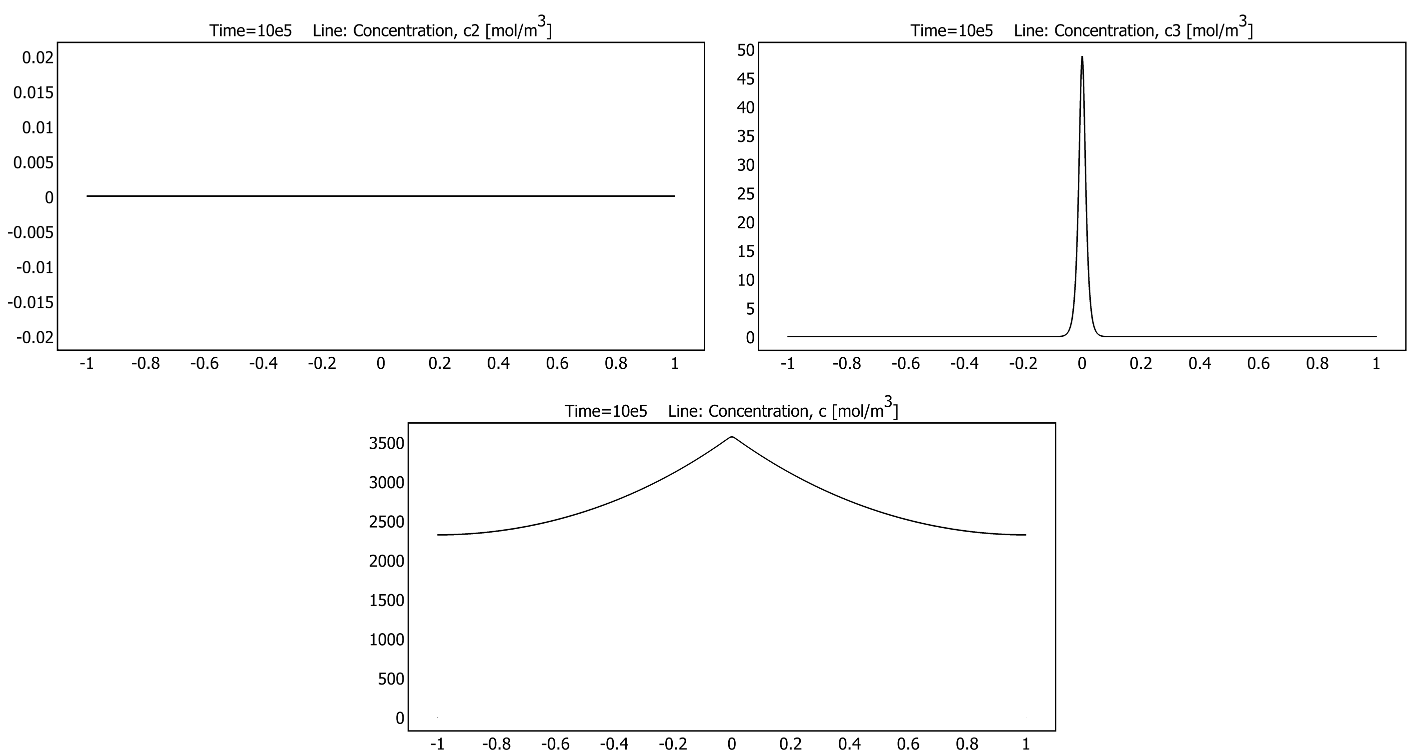

Figure 4. Type 3 Solution ($u_1 = 0$, $u_2 > 0$), Color Yellow, Single Spike,

$$\epsilon^2 = 0.0001,\ D = 1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 2.$$

