

HIGH ORDER VANISHING THEOREMS FOR NONSIMPLE BLOWUP SOLUTIONS OF SINGULAR LIOUVILLE EQUATION

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ABSTRACT. For a singular Liouville equation, it is plausible that a non-simple blowup phenomenon occurs around a quantized singular pole. The presence of complex blowup profiles of bubbling solutions presents substantial challenges in applications. In this article, we demonstrate that under natural assumptions, non-simple blowup takes place only when the derivatives of certain coefficient functions approach zero. Our main result encompasses all previous findings and determines the vanishing order for any specific quantized singular source. Our theorems can be utilized not only to eliminate multiple non-simple blowup scenarios in applications but also to investigate blowup solutions with moving poles.

1. INTRODUCTION

It is well known that certain partial differential equations serve as bridges that connect different fields of mathematics and physics. In particular a mean-field type equation defined on a Riemann surface (M, g) of the form

$$\Delta_g u + 2K_g(x) = 2K(x)e^v - 4\pi \sum_{j=1}^m \alpha_j \delta_{p_j}$$

connects conformal geometry and physics. A singular source p_j is called quantized if the corresponding α_j is a positive integer. One essential difficulty is to study the profile of blow-up solutions if the blowup point happens to be a quantized singular source. When we focus on the locally defined equation, the purpose of this article is to study the blowup solutions of

$$(1.1) \quad \Delta u + |x|^{2N} H(x) e^u = 0,$$

in a neighborhood of the origin in \mathbb{R}^2 . Here, H is a positive smooth function and $N \in \mathbb{N}$ is a positive integer. Since the analysis is local in nature, we focus the discussion in a neighborhood of the origin: Let u_k be a sequence of solutions of

$$(1.2) \quad \Delta u_k(x) + |x|^{2N} H_k(x) e^{u_k} = 0, \quad \text{in } B_\tau$$

for some $\tau > 0$ independent of k . B_τ is the ball centered at the origin with radius τ . N in this article is a positive integer and we seek to study the profile of blow-up

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solutions if 0 is the only blowup point. Now we state the usual assumptions on u_k and H_k : For a positive constant C independent of k , the following holds:

$$(1.3) \quad \begin{cases} \|H_k\|_{C^{N+2}(\bar{B}_\tau)} \leq C, & \frac{1}{C} \leq H_k(x) \leq C, & x \in \bar{B}_\tau, \\ \int_{B_\tau} H_k e^{u_k} \leq C, \\ |u_k(x) - u_k(y)| \leq C, & \forall x, y \in \partial B_\tau, \end{cases}$$

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

$$(1.4) \quad \max_{K \subset \subset B_\tau \setminus \{0\}} u_k \leq C(K).$$

If a sequence of solutions $\{u^k\}_{k=1}^\infty$ of (1.1) satisfies

$$\lim_{k \rightarrow \infty} u^k(x_k) = \infty, \quad \text{for some } \bar{x} \in B_\tau \text{ and } x_k \rightarrow \bar{x},$$

we say $\{u^k\}$ is a sequence of bubbling solutions or blowup solutions, \bar{x} is called a blowup point. The question we consider in this work is, when 0 is the only blow-up point in a neighborhood of the origin, what vanishing theorems will the coefficient functions H_k satisfy?

One indispensable assumption is that the blowup solutions violate the spherical Harnack inequality around the origin:

$$(1.5) \quad \max_{x \in B_\tau} u_k(x) + 2(1+N) \log|x| \rightarrow \infty,$$

It is also mentioned in literature (see [26, 34]) that 0 is called a non-simple blowup point. The authors have made progress in the study of non-simple blow-up solutions. In particular, Wei and Zhang proved in [36] the following Laplacian vanishing theorem:

Theorem A: (Wei-Zhang): Let $\{u_k\}$ be a sequence of solutions of (1.2) such that (1.3),(1.4) hold and the spherical Harnack inequality is violated as in (1.5). Then along a sub-sequence

$$\lim_{k \rightarrow \infty} \Delta(\log H_k)(0) = 0.$$

and in [35] the first-order vanishing theorem :

Theorem B (Wei-Zhang, [35]): Let $\{u_k\}$ be a sequence of solutions of (1.2) such that (1.3),(1.4) and (1.5) hold. Then along a subsequence

$$\lim_{k \rightarrow \infty} \nabla(\log H_k + \phi_k)(0) = 0$$

where ϕ_k is defined as

$$(1.6) \quad \begin{cases} \Delta \phi_k(x) = 0, & \text{in } B_\tau, \\ \phi_k(x) = u_k(x) - \frac{1}{2\pi\tau} \int_{\partial B_\tau} u_k dS, & x \in \partial B_\tau. \end{cases}$$

The equation (1.1) comes from its equivalent form

$$\Delta v + H e^v = 4\pi N \delta_0$$

by using a logarithmic function to eliminate the Dirac mass on the right-hand side. The study of blowup solutions for (1.1) gives precise description of bubbling profile for global equations such as the following mean field equation defined on a Riemann surface (M, g) :

$$(1.7) \quad \Delta_g u + \rho \left(\frac{h(x)e^{u(x)}}{\int_M h e^u} - 1 \right) = 4\pi \sum_{t=1}^M \alpha_t (\delta_{p_t} - 1).$$

Equation (1.7) describes a conformal metric with prescribed conic singularities (see [21, 32, 33]). In this context, h is a positive smooth function, $\rho > 0$ is a constant, and the volume of M is assumed to be 1 for convenience. Additionally, $\alpha_j > -1$ are constants. When the singular source is quantized ($\alpha \in \mathbb{N}$), the equation is deeply connected to Algebraic geometry, integrable systems, number theory, and complex Monge-Ampère equations (see [18]). In physics, this main equation reveals critical features of the mean field limits of point vortices in Euler flow [10, 11], models in the Chern-Simons-Higgs theory [25], and the electroweak theory [2], among others.

The non-simple bubbling situation has been observed in the Liouville equation [26, 4], Liouville systems [23, 24, 37], and fourth-order equations [1]. Although it has been established in [35, 36] that the first derivatives and the Laplace of the coefficient function vanish at the quantized singular source, it is still highly desirable to prove even higher-order vanishing theorems. The elimination of nonsimple blow-up situations significantly simplifies the profiles of bubbling solutions and is crucial for various subsequent applications. This article aims to demonstrate the vanishing of coefficients for any high order. Our first main result is to prove the vanishing of all second derives of the coefficient function for any $N \geq 1$.

Theorem 1.1. *Let $\{u_k\}$ be a sequence of solutions of (1.2) such that (1.3), (1.4) and (1.5) hold. Then along a sub-sequence*

$$|D^2(\log H_k + \phi_k)(0)| = o(1).$$

We also have an extension of Theorem 1.1, which is concerned about vanishing estimates for higher order derivatives. Our second main result is

Theorem 1.2. *Let $\{u_k\}$ be a sequence of solutions of (1.2) such that (1.3), (1.4) and (1.5) hold. Then for $N \geq 2^{M+1}$ where $M \in \mathbb{N} \cup \{0\}$ is a nonnegative integer, we have*

$$|D^\alpha(\log H_k + \phi_k)(0)| = o(1), \quad \forall |\alpha| \leq 2^M + 1,$$

along a subsequence.

Remark 1.1. *Theorem 1.1 holds for all $N \geq 1$. Theorem 1.2 proves that any order of derivatives of the coefficient function vanishes as long as N is large. Both theorems improve the previous Laplacian vanishing Theorem in [36]. For some applications such as the study of the blow-up profile of singular Liouville equations with moving poles, the Laplacian vanishing theorem in [36] cannot be applied. It is crucial to have Theorem 1.1 and Theorem 1.2.*

Notation: We will use $B(x_0, r)$ to denote a ball centered at x_0 with radius r . If x_0 is the origin, we use B_r . C represents a positive constant that can change from place to place.

The proof of the main theorem requires delicate point-wise estimates of the blow-up solutions around each local maximum point. This was the case in the proofs of the previous work of the second and third authors in [34, 35, 36]. The difference in this work is that we need more refined estimates, which require a subtler analysis. In general, the nature of the proof demands estimates of different levels of precision in a progressive manner. One level of estimate leads to a higher-order estimate later. Identifying the error threshold in each stage is crucial for the completion of the whole proof eventually. If we describe the entire scheme of the proof from a general viewpoint, we use Fourier analysis for the pointwise estimate around each local maximum, Harnack inequality to pass certain smallness information to regions away from local maximums. Moreover we use Pohozaev identities to capture crucial vanishing information on coefficient functions. The approach we employ in this work should be useful for other related situations as well.

The organization of this article is the following. Sections 2-5 contain the complete proof of Theorem 1.1. Section six is saved for the proof of Theorem 1.2. Finally in section seven we provide a different perspective by discussing the Dirichlet problem and giving a second proof of the Hessian vanishing estimate for $N = 1$ using new Pohozaev identities.

2. PRELIMINARY DISCUSSIONS FOR BLOWUP ANALYSIS AND THE LOCALLY DEFINED EQUATION

In the first stage of the proof of the main Theorem 1.1 we set up some notations and cite some preliminary results. Let

$$(2.1) \quad u_k(x) = u_k(x) - \phi_k(x), \quad \text{where } \phi_k \text{ is defined in (1.6) and}$$

$$(2.2) \quad h_k(x) = H_k(x)e^{\phi_k(x)}.$$

Now we write the equation of u_k as

$$(2.3) \quad \Delta u_k(x) + |x|^{2N} h_k(x) e^{u_k} = 0, \quad \text{in } B_\tau$$

Without loss of generality we assume

$$(2.4) \quad \lim_{k \rightarrow \infty} h_k(0) = 1.$$

Obviously (1.5) is equivalent to

$$(2.5) \quad \max_{x \in B_\tau} u_k(x) + 2(1+N) \log |x| \rightarrow \infty,$$

It is well known [26, 4] that u_k exhibits a non-simple blow-up profile. It is established in [26, 4] that there are $N + 1$ local maximum points of u_k : p_0^k, \dots, p_N^k

and they are evenly distributed on \mathbb{S}^1 after scaling according to their magnitude: Suppose along a subsequence

$$\lim_{k \rightarrow \infty} p_0^k / |p_0^k| = e^{i\theta_0},$$

then

$$\lim_{k \rightarrow \infty} \frac{p_l^k}{|p_0^k|} = e^{i(\theta_0 + \frac{2\pi l}{N+1})}, \quad l = 1, \dots, N.$$

For many reasons it is convenient to denote $|p_0^k|$ as δ_k and define μ_k as follows:

$$(2.6) \quad \delta_k = |p_0^k| \quad \text{and} \quad \mu_k = u_k(p_0^k) + 2(1+N) \log \delta_k.$$

Also we use

$$(2.7) \quad \varepsilon_k = e^{-\frac{1}{2}\mu_k}$$

to be the scaling factor most of the time. Since p_l^k 's are evenly distributed around ∂B_{δ_k} , standard results for Liouville equations around a regular blow-up point can be applied to have $u_k(p_l^k) = u_k(p_0^k) + o(1)$. Also, (1.5) gives $\mu_k \rightarrow \infty$. The interested readers may look into [26, 4] for more detailed information.