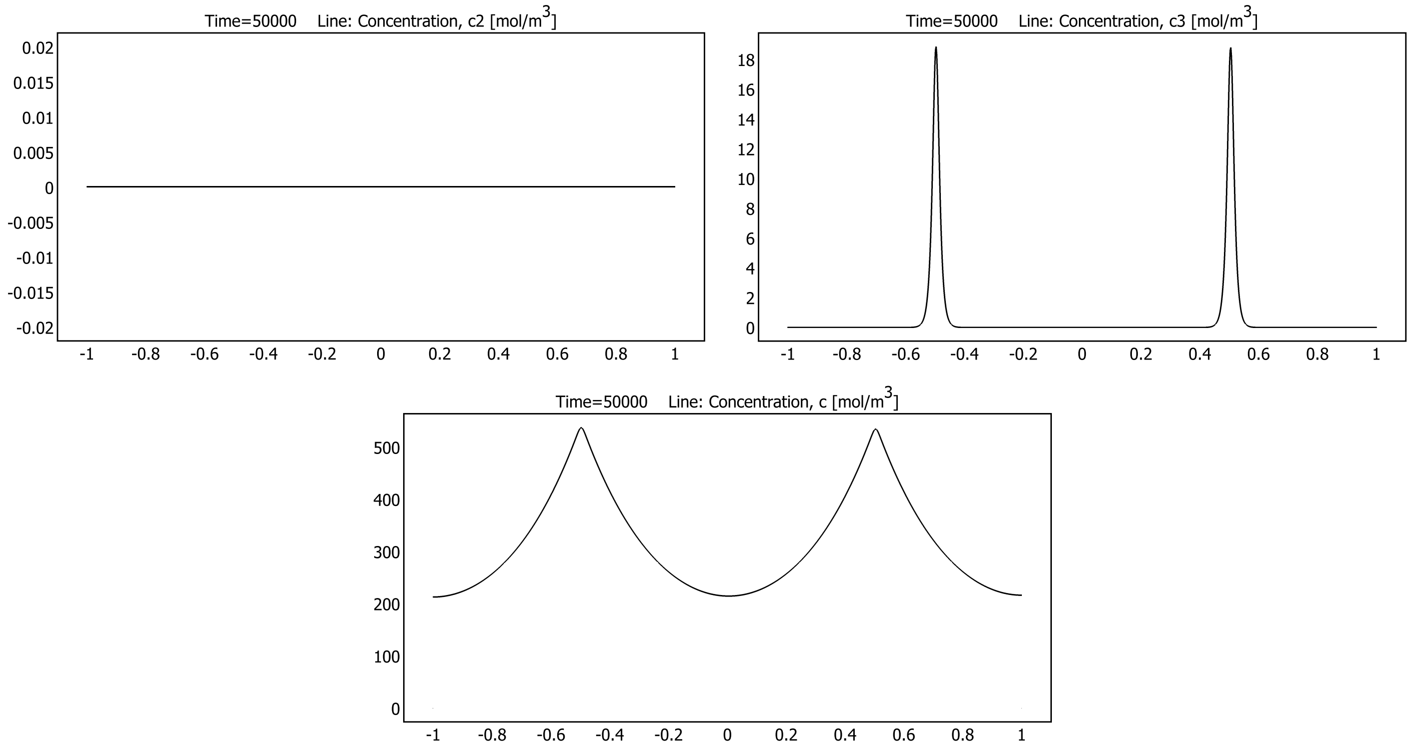

Figure 5. Type 3 Solution ($u_1 = 0,\ u_2 > 0$), Color Yellow, 2 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 2.$$