3. APPROXIMATING BUBBLING SOLUTIONS BY GLOBAL SOLUTIONS

We write p_0^k as $p_0^k = \delta_k e^{i\theta_k}$ and define v_k as

$$(3.1) \quad v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1) \log \delta_k, \quad |y| < \tau \delta_k^{-1}.$$

If we write out each component, (3.1) is

$$v_k(y_1, y_2) = u_k(\delta_k(y_1 \cos \theta_k - y_2 \sin \theta_k), \delta_k(y_1 \sin \theta_k + y_2 \cos \theta_k)) + 2(1+N) \log \delta_k.$$

Then it is standard to verify that v_k solves

$$(3.2) \quad \Delta v_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k y) e^{v_k(y)} = 0, \quad |y| < \tau / \delta_k,$$

where

$$(3.3) \quad \mathfrak{h}_k(x) = h_k(x e^{i\theta_k}), \quad |x| < \tau.$$

Thus the image of p_0^k after scaling is $Q_1^k = e_1 = (1, 0)$. Let $Q_1^k, Q_2^k, \dots, Q_N^k$ be the images of p_l^k ($l = 1, \dots, N$) after the scaling:

$$Q_l^k = \frac{p_l^k}{\delta_k} e^{-i\theta_k}, \quad l = 1, \dots, N.$$

It is established by Kuo-Lin in [26] and independently by Bartolucci-Tarantello in [4] that

$$(3.4) \quad \lim_{k \rightarrow \infty} Q_l^k = \lim_{k \rightarrow \infty} p_l^k / \delta_k = e^{\frac{2i\pi l}{N+1}}, \quad l = 0, \dots, N.$$

Then it is proved in [34] that (see (3.13) in [34])

$$(3.5) \quad Q_l^k - e^{\frac{2i\pi l}{N+1}} = O(\mu_k e^{-\mu_k}) + O(|\nabla \log \mathfrak{h}_k(0)| \delta_k).$$

Using the rate of $\nabla \mathfrak{h}_k(0)$ in [34] we have

$$(3.6) \quad Q_l^k - e^{\frac{2\pi l i}{N+1}} = O(\mu_k e^{-\mu_k}) + O(\delta_k^2).$$

Choosing $\varepsilon > 0$ small and independent of k , we can make disks centered at Q_l^k with radius 3ε (denoted as $B(Q_l^k, 3\varepsilon)$) mutually disjoint. Let

$$(3.7) \quad \mu_k = \max_{B(Q_l^k, \varepsilon)} v_k.$$

Since Q_l^k are evenly distributed around ∂B_1 , it is easy to use standard estimates for single Liouville equations ([39, 22, 17]) to obtain

$$\max_{B(Q_l^k, \varepsilon)} v_k = \mu_k + o(1), \quad l = 1, \dots, N.$$

Let

$$(3.8) \quad V_k(x) = \log \frac{e^{\mu_k}}{\left(1 + \frac{e^{\mu_k} \mathfrak{h}_k(\delta_k e_1)}{8(1+N)^2} |y^{N+1} - e_1|^2\right)^2}.$$

Clearly V_k is a solution of

$$(3.9) \quad \Delta V_k + \mathfrak{h}_k(\delta_k e_1) |y|^{2N} e^{V_k} = 0, \quad \text{in } \mathbb{R}^2, \quad V_k(e_1) = \mu_k.$$

This expression is based on the classification theorem of Prajapat-Tarantello [29]. For convenience we use

$$\beta_l = \frac{2\pi l}{N+1}, \quad \text{so } e_1 = e^{i\beta_0} = Q_0^k, \quad e^{i\beta_l} = Q_l^k + E, \quad \text{for } l = 1, \dots, N.$$

The following expansion of V_k on $|y| = \tau \delta_k^{-1}$ shows that the oscillation of V_k is $O(\delta_k^{N+1})$ on $\partial B(0, \tau \delta_k^{-1})$:

$$(3.10) \quad V_k(x) = -\mu_k + 2 \log \frac{8(1+N)^2}{\mathfrak{h}_k(\delta_k e_1)} - 4 \log L_k + 4L_k^{-N-1} \cos((N+1)\theta) + O(\delta_k^{2N+2})$$

where $L_k = \tau \delta_k^{-1}$.

4. VANISHING OF THE FIRST DERIVATIVES

Our first goal is to prove the following vanishing rate for $\nabla \mathfrak{h}_k(0)$:

Theorem 4.1.

$$(4.1) \quad \nabla(\log \mathfrak{h}_k)(0) = O(\delta_k)$$

Proof of Theorem 4.1:

Note that we have proved in [34] that

$$\nabla(\log \mathfrak{h}_k)(0) = O(\delta_k^{-1} \mu_k e^{-\mu_k}) + O(\delta_k).$$

From (2.7) we see that if $\delta_k \geq C \varepsilon_k \mu_k^{\frac{1}{2}}$, there is nothing to prove. So we assume that

$$(4.2) \quad \delta_k = o(\varepsilon_k \mu_k^{\frac{1}{2}}).$$

By way of contradiction we assume that there exists $C > 0$ independent of k , such that

$$(4.3) \quad |\nabla \mathfrak{h}_k(0)|/\delta_k \rightarrow \infty.$$

Another observation is that based on (3.6) we have

$$(4.4) \quad \varepsilon_k^{-1} |Q_l^k - e^{i\beta_l}| \rightarrow 0, \quad l = 0, \dots, N.$$

Thus ξ_k tends to U after scaling.

Under assumption (4.2) we cite Proposition 3.1 of [35]:

Proposition 3.1 of [35]: Let $l = 0, \dots, N$ and τ_1 be small so that $B(e^{i\beta_l}, \delta) \cap B(e^{i\beta_s}, \delta) = \emptyset$ for $l \neq s$. In each $B(e^{i\beta_l}, \delta)$

$$(4.5) \quad |v_k(x) - V_k(x)| \leq \begin{cases} C\mu_k e^{-\mu_k/2}, & |x - e^{i\beta_l}| \leq C e^{-\mu_k/2}, \\ C \frac{\mu_k e^{-\mu_k}}{|x - e^{i\beta_l}|} + O(\mu_k^2 e^{-\mu_k}), & C e^{-\mu_k/2} \leq |x - e^{i\beta_l}| \leq \tau_1. \end{cases}$$

One major step in the proof of Theorem 4.1 is the following estimate:

Proposition 4.1. Let $w_k = v_k - V_k$, then

$$|w_k(y)| \leq C \tilde{\delta}_k, \quad y \in \Omega_k := B(0, \tau \delta_k^{-1}),$$

where $\tilde{\delta}_k = |\nabla \mathfrak{h}_k(0)| \delta_k + \delta_k^2$.

Proof of Proposition 4.1:

Obviously based on (4.3) we have $|\nabla \mathfrak{h}_k(0)| \delta_k / \delta_k^2 \rightarrow \infty$. Now we recall the equation for v_k is (3.2), v_k is a constant on $\partial B(0, \tau \delta_k^{-1})$. Moreover $v_k(e_1) = \mu_k$. Recall that V_k defined in (3.8) satisfies

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1) |y|^{2N} e^{V_k} = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N} e^{V_k} < \infty,$$

V_k has its local maximums at $e^{i\beta_l}$ for $l = 0, \dots, N$ and $V_k(e_1) = \mu_k$. For $|y| \sim \delta_k^{-1}$, the oscillation of V_k is given by (3.10).

Let $\Omega_k = B(0, \tau_1 \delta_k^{-1})$, we shall derive a precise, point-wise estimate of w_k in $B_3 \setminus \cup_{l=1}^N B(Q_l^k, \tau_1)$ where $\tau_1 > 0$ is a small number independent of k . Here we note that among $N + 1$ local maximum points, we already have e_1 as a common local maximum point for both v_k and V_k and we shall prove that w_k is very small in B_3 if we exclude all bubbling disks except the one around e_1 . Before we carry out more specific computation we emphasize the importance of

$$(4.6) \quad w_k(e_1) = |\nabla w_k(e_1)| = 0.$$

Now we write the equation of w_k as

$$(4.7) \quad \Delta w_k + \mathfrak{h}_k(\delta_k y) |y|^{2N} e^{\xi_k} w_k = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y)) |y|^{2N} e^{V_k}$$

in Ω_k , where ξ_k is obtained from the mean value theorem:

$$e^{\xi_k(x)} = \begin{cases} \frac{e^{v_k(x)} - e^{V_k(x)}}{v_k(x) - V_k(x)}, & \text{if } v_k(x) \neq V_k(x), \\ e^{V_k(x)}, & \text{if } v_k(x) = V_k(x). \end{cases}$$

An equivalent form is

$$(4.8) \quad e^{\xi_k(x)} = \int_0^1 \frac{d}{dt} e^{tv_k(x) + (1-t)V_k(x)} dt = e^{V_k(x)} \left(1 + \frac{1}{2}w_k(x) + O(w_k(x)^2)\right).$$

For convenience we write the equation for w_k as

$$(4.9) \quad \Delta w_k + \mathfrak{h}_k(\delta_k y) |y|^{2N} e^{\xi_k} w_k = \delta_k \nabla \mathfrak{h}_k(\delta_k e_1) \cdot (e_1 - y) |y|^{2N} e^{V_k} + E_1$$

where

$$E_1 = O(\delta_k^2) |y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k.$$

Note that the oscillation of w_k on $\partial\Omega_k$ is $O(\delta_k^{N+1})$, which all comes from the oscillation of V_k .