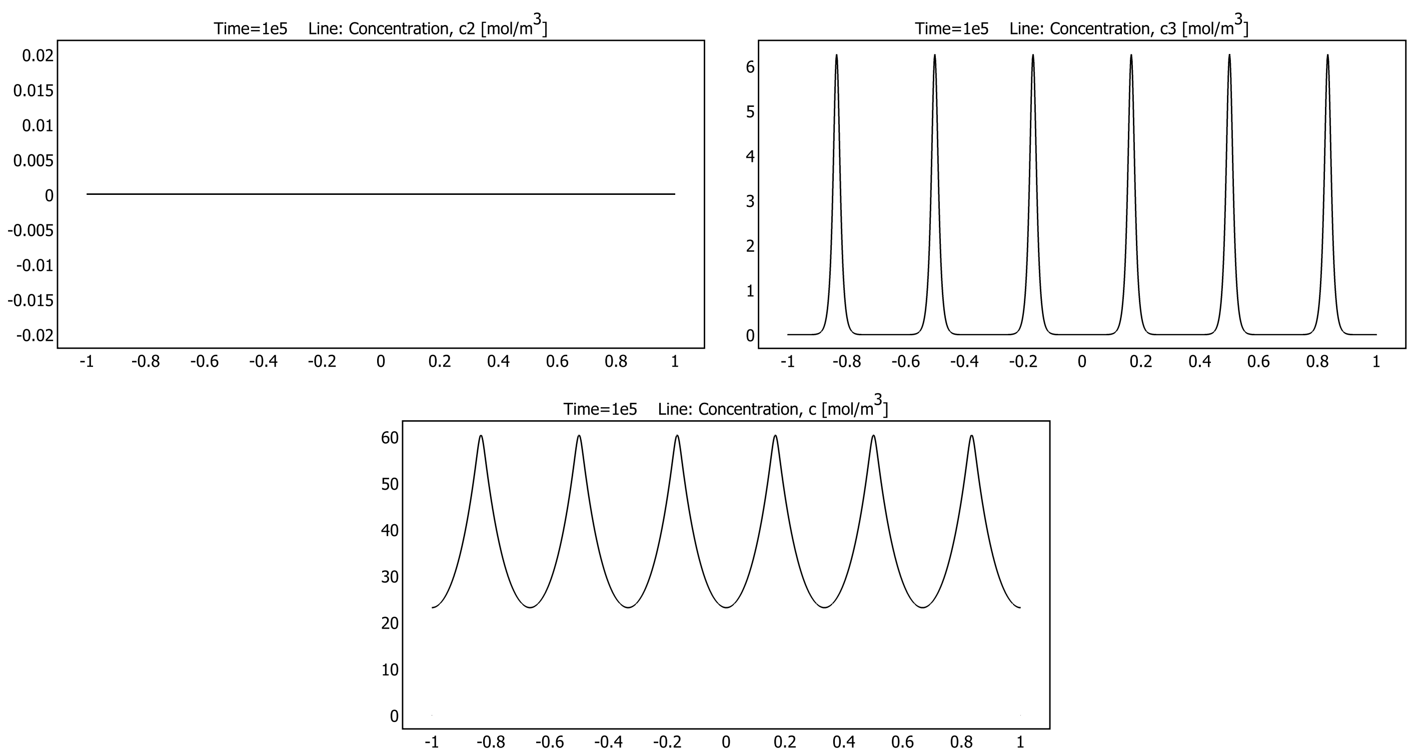

Figure 6. Type 3 Solution ($u_1 = 0,\ u_2 > 0$), Color Yellow, 6 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.01,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 2.$$