Let $M_k = \max_{x \in \bar{\Omega}_k} |w_k(x)|$. We shall get a contradiction by assuming $M_k/\tilde{\delta}_k \rightarrow \infty$. This assumption implies

$$(4.10) \quad M_k/\delta_k^2 \rightarrow \infty.$$

Set

$$\tilde{w}_k(y) = w_k(y)/M_k, \quad x \in \Omega_k.$$

Clearly $\max_{x \in \Omega_k} |\tilde{w}_k(x)| = 1$. The equation for \tilde{w}_k is

$$(4.11) \quad \Delta \tilde{w}_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k} \tilde{w}_k(y) = a_k \cdot (e_1 - y) |y|^{2N} e^{V_k} + \tilde{E}_1,$$

in Ω_k , where $a_k = \delta_k \nabla \mathfrak{h}_k(0)/M_k \rightarrow 0$,

$$(4.12) \quad \tilde{E}_1 = o(1) |y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k.$$

Also on the boundary, since $M_k/\delta_k^2 \rightarrow \infty$, we have

$$(4.13) \quad \tilde{w}_k = C + o(1), \quad \text{on } \partial\Omega_k.$$

Let

$$(4.14) \quad W_k(z) = \tilde{w}_k(e_1 + \varepsilon_k z)$$

and we recall that $\varepsilon_k = e^{-\frac{1}{2}\mu_k}$. then if we use W to denote the limit of W_k , we have

$$\Delta W + e^U W = 0, \quad \mathbb{R}^2, \quad |W| \leq 1,$$

and U is a solution of $\Delta U + e^U = 0$ in \mathbb{R}^2 with $\int_{\mathbb{R}^2} e^U < \infty$. Since 0 is the local maximum of U , we know from the classification theorem of Caffarelli-Gidas-Spruck [9] that

$$U(z) = \log \frac{1}{(1 + \frac{1}{8}|z|^2)^2}.$$

If we use $\bar{\xi}_{k,0}(|z|)$ to be the radial part of

$$\log(|e_1 + \varepsilon_k z|^{2N} \mathfrak{h}_k(\delta_k e_1)) + \xi_k(e_1 + \varepsilon_k z) + 2 \log \varepsilon_k$$

with respect to e_1 , which satisfies $\bar{\xi}_{k,0} \rightarrow U$ in $C_{loc}^2(\mathbb{R}^2)$ and

$$e^{\bar{\xi}_{k,0}(z)} \leq C(1 + |z|)^{-4}, \quad |z| \leq \tau_k \varepsilon_k^{-1}.$$

We write the equation of W_k as

$$(4.15) \quad \Delta W_k(z) + e^{\bar{\xi}_{k,0}} W_k = E_2^k, \quad |z| \leq \tau_1 \varepsilon_k^{-1}$$

where

$$E_2^k(z) = O(\varepsilon_k)(1 + |z|)^{-3}.$$

In the following we shall put the proof of Proposition 4.1 into a few estimates. In the first estimate we prove

Lemma 4.1. *For $\delta > 0$ small and independent of k ,*

$$(4.16) \quad \tilde{w}_k(y) = o(1), \quad \nabla \tilde{w}_k = o(1) \quad \text{in } B(e_1, \delta) \setminus B(e_1, \delta/8)$$

where $B(e_1, 3\delta)$ does not include other blowup points.

Proof of Lemma 4.1:

If (4.16) is not true, we have, without loss of generality that

$$(4.17) \quad \tilde{w}_k \rightarrow c > 0.$$

This is based on the fact that \tilde{w}_k tends to a global harmonic function with removable singularity. So \tilde{w}_k tends to constant. Here we assume $c > 0$ but the argument for $c < 0$ is the same. Recall that W_k is defined in (4.14). Here we further claim that $W \equiv 0$ in \mathbb{R}^2 because $W(0) = |\nabla W(0)| = 0$, a fact well known based on the classification of the kernel of the linearized operator. Going back to W_k , we have

$$W_k(z) = o(1), \quad |z| \leq R_k \text{ for some } R_k \rightarrow \infty.$$

Let

$$(4.18) \quad g_0^k(r) = \frac{1}{2\pi} \int_0^{2\pi} W_k(r, \theta) d\theta.$$

Then clearly $g_0^k(r) \rightarrow c > 0$ for $r \sim \varepsilon_k^{-1}$. The equation for g_0^k is

$$\begin{aligned} \frac{d^2}{dr^2} g_0^k(r) + \frac{1}{r} \frac{d}{dr} g_0^k(r) + \mathfrak{h}_k(\delta_k e_1) e^{\tilde{\xi}_{k,0}} g_0^k(r) &= \tilde{E}_0^k(r) \\ g_0^k(0) = \frac{d}{dr} g_0^k(0) &= 0. \end{aligned}$$

where

$$|\tilde{E}_0^k(r)| \leq O(\varepsilon_k)(1+r)^{-3}.$$

For the homogeneous equation, the two fundamental solutions are known: g_{01}^k , g_{02}^k , where, by elementary analysis, we obtain that g_{01}^k tends to

$$\frac{1 - \frac{1}{8}r^2}{1 + \frac{1}{8}r^2}.$$

By the standard reduction of order process, $g_{02}^k(r) = O(\log r)$ for $r > 1$ with bounds independent of k . Then it is easy to obtain, assuming $|W_k(z)| \leq 1$, that

$$\begin{aligned} |g_0^k(r)| &\leq C|g_{01}^k(r)| \int_0^r s|\tilde{E}_0^k(s)g_{02}^k(s)|ds + C|g_{02}^k(r)| \int_0^r s|g_{01}^k(s)\tilde{E}_0^k(s)|ds \\ &\leq C\varepsilon_k \log(2+r). \quad 0 < r < \delta_0 \varepsilon_k^{-1}. \end{aligned}$$

Clearly this is a contradiction to (4.17). We have proved $c = 0$, which means $\tilde{w}_k = o(1)$ in $B(e_1, \delta_0) \setminus B(e_1, \delta_0/8)$. Then it is easy to use the equation for \tilde{w}_k and

standard Harnack inequality to prove $\nabla \tilde{w}_k = o(1)$ in the same region. Lemma 4.1 is established. \square

The second estimate is a more precise description of \tilde{w}_k around e_1 :

Lemma 4.2. *For any given $\sigma \in (0, 1)$ there exists $C > 0$ such that*

$$(4.19) \quad |\tilde{w}_k(e_1 + \varepsilon_k z)| \leq C \varepsilon_k^\sigma (1 + |z|)^\sigma, \quad 0 < |z| < \tau \varepsilon_k^{-1}.$$

for some $\tau > 0$.

Remark 4.1. *Lemma 4.2 is an intermediate estimate for \tilde{w}_k . We eventually need to have an estimate starting with $o(\varepsilon_k)$. The reason that we cannot obtain more precise estimate is because we only know $\tilde{w} = o(1)$ on $\partial B(e_1, \tau)$. More precise information is needed to have a better estimate.*

Proof of Lemma 4.2: Let W_k be defined as in (4.14). In order to obtain a better estimate we need to write the equation of W_k more precisely than (4.15):

$$(4.20) \quad \Delta W_k + \mathfrak{h}_k(\delta_k e_1) e^{\Theta_k} W_k = E_3^k(z), \quad z \in \Omega_{W_k}$$

where Θ_k is defined by

$$e^{\Theta_k(z)} = |e_1 + \varepsilon_k z|^{2N} e^{\xi_k(e_1 + \varepsilon_k z) + 2 \log \varepsilon_k},$$

$\Omega_{W_k} = B(0, \tau \varepsilon_k^{-1})$ and $E_3^k(z)$ satisfies

$$E_3^k(z) = O(\varepsilon_k)(1 + |z|)^{-3}, \quad z \in \Omega_{W_k}.$$

Here we observe that by Lemma 4.1 $W_k = o(1)$ on $\partial \Omega_{W_k}$. Let

$$\Lambda_k = \max_{z \in \Omega_{W_k}} \frac{|W_k(z)|}{\varepsilon_k^\sigma (1 + |z|)^\sigma}.$$

If (4.19) does not hold, $\Lambda_k \rightarrow \infty$ and we use z_k to denote where Λ_k is attained. Note that because of the smallness of W_k on $\partial \Omega_{W_k}$, z_k is an interior point. Let

$$g_k(z) = \frac{W_k(z)}{\Lambda_k (1 + |z_k|)^\sigma \varepsilon_k^\sigma}, \quad z \in \Omega_{W_k},$$

we see immediately that

$$(4.21) \quad |g_k(z)| = \frac{|W_k(z)|}{\varepsilon_k^\sigma \Lambda_k (1 + |z|)^\sigma} \cdot \frac{(1 + |z|)^\sigma}{(1 + |z_k|)^\sigma} \leq \frac{(1 + |z|)^\sigma}{(1 + |z_k|)^\sigma}.$$

Note that σ can be as close to 1 as needed. The equation of g_k is

$$\Delta g_k(z) + \mathfrak{h}_k(\delta_k e_1) e^{\Theta_k} g_k = o(\varepsilon_k^{1-\sigma}) \frac{(1 + |z|)^{-3}}{(1 + |z_k|)^\sigma}, \quad \text{in } \Omega_{W_k}.$$

Then we can obtain a contradiction to $|g_k(z_k)| = 1$ as follows: If $\lim_{k \rightarrow \infty} z_k = P \in \mathbb{R}^2$, this is not possible because that fact that $g_k(0) = |\nabla g_k(0)| = 0$ and the sub-linear growth of g_k in (4.21) implies that $g_k \rightarrow 0$ over any compact subset of \mathbb{R}^2

(see [17, 39]). So we have $|z_k| \rightarrow \infty$. But this would lead to a contradiction again by using the Green's representation of g_k :

(4.22)

$$\begin{aligned} \pm 1 &= g_k(z_k) = g_k(z_k) - g_k(0) \\ &= \int_{\Omega_{k,1}} (G_k(z_k, \eta) - G_k(0, \eta)) (\mathfrak{h}_k(\delta_k e_1) e^{\Theta_k} g_k(\eta) + o(\varepsilon_k^{1-\sigma}) \frac{(1+|\eta|)^{-3}}{(1+|z_k|)^\sigma}) d\eta + o(1). \end{aligned}$$

where $G_k(y, \eta)$ is the Green's function on Ω_{W_k} and $o(1)$ in the equation above comes from the smallness of W_k on $\partial\Omega_{W_k}$. Let $L_k = \tau\varepsilon_k^{-1}$, the expression of G_k is

$$G_k(y, \eta) = -\frac{1}{2\pi} \log|y - \eta| + \frac{1}{2\pi} \log\left(\frac{|\eta|}{L_k} \left| \frac{L_k^2 \eta}{|\eta|^2} - y \right|\right).$$

$$G_k(z_k, \eta) - G_k(0, \eta) = -\frac{1}{2\pi} \log|z_k - \eta| + \frac{1}{2\pi} \log\left|\frac{z_k}{|z_k|} - \frac{\eta z_k}{L_k^2}\right| + \frac{1}{2\pi} \log|\eta|.$$

Using this expression in (4.22) we obtain from elementary computation that the right hand side of (4.22) is $o(1)$, a contradiction to $|g_k(z_k)| = 1$. Lemma 4.2 is established. \square

The smallness of \tilde{w}_k around e_1 can be used to obtain the following third key estimate:

Lemma 4.3.

$$(4.23) \quad \tilde{w}_k = o(1) \quad \text{in} \quad B(e^{i\beta_l}, \tau) \quad l = 1, \dots, N.$$

Proof of Lemma 4.3: We abuse the notation W_k by defining it as

$$W_k(z) = \tilde{w}_k(e^{i\beta_l} + \varepsilon_k z), \quad z \in \Omega_{k,l} := B(0, \tau\varepsilon_k^{-1}).$$

Here we point out that based on (3.6) and (4.2) we have $\varepsilon_k^{-1}|Q_l^k - e^{i\beta_l}| \rightarrow 0$. So the scaling around $e^{i\beta_l}$ or Q_l^k does not affect the limit function.

$$\varepsilon_k^2 |e^{i\beta_l} + \varepsilon_k z|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k(e^{i\beta_l} + \varepsilon_k z)} \rightarrow e^{U(z)}$$

where $U(z)$ is a solution of

$$\Delta U + e^U = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty.$$

Here we recall that $\lim_{k \rightarrow \infty} \mathfrak{h}_k(\delta_k e_1) = 1$. Since W_k converges to a solution of the linearized equation:

$$\Delta W + e^U W = 0, \quad \text{in} \quad \mathbb{R}^2.$$

If the growth of W at infinity is sub-linear: $|W(x)| = o(1)|x|$ when $|x| \rightarrow \infty$, W can be written as a linear combination of three functions:

$$W(x) = c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2,$$

where

$$\phi_0 = \frac{1 - \frac{1}{8}|x|^2}{1 + \frac{1}{8}|x|^2}$$

$$\phi_1 = \frac{x_1}{1 + \frac{1}{8}|x|^2}, \quad \phi_2 = \frac{x_2}{1 + \frac{1}{8}|x|^2}.$$

The remaining part of the proof consisting of proving $c_0 = 0$ and $c_1 = c_2 = 0$. First we prove $c_0 = 0$.

Step one: $c_0 = 0$. First we write the equation for W_k in a convenient form. Since

$$|e^{i\beta_l} + \varepsilon_k z|^{2N} \mathfrak{h}_k(\delta_k e_1) = \mathfrak{h}_k(\delta_k e_1) + O(\varepsilon_k |z|),$$

and

$$\varepsilon_k^2 e^{\xi_k(e^{i\beta_l} + \varepsilon_k z)} = e^{U_k(z)} + O(\varepsilon_k^\varepsilon)(1 + |z|)^{-3}.$$

Based on (4.11) we write the equation for W_k as

$$(4.24) \quad \Delta W_k(z) + \mathfrak{h}_k(\delta_k e_1) e^{U_k} W_k = E_l^k(z)$$

where

$$E_l^k(z) = O(\varepsilon_k^\varepsilon)(1 + |z|)^{-3} \quad \text{in } \Omega_{k,l}.$$

In order to prove $c_0 = 0$, the key is to control the derivative of $W_0^k(r)$ where

$$W_0^k(r) = \frac{1}{2\pi r} \int_{\partial B_r} W_k(re^{i\theta}) dS, \quad 0 < r < \tau \varepsilon_k^{-1}.$$

To obtain a control of $\frac{d}{dr} W_0^k(r)$ we use $\phi_0^k(r)$ as the radial solution of

$$\Delta \phi_0^k + \mathfrak{h}_k(\delta_k e_1) e^{U_k} \phi_0^k = 0, \quad \text{in } \mathbb{R}^2.$$

When $k \rightarrow \infty$, $\phi_0^k \rightarrow c_0 \phi_0$. Thus using the equation for ϕ_0^k and W_k , we have

$$(4.25) \quad \int_{\partial B_r} (\partial_\nu W_k \phi_0^k - \partial_\nu \phi_0^k W_k) = o(\varepsilon_k^\varepsilon).$$

Thus from (4.25) we have

$$(4.26) \quad \frac{d}{dr} W_0^k(r) = \frac{1}{2\pi r} \int_{\partial B_r} \partial_\nu W_k = o(\varepsilon_k^\varepsilon)/r + O(1/r^3), \quad 1 < r < \tau \varepsilon_k^{-1}.$$

Since we have known that

$$W_0^k(\tau \varepsilon_k^{-1}) = o(1).$$

By the fundamental theorem of calculus we have

$$W_0^k(r) = W_0^k(\tau \varepsilon_k^{-1}) + \int_{\tau \varepsilon_k^{-1}}^r \left(\frac{o(\varepsilon_k^\varepsilon)}{s} + O(s^{-3}) \right) ds = O(1/r^2) + O(\varepsilon_k^\varepsilon \log \frac{1}{\varepsilon_k})$$

for $r \geq 1$. Thus $c_0 = 0$ because $W_0^k(r) \rightarrow c_0 \phi_0$, which means when r is large, it is $-c_0 + O(1/r^2)$.

Step two: v_k is close to a global solution near each Q_l^k . We first observe once we have proved $c_1 = c_2 = c_0 = 0$ around each $e^{i\beta_l}$, it is easy to use maximum principle to prove $\tilde{w}_k = o(1)$ in B_3 using $\tilde{w}_k = o(1)$ on ∂B_3 and the Green's representation of \tilde{w}_k . The smallness of \tilde{w}_k immediately implies $\tilde{w}_k = o(1)$ in B_R for any fixed $R \gg 1$. Outside B_R , a crude estimate of v_k is

$$v_k(y) \leq -\mu_k - 4(N+1) \log |y| + C, \quad 3 < |y| < \tau \delta_k^{-1}.$$

Using this and the Green's representation of w_k we can first observe that the oscillation on each ∂B_r is $o(1)$ ($R < r < \tau \delta_k^{-1}/2$) and then by the Green's representation

of \tilde{w}_k and fast decay rate of e^{V_k} we obtain $\tilde{w}_k = o(1)$ in $\overline{B(0, \tau\delta_k^{-1})}$. A contradiction to $\max |\tilde{w}_k| = 1$.

There are $N + 1$ local maximums with one of them being e_1 . Correspondingly there are $N + 1$ global solutions $V_{l,k}$ that approximate v_k accurately near Q_l^k for $l = 0, \dots, N$. Note that $Q_0^k = e_1$. For $V_{l,k}$ the expression is

$$V_{l,k} = \log \frac{e^{\mu_l^k}}{\left(1 + \frac{e^{\mu_l^k}}{D_l^k} |y|^{N+1} - (e_1 + p_l^k)|^2\right)^2}, \quad l = 0, \dots, N,$$

where $p_l^k = E$ and

$$(4.27) \quad D_l^k = 8(N+1)^2 / \mathfrak{h}_k(\delta_k Q_l^k).$$

The equation that $V_{l,k}$ satisfies is

$$\Delta V_{l,k} + |y|^{2N} \mathfrak{h}_k(\delta_k Q_l^k) e^{V_{l,k}} = 0, \quad \text{in } \mathbb{R}^2.$$

Since v_k and $V_{l,k}$ have the same common local maximum at Q_l^k , it is easy to see that

$$(4.28) \quad Q_l^k = e^{i\beta_l} + \frac{p_l^k e^{i\beta_l}}{N+1} + O(|p_l^k|^2), \quad \beta_l = \frac{2l\pi}{N+2}.$$

Let $M_{l,k}$ be the maximum of $|v_k - V_{l,k}|$ and we claim that all these $M_{l,k}$ are comparable:

$$(4.29) \quad M_{l,k} \sim M_{s,k}, \quad \forall s \neq l.$$

The proof of (4.29) is as follows: We use $L_{s,l}$ to denote the limit of $(v_k - V_{l,k})/M_{l,k}$ around Q_s^k :

$$(4.30) \quad \frac{(v_k - V_{l,k})(Q_s^k + \varepsilon_k z)}{M_{l,k}} = L_{s,l} + o(1), \quad |z| \leq \tau \varepsilon_k^{-1}$$

where

$$L_{s,l} = c_{1,s,l} \frac{z_1}{1 + \frac{1}{8}|z|^2} + c_{2,s,l} \frac{z_2}{1 + \frac{1}{8}|z|^2}, \quad \text{and } L_{l,l} = 0, \quad s = 0, \dots, N.$$

If all $c_{1,s,l}$ and $c_{2,s,l}$ are zero for a fixed l , we can obtain a contradiction just like the beginning of step two. So at least one of them is not zero. For each $s \neq l$, by Lemma 4.2 we have

$$(4.31) \quad v_k(Q_s^k + \varepsilon_k z) - V_{s,k}(Q_s^k + \varepsilon_k z) = O(\varepsilon_k^\sigma)(1 + |z|)^\sigma M_{s,k}, \quad |z| < \tau \varepsilon_k^{-1}.$$

Let $M_k = \max_i M_{i,k}$ ($i = 0, \dots, N$) and we suppose $M_k = M_{l,k}$. Then to determine $L_{s,l}$ we see that

$$(4.32) \quad \frac{v_k(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z)}{M_k} = o(\varepsilon_k^\sigma)(1 + |z|)^\sigma + \frac{V_{s,k}(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z)}{M_k}.$$

Step 3: Use global solutions to determine $L_{s,l}$. (4.30) and (4.32) provide crucial information to determine the coefficients of $L_{s,l}$. From them we know that $L_{s,l}$ is mainly determined by the difference of two global solutions $V_{s,k}$ and $V_{l,k}$. In order

to obtain a contradiction to our assumption we will put the difference in several terms. The main idea in this part of the reasoning is that “first order terms” tell us what the kernel functions should be, then the “second order terms” tell us where the pathology is.

We write $V_{s,k}(y) - V_{l,k}(y)$ as

$$V_{s,k}(y) - V_{l,k}(y) = \mu_s^k - \mu_l^k + 2A - A^2 + O(|A|^3)$$

where

$$A(y) = \frac{\frac{e^{\mu_l^k}}{D_l^k} |y^{N+1} - e_1 - p_l^k|^2 - \frac{e^{\mu_s^k}}{D_s^k} |y^{N+1} - e_1 - p_s^k|^2}{1 + \frac{e^{\mu_s^k}}{D_s^k} |y^{N+1} - e_1 - p_s^k|^2}.$$