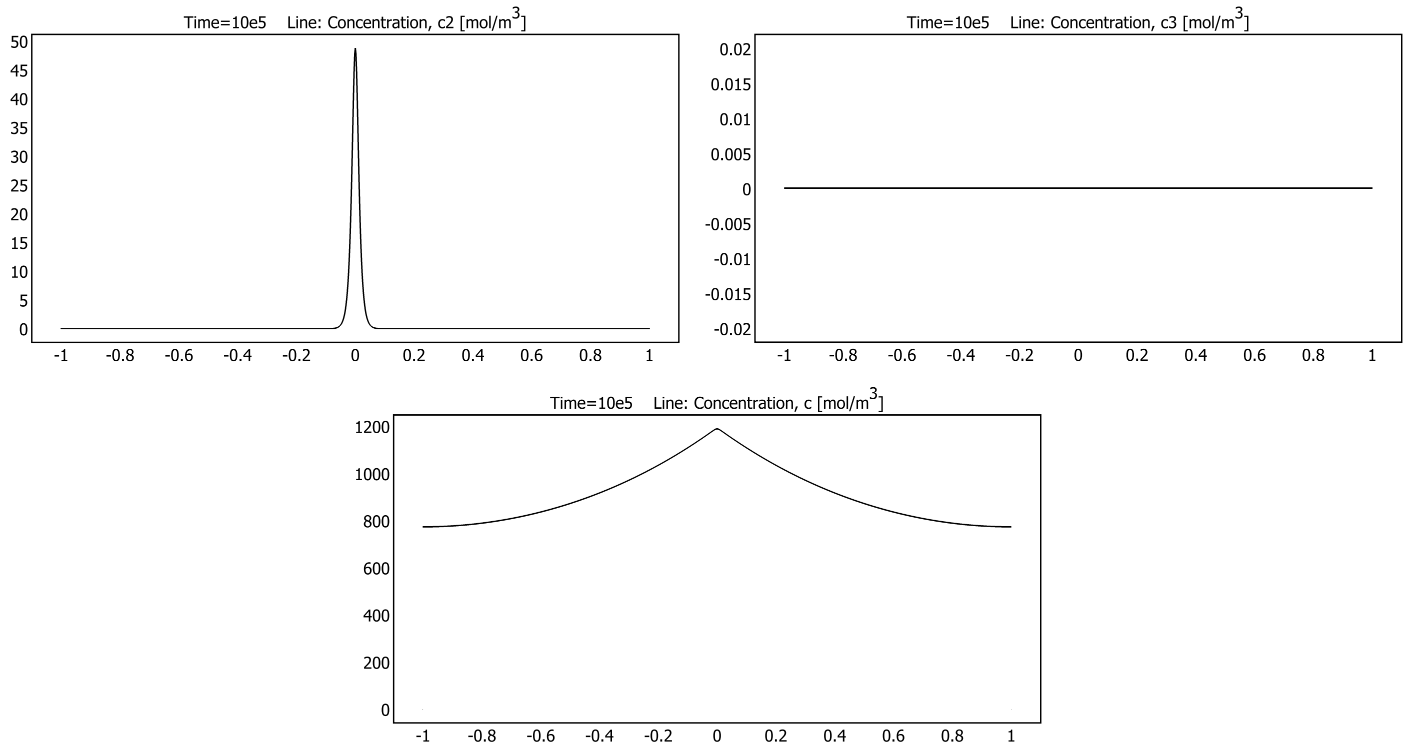

Figure 7. Type 2 Solution ($u_1 > 0$, $u_2 = 0$), Color Blue, Single Spike,

$$\epsilon^2 = 0.0001,\ D = 1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 0.5.$$

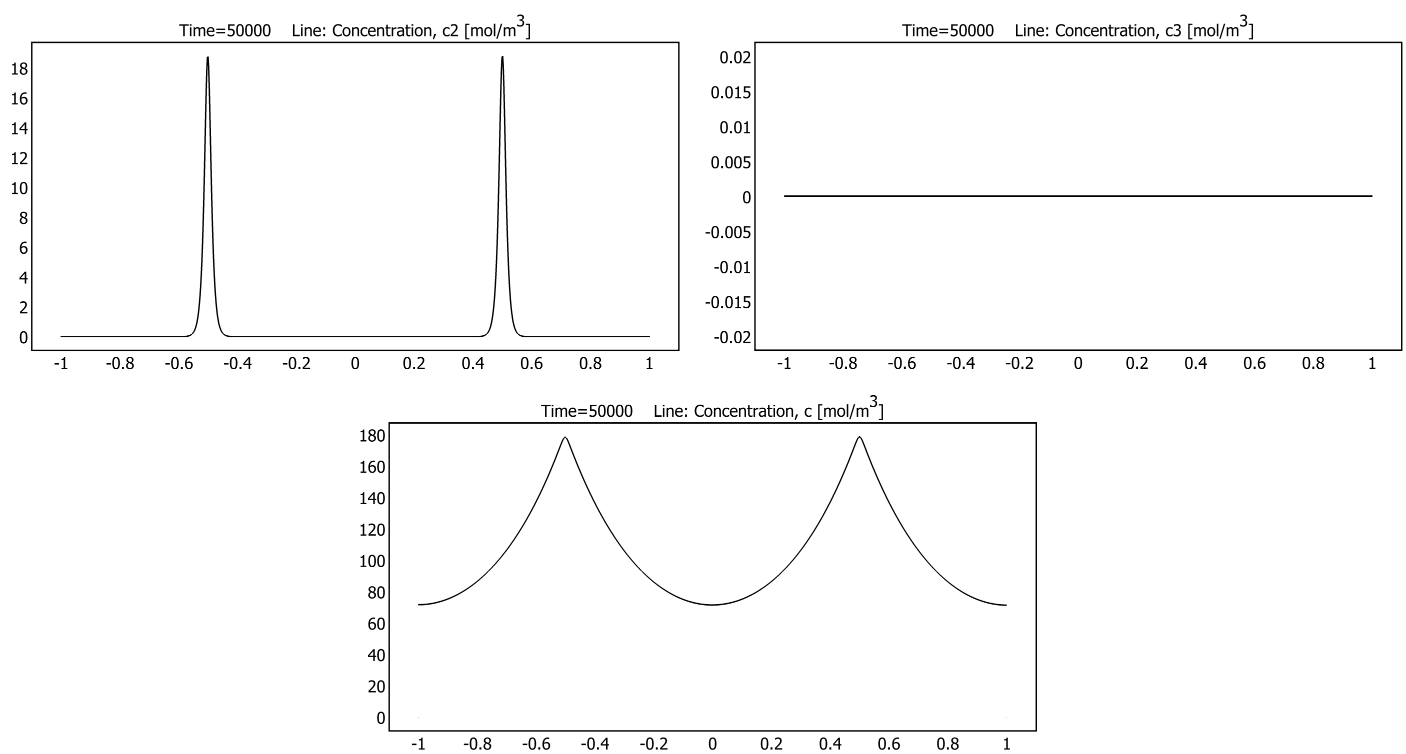

Figure 8. Type 2 Solution ($u_1 > 0$, $u_2 = 0$), Color Blue, 2 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.1,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 0.5.$$