Here for convenience we abuse the notation ε_k by assuming $\varepsilon_k = e^{-\mu_s^k/2}$. Note that $\varepsilon_k = e^{-\mu_t^k/2}$ for some t , but it does not matter which t it is. From A we claim that

$$(4.33) \quad \begin{aligned} & V_{s,k}(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z) \\ &= \phi_1 + \phi_2 + \phi_3 + \phi_4 + \mathfrak{R}, \end{aligned}$$

where

$$\begin{aligned} \phi_1 &= (\mu_s^k - \mu_l^k) \frac{1 - \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{8} |z + \frac{N}{2} \varepsilon_k z^2 e^{-i\beta_s} + O(\varepsilon_k^2)(1 + |z|)^3|^2}{\mathbf{B}}, \\ \phi_2 &= \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{4\mathbf{B}} \delta_k \nabla \log \mathfrak{h}_k(\delta_k Q_s^k) (Q_l^k - Q_s^k) \\ &\quad \cdot \left| z + \frac{(p_s^k - p_l^k) e^{i\beta_s}}{(N+1)\varepsilon_k} + \frac{N}{2} \varepsilon_k z^2 e^{-i\beta_s} + O(\varepsilon_k^2)(1 + |z|)^3 \right|^2 \\ \phi_3 &= \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{2\mathbf{B}} \operatorname{Re} \left(\left(z + \frac{N}{2} \varepsilon_k e^{-i\beta_s} z^2 + O(\varepsilon_k^2)(1 + |z|)^3 \right) \left(\frac{\bar{p}_s^k - \bar{p}_l^k}{(N+1)\varepsilon_k} e^{-i\beta_s} \right) \right) \\ \phi_4 &= \frac{\mathfrak{h}_k(\delta_k Q_s)}{4} \frac{|p_s^k - p_l^k|^2}{(N+1)^2 \varepsilon_k^2} \frac{1}{\mathbf{B}^2} \left(1 - \frac{\mathfrak{h}_k(\delta_k Q_s)}{8} |z|^2 \cos(2\theta - 2\theta_{st} - 2\beta_s) \right), \\ \mathbf{B} &= 1 + \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{8} |z + \frac{N}{2} \varepsilon_k e^{-i\beta_s} z^2 + O(\varepsilon_k^2)(1 + |z|)^3|^2, \end{aligned}$$

and \mathfrak{R}_k is the collections of other insignificant terms. Here we briefly explain the roles of each term. ϕ_1 corresponds to the radial solution in the kernel of the linearized operator of the global equation. In other words, ϕ_1^k/M_k should tend to zero because in step one we have proved $c_0 = 0$. ϕ_2^k/M_k is the combination of the two other functions in the kernel. ϕ_4 is the second order term which will play a leading role later. ϕ_3^k comes from the difference of \mathfrak{h}_k at Q_l^k and Q_s^k . The derivation of (4.33) is as follows: Here we use simplified notations for convenience. First we list the following elementary expressions:

$$(4.34) \quad e^{\mu_l} = e^{\mu_s} (1 + (\mu_l - \mu_s) + O((\mu_l - \mu_s)^2)).$$

By the definition of D_s^k in (4.27)

$$(4.35) \quad \frac{1}{D_l} = \frac{1}{D_s} \left(1 + \frac{D_s - D_l}{D_l} \right) = \frac{1}{D_s} \left(1 + \delta_k \nabla \log \mathfrak{h}_k(\delta_k Q_s) (Q_l - Q_s) \right) + O(\delta_k^2)$$

$$\begin{aligned}
(4.36) \quad & |y^{N+1} - e_1 - p_l|^2 - |y^{N+1} - e_1 - p_s|^2 \\
& = |y^{N+1} - 1 - p_s + (p_s - p_l)|^2 - |y^{N+1} - 1 - p_s|^2 \\
& = 2\operatorname{Re} \left((y^{N+1} - 1 - p_s)(\bar{p}_s - \bar{p}_l) \right) + |p_l - p_s|^2.
\end{aligned}$$

Using (4.34),(4.35) and (4.36) we have

$$\begin{aligned}
(4.37) \quad & \frac{e^{\mu_l}}{D_l} |y^{N+1} - e_1 - p_l|^2 - \frac{e^{\mu_s}}{D_s} |y^{N+1} - e_1 - p_s|^2 \\
& = \frac{e^{\mu_s}}{D_s} \left(2\operatorname{Re} \left((y^{N+1} - 1 - p_s)(\bar{p}_s - \bar{p}_l) \right) + |p_s - p_l|^2 \right) \\
& \quad + \frac{e^{\mu_s}}{D_s} |y^{N+1} - 1 - p_l|^2 (\delta_k \nabla \log \mathfrak{h}_k(\delta_k Q_s))(Q_l - Q_s) + \mu_l - \mu_s + E_{c,k}
\end{aligned}$$

where $E_{c,k}$ is a constant of the size $O((\mu_l - \mu_s)^2) + O(\delta_k^2)$. By the expression of Q_s^k in (4.28) we have, for $y = Q_s^k + \varepsilon_k z$,

$$y^{N+1} = 1 + p_s + (N+1)\varepsilon_k z e^{-i\beta_s} + \frac{N(N+1)}{2} \varepsilon_k^2 z^2 e^{-2i\beta_s} + O(\varepsilon_k^3)(1+|z|)^3,$$

which yields

$$(4.38) \quad |y^{N+1} - e_1 - p_s|^2 = (N+1)^2 \varepsilon_k^2 |z + \frac{N}{2} \varepsilon_k e^{-i\beta_s} z^2 + O(\varepsilon_k^2)(1+|z|)^3|^2$$

$$\begin{aligned}
(4.39) \quad & |y^{N+1} - e_1 - p_l|^2 \\
& = (N+1)^2 \varepsilon_k^2 \left| z + \frac{(p_s - p_l)e^{i\beta_s}}{(N+1)\varepsilon_k} + \frac{N}{2} \varepsilon_k z^2 e^{-i\beta_s} + O(\varepsilon_k^2)(1+|z|)^3 \right|^2.
\end{aligned}$$

Using (4.38) and (4.39) in (4.37) we have

$$\begin{aligned}
(4.40) \quad A & = \frac{\frac{e^{\mu_l^k}}{D_l^k} |y^{N+1} - e_1 - p_l|^2 - \frac{e^{\mu_s^k}}{D_s^k} |y^{N+1} - e_1 - p_s|^2}{1 + \frac{e^{\mu_s^k}}{D_s^k} |y^{N+1} - e_1 - p_s|^2} \\
& = \frac{\mathfrak{h}_k(\delta_k Q_s)}{8} \left(2\operatorname{Re} \left[\left(z + \frac{N}{2} \varepsilon_k e^{-i\beta_s} z^2 + O(\varepsilon_k^2)(1+|z|)^3 \right) \frac{\bar{p}_s - \bar{p}_l}{(N+1)\varepsilon_k} e^{-i\beta_s} \right] \right. \\
& \quad \left. + \left| \frac{p_s - p_l}{(N+1)\varepsilon_k} \right|^2 + \left| z + \frac{(p_s - p_l)e^{i\beta_s}}{(N+1)\varepsilon_k} + \frac{N}{2} \varepsilon_k z^2 e^{-i\beta_s} + O(\varepsilon_k^2)(1+|z|)^3 \right|^2 \right) * \\
& \quad \left(\delta_k \nabla (\log \mathfrak{h}_k)(\delta_k Q_s)(Q_l - Q_s) + \mu_l - \mu_s + E_{c,k} \right) / B.
\end{aligned}$$

where the expression of B is

$$B = 1 + \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{8} \left| z + \frac{N}{2} \varepsilon_k e^{-i\beta_s} z^2 + O(\varepsilon_k^2)(1+|z|)^3 \right|^2.$$

Here we point out the expression of ϕ_1 is a combination of the $\mu_s^k - \mu_l^k$ outside $2A - A^2$ and the $(\mu_l^k - \mu_s^k)$ term in the expression of A in (4.40).

For A^2 the leading term, which is the only term that matters in the computation later is

$$(4.41) \quad A^2 = \left(\frac{\mathfrak{h}_k(\delta_k Q_s)^2}{32} |z|^2 \left| \frac{p_s - p_l}{(N+1)\varepsilon_k} \right|^2 (1 + \cos(2\theta - 2\theta_{sl} - \beta_s)) \right) / B^2 + \mathfrak{R}$$

where $z = |z|e^{i\theta}$, $p_s - p_l = |p_s - p_l|e^{i\theta_{sl}}$ and \mathfrak{R} represents the sum of other terms. Using these expressions we can obtain (4.33) by direct computation. Here ϕ_1, ϕ_3 correspond to solutions to the linearized operator. Here we note that if we set $\varepsilon_{l,k} = e^{-\mu_l^k/2}$, there is no essential difference between $\varepsilon_{l,k}$ and $\varepsilon_k = e^{-\frac{1}{2}\mu_{l,k}}$ because $\varepsilon_{l,k} = \varepsilon_k(1 + o(1))$. If $|\mu_{s,k} - \mu_{l,k}|/M_k \geq C$ we get a contradiction to $\tilde{w}_k = o(1)$ outside the bubble disks. Thus, we must have $|\mu_{s,k} - \mu_{l,k}|/M_k \rightarrow 0$. After simplification (see ϕ_3 of (4.33)) we have

$$(4.42) \quad \begin{aligned} c_{1,s,l} &= \lim_{k \rightarrow \infty} \frac{|p_s^k - p_l^k|}{2(N+1)M_k \varepsilon_k} \cos(\beta_s + \theta_{sl}), \\ c_{2,s,l} &= \lim_{k \rightarrow \infty} \frac{|p_s^k - p_l^k|}{2(N+1)\varepsilon_k M_k} \sin(\beta_s + \theta_{sl}) \end{aligned}$$

It is also important to observe that even if $M_k = o(\varepsilon_k)$ we still have $M_k \sim \max_s |p_s^k - p_l^k|/\varepsilon_k$. Since each $|p_l^k| = E$, an upper bound for M_k is

$$(4.43) \quad M_k \leq C\mu_k \varepsilon_k + C\delta_k^2 \varepsilon_k^{-1} \leq C\mu_k \varepsilon_k.$$

Equation (4.42) gives us a key observation: $|c_{1,s,l}| + |c_{2,s,l}| \sim |p_s^k - p_l^k|/(\varepsilon_k M_k)$. So whenever $|c_{1,s,l}| + |c_{2,s,l}| \neq 0$ we have $\frac{|p_s^k - p_l^k|}{\varepsilon_k} \sim M_k$. In other words for each l , $M_{l,k} \sim \max_{t \neq l} \frac{|p_t^k - p_l^k|}{\varepsilon_k}$. Hence for any t , if $\frac{|p_t^k - p_l^k|}{\varepsilon_k} \sim M_k$, let $M_{t,k}$ be the maximum of $|v_k - V_{t,k}|$, we have $M_{t,k} \sim M_k$. If all $\frac{|p_t^k - p_l^k|}{\varepsilon_k} \sim M_k$ (4.29) is proved. So we prove that even if some p_t^k is very close to p_l^k , M_t^k is still comparable to M_k . Here is the reason, without loss of generality, $M_k = M_{1,k}$ and corresponding to M_k , there exist p_s^k such that

$$\frac{|p_1^k - p_s^k|}{\varepsilon_k} \sim M_k.$$

For any given p_t^k , if p_t^k is too close to p_1^k : $|p_t^k - p_1^k| < \frac{1}{5}|p_s^k - p_1^k|$, then $|p_t^k - p_s^k| \geq \frac{1}{2}|p_s^k - p_1^k|$. Thus $\frac{|p_t^k - p_s^k|}{\varepsilon_k} \sim M_k$ and $M_t^k \sim M_k$. (4.29) is established. From now on for convenience we shall just use M_k . Since $M_k \sim \max_{s,t} |p_s^k - p_t^k|/\varepsilon_k$, (4.43) holds for M_k .

Now we set

$$(4.44) \quad w_{l,k} = (v_k - V_{l,k}).$$

and

$$\tilde{w}_{l,k} = w_{l,k}/M_k.$$

The equation of $w_{l,k}$ can be written as

$$(4.45) \quad \begin{aligned} \Delta w_{l,k} + |y|^{2N} \mathfrak{h}_k(\delta_k Q_l) e^{\xi_l} w_{l,k} \\ = -\delta_k \nabla \mathfrak{h}_k(\delta_k Q_l) (y - Q_l) |y|^{2N} e^{V_{l,k}} - \delta_k^2 \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l)}{\alpha!} (y - Q_l)^\alpha |y|^{2N} e^{V_{l,k}} \\ + O(\delta_k^3) |y - Q_l|^3 |y|^{2N} e^{V_{l,k}} \end{aligned}$$

where we omitted k in Q_l and ξ_l . ξ_l comes from the Mean Value Theorem and satisfies

$$(4.46) \quad e^{\xi_l} = e^{V_{l,k}} \left(1 + \frac{1}{2} w_{l,k} + O(w_{l,k}^2)\right).$$

The function $\tilde{w}_{l,k}$ satisfies

$$(4.47) \quad \lim_{k \rightarrow \infty} \tilde{w}_{l,k}(Q_s^k + \varepsilon_k z) = \frac{c_{1,s,l} z_1 + c_{2,s,l} z_2}{1 + \frac{1}{8} |z|^2}$$

and around each Q_s^k (4.31) holds with $M_{s,k}$ replaced by M_k . The equation of $\tilde{w}_{l,k}$ is

$$(4.48) \quad \begin{aligned} \Delta \tilde{w}_{l,k} + |y|^{2N} \mathfrak{h}_k(\delta_k Q_l^k) e^{\xi_l^k} \tilde{w}_{l,k} \\ = o(1) (y - Q_l^k) |y|^{2N} e^{V_{l,k}} + o(1) \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l^k)}{\alpha!} (y - Q_l^k)^\alpha |y|^{2N} e^{V_{l,k}} \\ + o(\delta_k) |y - Q_l^k|^3 |y|^{2N} e^{V_{l,k}} \end{aligned}$$

Step four: Better estimate of $\tilde{w}_{l,k}$ away from local maximums. Now for $|y| \sim 1$, we use $w_{l,k}(Q_l^k) = 0$ to write $w_{l,k}(y)$ as

$$(4.49) \quad \begin{aligned} w_{l,k}(y) = \int_{\Omega_k} (G_k(y, \eta) - G_k(Q_l, \eta)) \left(\mathfrak{h}_k(\delta_k Q_l) |\eta|^{2N} e^{\xi_l} w_{l,k}(\eta) \right. \\ \left. + \delta_k \nabla \mathfrak{h}_k(\delta_k Q_l) (\eta - Q_l) |\eta|^{2N} e^{V_{l,k}} \right. \\ \left. + \delta_k^2 \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l)}{\alpha!} (\eta - Q_l)^\alpha |\eta|^{2N} e^{V_{l,k}} \right) + o(\delta_k^2). \end{aligned}$$

Note that the the oscillation of $w_{l,k}$ on $\partial\Omega_k$ is $O(\delta_k^{N+1})$. The harmonic function defined by the boundary value of $w_{l,k}$ has an oscillation of $O(\delta_k^{N+1})$ on $\partial\Omega_k$. The oscillation of this harmonic function in B_R (for any fixed $R > 1$) is $O(\delta_k^{N+2})$. The regular part of the Green's function brings little error in the computation, indeed

$$\begin{aligned} G_k(y, \eta) - G_k(Q_l^k, \eta) \\ = \frac{1}{2\pi} \log \frac{|Q_l - \eta|}{|y - \eta|} + \frac{1}{2\pi} \log \left| \frac{\frac{\eta}{|\eta|} - L_k^{-2} y |\eta|}{\frac{\eta}{|\eta|} - L_k^{-2} Q_l |\eta|} \right|. \\ = \frac{1}{2\pi} \log \frac{|Q_l - \eta|}{|y - \eta|} + O(\delta_k^2) |y| |\eta|, \quad \text{for } |y| \sim 1 \end{aligned}$$

where $L_k = \tau \delta_k^{-1}$. When we consider the integration in (4.49) from the last term, we have the order is $O(\delta_k^2)$, which is $o(\varepsilon_k^2 \mu_k) = o(\varepsilon_k)$. So we have

$$\begin{aligned}
(4.50) \quad \tilde{w}_{l,k}(y) &= -\frac{1}{2\pi} \int_{\Omega_k} \log \frac{|y-\eta|}{|Q_l^k-\eta|} \left(\tilde{w}_{l,k}(\eta) \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{\xi_l} \right. \\
&\quad + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) (\eta - Q_l^k) |\eta|^{2N} e^{V_{l,k}} \\
&\quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l^k)}{\alpha!} (\eta - Q_l^k)^\alpha |\eta|^{2N} e^{V_{l,k}} \right) d\eta + o(\varepsilon_k), \\
&= \tilde{H}_{l,k} + o(\varepsilon_k) \quad \text{for } |y| \sim 1.
\end{aligned}$$

Here we note that it is important to make the error $o(\varepsilon_k)$. A larger error than $o(\varepsilon_k)$ would cause major problems. Now we claim a better estimate of $\tilde{w}_{l,k}$ around Q_l^k :

$$(4.51) \quad |\tilde{w}_{l,k}(Q_l^k + \varepsilon_k y)| = o(\varepsilon_k)(1 + |y|), \quad |y| \leq \tau \varepsilon_k^{-1}.$$

Proof of (4.51): First we need the following crude identity based on (4.33):

$$\begin{aligned}
(4.52) \quad \int_{B(Q_s^k, \tau)} (\tilde{w}_{l,k}(\eta) \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{V_{s,k}} + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) (\eta - Q_l^k) |\eta|^{2N} e^{V_{l,k}}) d\eta \\
= O(\varepsilon_k^\delta), \quad s = 0, \dots, N,
\end{aligned}$$

for some $\delta \in (0, 1)$. When we compare the first term on the right hand side of (4.50) and the first term of (4.52), e^{ξ_l} is replaced by $e^{V_{s,k}}$, this replacement is minor, as one can check from (4.43) and (4.46) that

$$|e^{\xi_l(Q_s^k + \varepsilon_k z)} - e^{V_{s,k}(Q_s^k + \varepsilon_k z)}| = o(\varepsilon_k^\delta)(1 + |z|)^{-4}, \quad |z| \leq \tau_1 \varepsilon_k^{-1}.$$

(4.33) is mainly used in the evaluation of the first term. In order not to disturb the main stream of the proof, we put the proof of (4.52) at the end of this step. The reason it is called a crude estimate is because its actual leading term is of the order $O(\varepsilon_k)$, but it is sufficient to have $O(\varepsilon_k^\delta)$ for the proof of (4.51). Next we observe from (4.50) that $\tilde{H}_{l,k}$ is a harmonic function in $B(Q_l^k, \tau_1)$ and $\tilde{H}_{l,k}(Q_l^k) = 0$. Now we evaluate $H_{l,k}$ on $|y - Q_l^k| = \tau_1$. It is easy to see that the integral outside $\cup_{s=0}^N B(Q_s^k, \tau_1)$ is $O(\varepsilon_k^2)$. Next we see that the integral over $B(Q_l^k, \tau_1)$ is $o(\varepsilon_k^\delta)$ for some $\delta > 0$ because for the first term and the second term we use (4.52), for the third terms we have $|\eta - Q_l^k|^2$, which contributes ε_k^2 after scaling. The integration over other disks gives

$$\sum_{s \neq l} \sigma_{s,k} \log \frac{|y - Q_s^k|}{|Q_l^k - Q_s^k|}$$

where $\sigma_{s,k} \rightarrow 0$ as $k \rightarrow \infty$. Since this is a harmonic function we have

$$\tilde{H}_{l,k}(Q_l^k + \varepsilon_k z) = O(\tilde{\sigma}_k) \varepsilon_k (1 + |z|), \quad |z| \leq \tau_1 \varepsilon_k^{-1},$$

for some $\tilde{\sigma}_k \rightarrow 0$. Thus we have (4.51).

At the end of this step we prove (4.52).

Here we recall that v_k is close to $V_{s,k}$ near Q_s^k (see 4.31)). That is why we shall use (4.33). We state (4.52) here:

$$\begin{aligned} \int_{B(Q_s^k, \tau)} (\tilde{w}_{l,k}(\eta) \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{V_{s,k}} + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) (\eta - Q_l^k) |\eta|^{2N} e^{V_{l,k}}) d\eta \\ = O(\varepsilon_k^\delta) \end{aligned}$$

for some $\delta \in (0, 1)$. Before the evaluation we recall the definition of $w_{l,k}$ in (4.44) that around Q_s^k ,

$$\tilde{w}_{l,k} = \frac{v_k - V_{s,k}}{M_k} + \frac{V_{s,k} - V_{l,k}}{M_k}.$$

The first term after scaling at Q_s^k is $O(\varepsilon_k^\delta)(1 + |y|)^\delta$, so the leading term in the second term. So our goal in this section is to prove

$$\begin{aligned} \int_{B(Q_s^k, \tau)} \left(\frac{V_{s,k} - V_{l,k}}{M_k} \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{V_{s,k}} + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) (\eta - Q_l^k) |\eta|^{2N} e^{V_{l,k}} \right) d\eta \\ (4.53) \quad = O(\varepsilon_k^\delta) \end{aligned}$$

Then we use (4.33) in $(V_{s,k} - V_{l,k})/M_k$. Later we shall see that the terms of ϕ_1 and ϕ_3 lead to $o(\varepsilon_k)$. We first look at the integration involving ϕ_2 :

$$\begin{aligned} (4.54) \quad \int_{B(Q_s^k, \tau)} \frac{\phi_2^k}{M_k} \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{V_{s,k}} d\eta \\ = \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{4} \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_s^k) (Q_l^k - Q_s^k) \int_{B(Q_s^k, \tau)} \frac{|z|^2}{(1 + \frac{\mathfrak{h}_k(\delta_k Q_l^k)}{8} |z|^2)^3} dz \\ = 8\pi \sigma_k \nabla(\log \mathfrak{h}_k(\delta_k Q_s^k)) (Q_l^k - Q_s^k) (1 + O(\varepsilon_k^2 \log 1/\varepsilon_k)) \end{aligned}$$

We see that this term almost cancels with the second term of (4.52). The computation of ϕ_2 is based on this equation:

$$(4.55) \quad \int_{\mathbb{R}^2} \frac{\frac{\mathfrak{h}_k(\delta_k Q_s^k)}{4} \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_s^k) (Q_l^k - Q_s^k) |z|^2}{(1 + \frac{\mathfrak{h}_k(\delta_k Q_s^k)}{8} |z|^2)^3} dz = 8\pi \sigma_k \nabla(\log \mathfrak{h}_k(\delta_k Q_s^k)) (Q_l^k - Q_s^k),$$

and by (4.2)

$$(4.56) \quad \nabla \log \mathfrak{h}_k(\delta_k Q_l^k) - \nabla \log \mathfrak{h}_k(\delta_k Q_s^k) = O(\delta_k) = o(\varepsilon_k \mu_k^{\frac{1}{2}}).$$