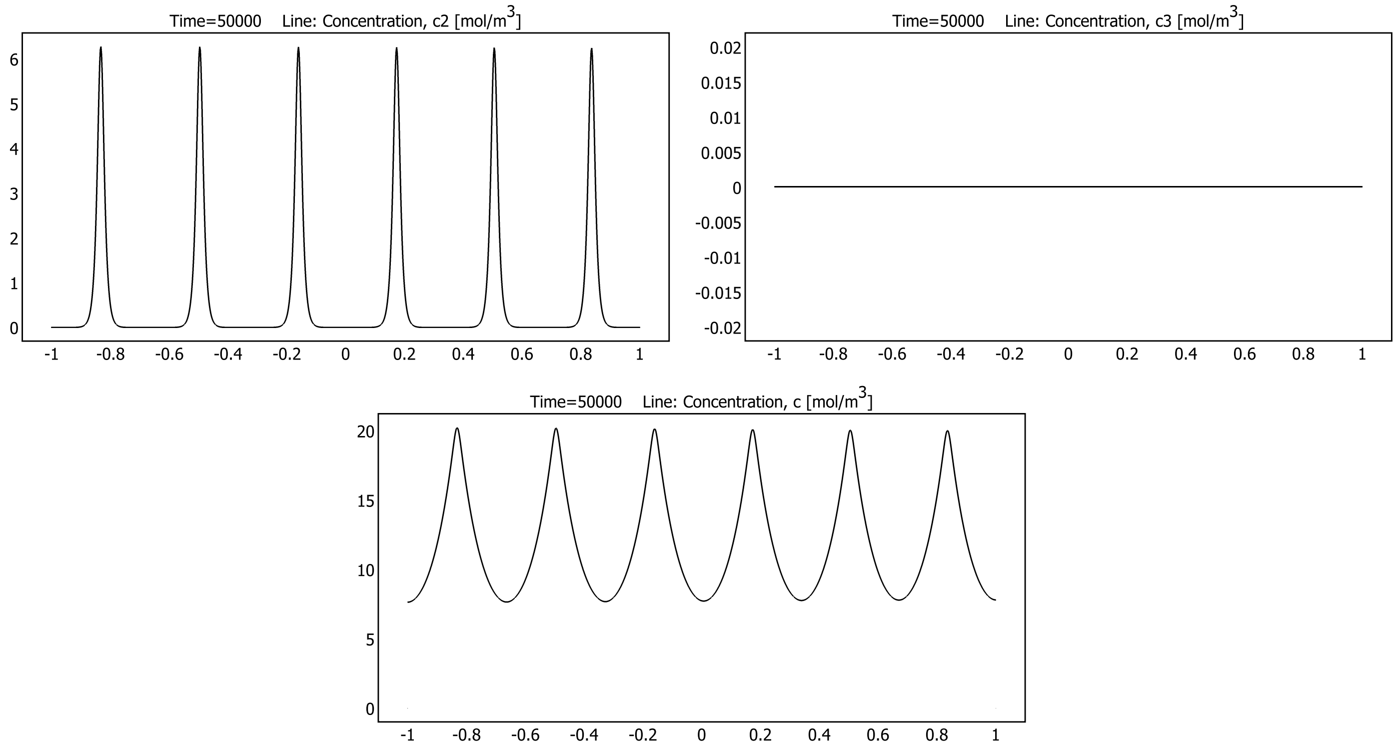

Figure 9. Type 2 Solution ($u_1 > 0$, $u_2 = 0$), Color Blue, 6 Spikes,

$$\epsilon^2 = 0.0001,\ D = 0.01,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 0.5.$$

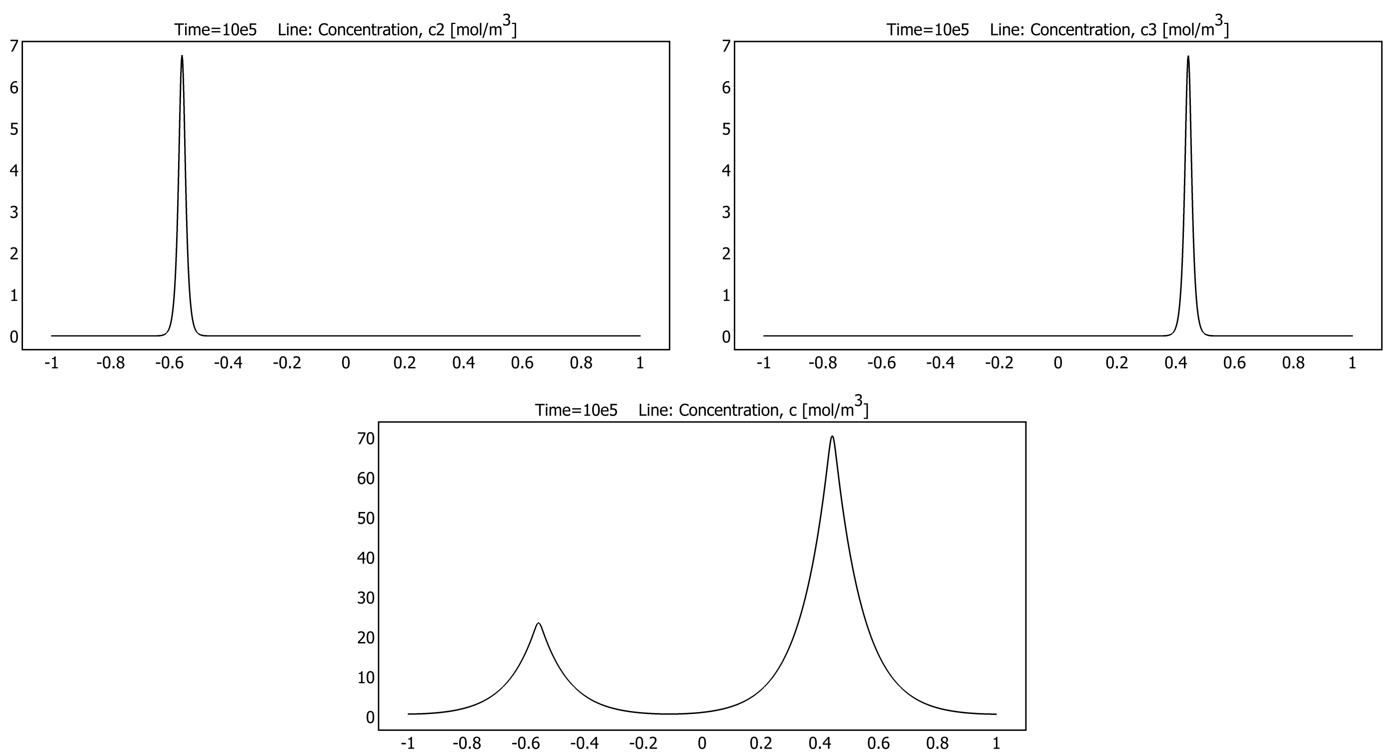

Figure 10. Type 2/3 Solution Combined, Colors Blue/Yellow, 1/1 Spikes in Different Locations,

$$\epsilon^2 = 0.0001,\ D = 0.001,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 0.5.$$

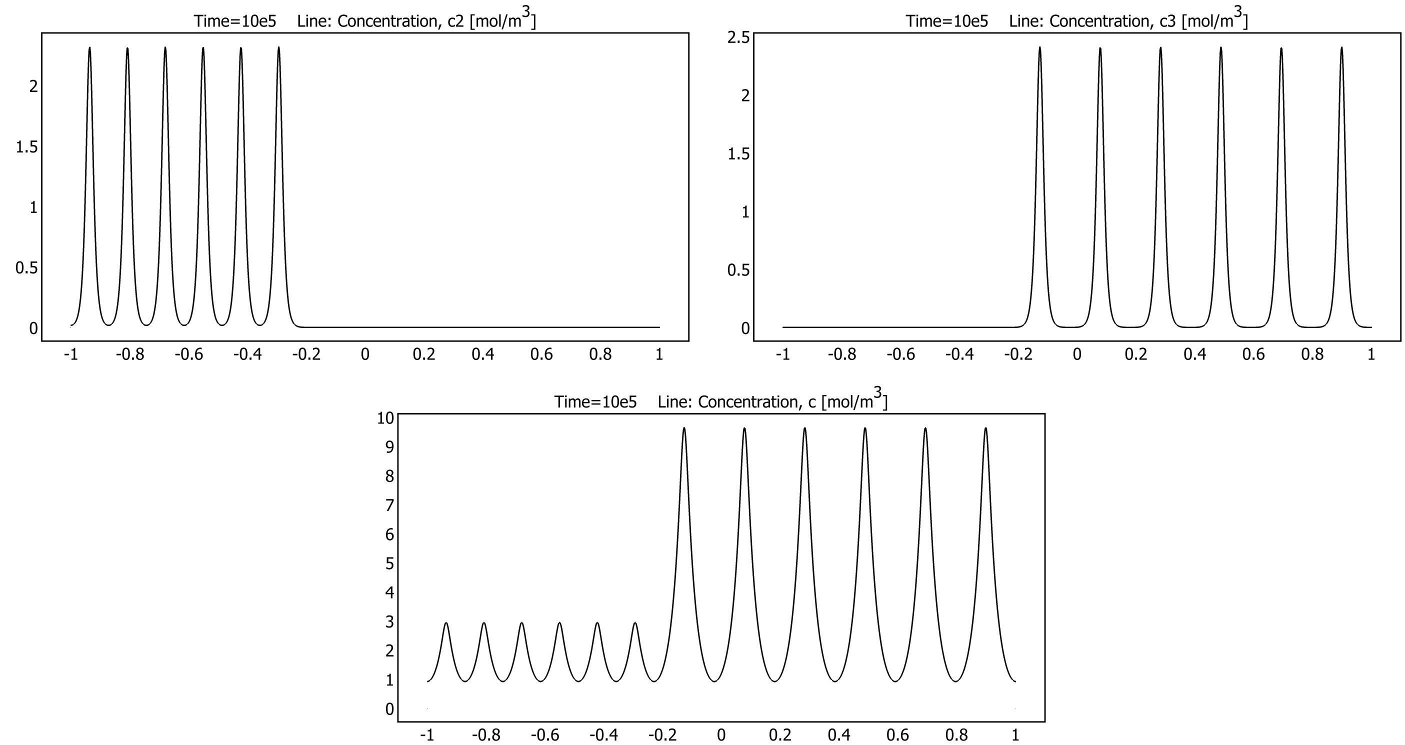

Figure 11. Type 2/3 Solution Combined, Colors Blue/Yellow, 6/6 Spikes in Different Locations,

$$\epsilon^2 = 0.0001,\ D = 0.01,\ \mu_1 = 1,\ \mu_2 = 3,\ \beta = 0.5.$$

7. Discussion

In this final section, we discuss some possible generalisations, extensions and related topics.