The integration involving ϕ_4 provides the leading term. More detailed information is the following: First for a global solution

$$V_{\mu,p} = \log \frac{e^\mu}{(1 + \frac{e^\mu}{\lambda} |z^{N+1} - p|^2)^2}$$

of

$$\Delta V_{\mu,p} + \frac{8(N+1)^2}{\lambda} |z|^{2N} e^{V_{\mu,p}} = 0, \quad \text{in } \mathbb{R}^2,$$

by differentiation with respect to μ we have

$$\Delta(\partial_\mu V_{\mu,p}) + \frac{8(N+1)^2}{\lambda} |z|^{2N} e^{V_{\mu,p}} \partial_\mu V_{\mu,p} = 0, \quad \text{in } \mathbb{R}^2.$$

By the expression of $V_{\mu,p}$ we see that

$$\partial_r \left(\partial_\mu V_{\mu,p} \right) (x) = O(|x|^{-2N-3}).$$

Thus we have

$$(4.57) \quad \int_{\mathbb{R}^2} \partial_\mu V_{\mu,p} |z|^{2N} e^{V_{\mu,p}} = \int_{\mathbb{R}^2} \frac{(1 - \frac{e^\mu}{\lambda} |z^{N+1} - P|^2) |z|^{2N}}{(1 + \frac{e^\mu}{\lambda} |z^{N+1} - P|^2)^3} dz = 0.$$

From $V_{\mu,p}$ we also have

$$\int_{\mathbb{R}^2} \partial_P V_{\mu,p} |y|^{2N} e^{V_{\mu,p}} = \int_{\mathbb{R}^2} \partial_{\bar{P}} V_{\mu,p} |y|^{2N} e^{V_{\mu,p}} = 0,$$

which gives

$$(4.58) \quad \int_{\mathbb{R}^2} \frac{\frac{e^\mu}{\lambda} (\bar{z}^{N+1} - \bar{P}) |z|^{2N}}{(1 + \frac{e^\mu}{\lambda} |z^{N+1} - P|^2)^3} = \int_{\mathbb{R}^2} \frac{\frac{e^\mu}{\lambda} (z^{N+1} - P) |z|^{2N}}{(1 + \frac{e^\mu}{\lambda} |z^{N+1} - P|^2)^3} = 0.$$

From (4.57) and (4.58) we use scaling and cancellation to have

$$(4.59) \quad \int_{B(0, \tau \varepsilon_k^{-1})} \frac{\phi_1}{M_k} B^{-2} = o(\varepsilon_k), \quad \int_{B(0, \tau \varepsilon_k^{-1})} \frac{\phi_3}{M_k} B^{-2} = o(\varepsilon_k).$$

Thus (4.52) holds.

Step five: Completion of the proof. We recall from (4.44) that around Q_s^k

$$(4.60) \quad \tilde{w}_{l,k} = \frac{v_k - V_{s,k}}{M_k} + \frac{V_{s,k} - V_{l,k}}{M_k}.$$

We define this quantity without giving a precise estimate of it:

$$\begin{aligned} D_{s,l}^k &:= \int_{B(Q_s^k, \tau)} \left(\tilde{w}_{l,k}(\eta) \mathfrak{h}_k(\delta_k Q_l^k) |\eta|^{2N} e^{\xi_l} + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) (\eta - Q_l^k) |\eta|^{2N} e^{V_{l,k}} \right. \\ &\quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l^k)}{\alpha!} (\eta - Q_l^k)^\alpha |\eta|^{2N} e^{V_{l,k}} \right) d\eta. \end{aligned}$$

By (4.51) we know $D_{s,l}^k = O(\varepsilon_k^\delta)$. Next we let

$$H_{y,l}(\eta) = \frac{1}{2\pi} \log \frac{|y - \eta|}{|Q_l^k - \eta|}.$$

Then in (4.50) we have

$$\begin{aligned} \tilde{w}_{l,k}(y) &= - \sum_{s \neq l} H_{y,l}(Q_s) D_{s,l}^k \\ &\quad - \sum_{s \neq l} \int_{B(Q_s, \tau)} (\partial_1 H_{y,l}(Q_s) (\eta_1 - Q_s^1) + \partial_2 H_{y,l}(Q_s) (\eta_2 - Q_s^2)) \cdot \mathfrak{h}_k(\delta_k Q_l) |\eta|^{2N} e^{\xi_l} \tilde{w}_{l,k}(\eta) \\ &\quad + o(\varepsilon_k). \end{aligned}$$

After evaluation we have

$$\begin{aligned} \tilde{w}_{l,k}(y) = & -\frac{1}{2\pi} \sum_{s \neq l} \log \frac{|y - Q_s^k|}{|Q_l^k - Q_s^k|} D_{s,l}^k + \sum_{s \neq l} \left(8 \left(\frac{y_1 - Q_s^1}{|y - Q_s|^2} - \frac{Q_l^1 - Q_s^1}{|Q_l - Q_s|^2} \right) c_{1,s,l} \right. \\ & \left. + 8 \left(\frac{y_2 - Q_s^2}{|y - Q_s|^2} - \frac{Q_l^2 - Q_s^2}{|Q_s - Q_l|^2} \right) c_{2,s,l} \right) \varepsilon_k + o(\varepsilon_k). \end{aligned}$$

where we used

$$\int_{\mathbb{R}^2} \frac{z_1^2}{(1 + \frac{1}{8}|z|^2)^3} dz = \int_{\mathbb{R}^2} \frac{z_2^2}{(1 + \frac{1}{8}|z|^2)^3} dz = 16\pi.$$

Recall that $c_{1,s,l}$ and $c_{2,s,l}$ are defined in (4.42).

For $|y| \sim 1$ but away from the $N+1$ bubbling disks, we have, for $l \neq s$,

$$v_k(y) = V_{l,k}(y) + M_k \tilde{w}_{l,k}(y)$$

and

$$v_k(y) = V_{s,k}(y) + M_k \tilde{w}_{s,k}(y).$$

Thus for $s \neq l$ we have

$$(4.61) \quad \frac{V_{s,k}(y) - V_{l,k}(y)}{M_k} = \tilde{w}_{l,k}(y) - \tilde{w}_{s,k}(y).$$

In (4.33) we consider $|z| \sim \varepsilon_k^{-1}$, then we see that if

$$\left| \frac{\mu_l^k - \mu_s^k}{M_k} + 2\sigma_k \nabla \log \mathfrak{h}_k(\delta_k Q_s^k)(Q_l^k - Q_s^k) \right| \geq C\varepsilon_k,$$

for a large C , it is easy to see that (4.61) does not hold. So we have

$$\left| \frac{\mu_l^k - \mu_s^k}{M_k} + 2\sigma_k \nabla \log \mathfrak{h}_k(\delta_k Q_s^k)(Q_l^k - Q_s^k) \right| = O(\varepsilon_k),$$

and we focus on the leading term ϕ_3 , which gives, for $|y| \sim 1$ away from bubbling disks,

$$\frac{V_{s,k}(y) - V_{l,k}(y)}{M_k} = D_\mu^k \varepsilon_k + 4Re \left(\frac{y^{N+1} - 1}{|y^{N+1} - 1|^2} \frac{\bar{p}_s - \bar{p}_l}{M_k \varepsilon_k} \right) \varepsilon_k + O(|y - Q_s^k|^2) \varepsilon_k + o(\varepsilon_k),$$

where

$$D_\mu^k = \frac{\mu_l^k - \mu_s^k}{M_k \varepsilon_k} + 2 \frac{\sigma_k}{\varepsilon_k} \nabla \log \mathfrak{h}_k(\delta_k Q_s^k)(Q_l^k - Q_s^k).$$

On the other hand, for $y \in B_5 \setminus (\cup_{l=1}^N B(Q_l^k, \tau_1))$,

$$\begin{aligned}
& \tilde{w}_{l,k}(y) - \tilde{w}_{s,k}(y) \\
&= -\frac{1}{2\pi} \sum_{m,m \neq l} \log \frac{|y - Q_m|}{|Q_l - Q_m|} D_{m,l}^k \\
&+ 8\varepsilon_k \sum_{m,m \neq l} \left(\left(\frac{y_1 - Q_m^1}{|y - Q_m|^2} - \frac{Q_l^1 - Q_m^1}{|Q_l - Q_m|^2} \right) \frac{|p_m - p_l|}{2(N+1)M_k \varepsilon_k} \cos(\beta_m + \theta_{ml}) \right. \\
&+ \left. \left(\frac{y_2 - Q_m^2}{|y - Q_m|^2} - \frac{Q_l^2 - Q_m^2}{|Q_l - Q_m|^2} \right) \frac{|p_m - p_l|}{2(N+1)M_k \varepsilon_k} \sin(\beta_m + \theta_{ml}) \right) \\
&+ \frac{1}{2\pi} \sum_{m,m \neq s} \log \frac{|y - Q_m|}{|Q_s - Q_m|} D_{m,s}^k \\
&- 8\varepsilon_k \sum_{m,m \neq s} \left(\left(\frac{y_1 - Q_m^1}{|y - Q_m|^2} - \frac{Q_s^1 - Q_m^1}{|Q_s - Q_m|^2} \right) \frac{|p_m - p_s|}{2(N+1)M_k \varepsilon_k} \cos(\beta_m + \theta_{ms}) \right. \\
&+ \left. \left(\frac{y_2 - Q_m^2}{|y - Q_m|^2} - \frac{Q_s^2 - Q_m^2}{|Q_s - Q_m|^2} \right) \frac{|p_m - p_s|}{2(N+1)M_k \varepsilon_k} \sin(\beta_m + \theta_{ms}) \right)
\end{aligned}$$

for all $l \neq s$. If we fix a set of l, s that corresponds to the largest $|D_{s,l}^k|$ and we consider y close to Q_s^k . If we use $y = e^{i\beta_s} + z$ by abusing the notation z , then we have

$$y^{N+1} = (e^{i\beta_s}(1 + ze^{-i\beta_s}))^{N+1} = 1 + (N+1)ze^{-i\beta_s} + O(|z|^2).$$

Therefore

$$\begin{aligned}
& 4Re \left(\frac{y^{N+1} - 1}{|y^{N+1} - 1|^2} \frac{\bar{p}_s - \bar{p}_l}{M_k \varepsilon_k} \right) \\
&= \frac{4|p_s - p_l|}{(N+1)|z|^2 M_k \varepsilon_k} \left(z_1 \cos(\beta_s + \beta_{sl}) + z_2 \sin(\beta_s + \beta_{sl}) + O(|z|^2) \right).
\end{aligned}$$