Remark 7. *The results can be generalized to the case of negative parameters. For the existence of Type 1 solutions, we need* $\max(\mu_1, \mu_2) < \beta$ *or* $\beta < \min(\mu_1, \mu_2)$.

If β *is negative, then* μ_1 *and* μ_2 *must both be positive, otherwise (2.3) is not possible. Thus we need to have* $\beta < 0 < \min(\mu_1, \mu_2)$. *To satisfy (2.10), it is further required that* $\beta > -\sqrt{\mu_1\mu_2}$. *In summary, we need* $-\sqrt{\mu_1\mu_2} < \beta < 0 < \min(\mu_1, \mu_2)$. *Under this assumption, a Type 1 solution exists. We have* $g(\beta) < 0$, *and so the solution is unstable.*

If μ_1 *is negative, then by (2.3),* β *must be positive and so* $\beta - \mu_1 > 0$. *From (2.10), we get* $\beta^2 - \mu_1\mu_2 > 0$ *and* $\beta - \mu_2 > 0$. *This implies* $\frac{\beta^2}{\mu_1} < \mu_2 < \beta$. *Note that* μ_2 *can have either sign. Under these conditions a Type 1 solution exists. Here* $g(\beta) > 0$, *and so the solution is stable.*

For $\beta < 0$ *and* $\mu_1 > 0$, *Type 2 solutions exist and they are stable. Analogously, for* $\beta < 0$ *and* $\mu_2 > 0$, *Type 3 solutions exist and they are stable.*

The following extensions will be interesting to consider: The activator interaction rates can be changed to quadratic, quartic, etc. instead of cubic activator growth rates for activator interaction.

We expect Hopf bifurcation to occur for sufficiently large τ, resulting in oscillating spikes. Our analysis covered only the case $\tau = 0$ and we did not observe oscillations numerically for $\tau = 1$. The instabilities of the spikes which we encountered in the numerical calculations were (i) disappearance of spikes when the amplitude becomes unstable (related to large eigenvalues) – this happens if the ratio of the diffusion constants $\frac{D}{\epsilon^2}$ is too large (ii) movement of the spikes to the boundary when their positions became unstable (related to small eigenvalues) – this occurs if D is too large.

Two-dimensional domains can be considered instead of an one-dimensional interval.

The analysis can be extended from single spikes to multiple spikes in different locations. If the inhibitor diffusivity is small enough then stable combinations of spikes are expected. Stable solutions should be possible for the following combinations: Type 1 multiple spikes only, Type 2 multiple spikes only, Type 3 multiple spikes only, combination of Type 2 and Type 3 multiple spikes. This would be in agreement with the case studies in the numerical computations (see Section 6). Combinations of Type 2 and Type 3 solutions in a different system with reaction kinetics of Klausmeier type have been studied in [8].

Activator-inhibitor systems with $N \geq 3$ activator components and a combination of self-activation and cross-activation are an important extension. In particular, the results can be extended to color pattern formation for more than two colors.

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WUHAN UNIVERSITY, DEPARTMENT OF MATHEMATICS, WUHAN, CHINA, 430072
Email address: `wwao@whu.edu.cn`

DEPARTMENT OF MATHEMATICS, CHINESE UNIVERSITY OF HONG KONG, HONG KONG, CHINA
Email address: `wei@math.cuhk.edu.hk`

BRUNEL UNIVERSITY LONDON, DEPARTMENT OF MATHEMATICS, UXBRIDGE UB8 3PH, UNITED KINGDOM
Email address: `matthias.winter@brunel.ac.uk`