In the expression of $\tilde{w}_{l,k}(y) - \tilde{w}_{s,k}(y)$, we identify the leading term, which is

$$\begin{aligned}
& 8\varepsilon_k \left(\left(\frac{y_1 - Q_s^1}{|y - Q_s|^2} - \frac{Q_l^1 - Q_s^1}{|Q_l - Q_s|^2} \right) \frac{|p_s - p_l|}{2(N+1)M_k \varepsilon_k} \cos(\beta_s + \theta_{sl}) \right. \\
&+ \left. \left(\frac{y_2 - Q_s^2}{|y - Q_s|^2} - \frac{Q_l^2 - Q_s^2}{|Q_l - Q_s|^2} \right) \frac{|p_s - p_l|}{2(N+1)M_k \varepsilon_k} \sin(\beta_s + \theta_{sl}) \right) \\
&\quad - \frac{1}{2\pi} \log \frac{|y - Q_s|}{|Q_l - Q_s|} D_{s,l}^k.
\end{aligned}$$

If we use $y = e^{i\beta_s} + z$ for $|z|$ small and replace Q_s by $e^{i\beta_s}$ because their difference is $o(\varepsilon_k)$. Then the expression above has this leading term:

$$\frac{4|p_s - p_l|}{(N+1)|z|^2 M_k \varepsilon_k} \left(z_1 \cos(\beta_s + \beta_{sl}) + z_2 \sin(\beta_s + \beta_{sl}) \right) - \frac{1}{2\pi} \log \frac{|z|}{|e^{i\beta_l} - e^{i\beta_s}|} D_{s,l}^k.$$

Thus we obtain $D_{s,l}^k/\varepsilon_k = o(1)$. Therefore

$$(4.62) \quad D_{s,l}^k = o(\varepsilon_k), \quad \forall s \neq l.$$

With this updated information we write $\tilde{w}_{l,k}(y) - \tilde{w}_{s,k}(y)$ as

$$\begin{aligned} & \tilde{w}_{l,k}(y) - \tilde{w}_{s,k}(y) \\ &= 8\varepsilon_k \sum_{m,m \neq l} \left(\left(\frac{y_1 - \cos \beta_m}{|y - e^{i\beta_m}|^2} - \frac{\cos \beta_l - \cos \beta_m}{|e^{i\beta_l} - e^{i\beta_m}|^2} \right) \frac{|p_m - p_l|}{2(N+1)M_k \varepsilon_k} \cos(\beta_m + \theta_{ml}) \right. \\ & \quad \left. + \left(\frac{y_2 - \sin \beta_m}{|y - e^{i\beta_m}|^2} - \frac{\sin \beta_l - \sin \beta_m}{|e^{i\beta_l} - e^{i\beta_m}|^2} \right) \frac{|p_m - p_l|}{2(N+1)M_k \varepsilon_k} \sin(\beta_m + \theta_{ml}) \right) \\ & - 8\varepsilon_k \sum_{m,m \neq s} \left(\left(\frac{y_1 - \cos \beta_m}{|y - e^{i\beta_m}|^2} - \frac{\cos \beta_s - \cos \beta_m}{|e^{i\beta_s} - e^{i\beta_m}|^2} \right) \frac{|p_m - p_s|}{2(N+1)M_k \varepsilon_k} \cos(\beta_m + \theta_{ms}) \right. \\ & \quad \left. + \left(\frac{y_2 - \sin \beta_m}{|y - e^{i\beta_m}|^2} - \frac{\sin \beta_s - \sin \beta_m}{|e^{i\beta_s} - e^{i\beta_m}|^2} \right) \frac{|p_m - p_s|}{2(N+1)M_k \varepsilon_k} \sin(\beta_m + \theta_{ms}) \right) + o(\varepsilon_k) \end{aligned}$$

for $|y| \in B_5 \setminus (\cup_{l=1}^N B(Q_l^k, \tau_1))$.

Now in particular we take $l = 0$ and we use the following notations: $\tilde{w}_k, V_k, c_{1,s}, c_{2,s}, \theta_s$, instead of $\tilde{w}_0^k, v_{0,k}, c_{1,s,0}, c_{2,s,0}, \theta_{s,0}$.

The expression of \tilde{w}_k (see (4.50) for example) gives

$$\begin{aligned} & \nabla \tilde{w}_k(y) \\ &= \int_{\Omega_k} \nabla_y G(y, \eta) \left(\mathfrak{h}_k(\delta_k e_1) |\eta|^{2N} e^{\xi_k} \tilde{w}_k(\eta) + \sigma_k \nabla \mathfrak{h}_k(\delta_k e_1) (\eta - e_1) |\eta|^{2N} e^{V_k(\eta)} \right. \\ & \quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k e_1) (\eta - e_1)^\alpha}{\alpha!} |\eta|^{2N} e^{V_k(\eta)} \right) d\eta + o(\varepsilon_k), \end{aligned}$$

for $y \in B_5 \setminus (\cup_{s=1}^N B(Q_s^k, \tau_1))$. Now we take $y = e_1$, we have

$$\begin{aligned} & 0 = \nabla \tilde{w}_k(e_1) \\ &= \int_{\Omega_k} \left(-\frac{1}{2\pi} \right) \frac{e_1 - \eta}{|e_1 - \eta|^2} \left(\mathfrak{h}_k(\delta_k e_1) |\eta|^{2N} e^{\xi_k} \tilde{w}_k(\eta) + \sigma_k \nabla \mathfrak{h}_k(\delta_k e_1) (\eta - e_1) |\eta|^{2N} e^{V_k(\eta)} \right. \\ & \quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k e_1) (\eta - e_1)^\alpha}{\alpha!} |\eta|^{2N} e^{V_k(\eta)} \right) d\eta + o(\varepsilon_k), \end{aligned}$$

Obviously we will integral each of the two components in $B(Q_s^k, \tau_1)$ for $\tau_1 > 0$ small. Then we observe from (4.62) that

$$\begin{aligned} & \int_{B(Q_l, \tau_1)} \left(\mathfrak{h}_k(\delta_k e_1) |\eta|^{2N} e^{\xi_k} \tilde{w}_k(\eta) + \sigma_k \nabla \mathfrak{h}_k(\delta_k e_1) (\eta - e_1) |\eta|^{2N} e^{V_k(\eta)} \right. \\ & \quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k e_1) (\eta - e_1)^\alpha}{\alpha!} |\eta|^{2N} e^{V_k(\eta)} \right) d\eta = o(\varepsilon_k). \end{aligned}$$

Based on this we use the following format: If f is a smooth function,

$$\begin{aligned} & \int_{B(Q_s, \tau_1)} f(\eta) \left(\mathfrak{h}_k(\delta_k e_1) |\eta|^{2N} e^{\tilde{\xi}_k} \tilde{w}_k(\eta) + \sigma_k \nabla \mathfrak{h}_k(\delta_k e_1) (\eta - e_1) |\eta|^{2N} e^{V_k(\eta)} \right. \\ & \quad \left. + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k e_1) (\eta - e_1)^\alpha}{\alpha!} |\eta|^{2N} e^{V_k(\eta)} \right) d\eta \\ & = \partial_1 f(e^{i\beta_s}) c_{1s} \cdot 16\pi \varepsilon_k + \partial_1 f(e^{i\beta_s}) c_{2s} \cdot 16\pi \varepsilon_k + o(\varepsilon_k). \end{aligned}$$

Then we replace $f(\eta_1, \eta_2)$ by

$$f_1(\eta_1, \eta_2) = \left(-\frac{1}{2\pi}\right) \frac{1 - \eta_1}{(1 - \eta_1)^2 + \eta_2^2}$$

and

$$f_2(\eta_1, \eta_2) = \frac{1}{2\pi} \frac{\eta_2}{(1 - \eta_1)^2 + \eta_2^2}.$$

Then we have, from the expressions of $c_{1,s}$, $c_{2,s}$ in (4.42), that

$$\begin{aligned} 0 & = \partial_1 \tilde{w}_k(e_1) \\ & = 16\pi \varepsilon_k \sum_{s=1}^N \left(\frac{\cos(\beta_s + \theta_s) \cos \beta_s + \sin \beta_s \sin(\beta_s + \theta_s)}{4\pi(1 - \cos \beta_s)} \frac{|p_s|}{2(N+1)M_k \varepsilon_k} \right) + o(\varepsilon_k) \\ & = 4\varepsilon_k \sum_{s=1}^N \frac{\cos \theta_s}{1 - \cos \beta_s} \frac{|p_s|}{2(N+1)M_k \varepsilon_k} + o(\varepsilon_k). \end{aligned}$$

Similarly

$$\begin{aligned} 0 & = \partial_2 \tilde{w}_k(e_1) \\ & = 16\pi \varepsilon_k \sum_{s=1}^N \left(\frac{\cos(\beta_s + \theta_s) \sin \beta_s - \cos \beta_s \sin(\beta_s + \theta_s)}{4\pi(1 - \cos \beta_s)} \frac{|p_s|}{2(N+1)M_k \varepsilon_k} \right) + o(\varepsilon_k) \\ & = -4\varepsilon_k \sum_{s=1}^N \frac{\sin \theta_s}{1 - \cos \beta_s} \frac{|p_s|}{2(N+1)M_k \varepsilon_k} + o(\varepsilon_k). \end{aligned}$$

If we use a_s to denote

$$a_s := \lim_{k \rightarrow \infty} \frac{|p_s^k|}{(1 - \cos \beta_s) M_k \varepsilon_k}, \quad s = 1, \dots, N,$$

then we have

$$(4.63) \quad \sum_{s=1}^N a_s \cos \theta_s = 0$$

$$(4.64) \quad \sum_{s=1}^N a_s \sin \theta_s = 0,$$

where $a_s \geq 0$ for all s . Taking the sum of the squares of (4.63) and (4.64) we obtain

$$\sum_{s=1}^N a_s^2 + \sum_{s < t} 2a_s a_t \cos(\theta_s - \theta_t) = 0.$$

Since $\sum_s a_s^2 \geq 2\sum_{s<t} a_s a_t$, we have

$$\sum_{s<t} 2a_s a_t (1 + \cos(\theta_s - \theta_t)) \leq 0.$$

Since each term on the left is obviously non-negative, we know each

$$a_s a_t (1 + \cos(\theta_s - \theta_t)) = 0, \quad \forall s < t.$$

If there is only one $a_s > 0$, it is easy to see that (4.63) and (4.64) cannot both hold. If there are three $a'_t > 0$, it is also elementary to see this is not possible: say $a_1, a_2, a_3 > 0$, then they have to be equal. Then we see that we must have

$$\theta_1 - \theta_2 = \pm\pi, \quad \theta_2 - \theta_3 = \pm\pi, \quad \theta_1 - \theta_3 = \pm\pi.$$

Obviously these three equations cannot hold at the same time. So the only situation left is there are exactly two a'_t s positive. All other a_t s are zero. Since $p_0 = 0$, this means there are exactly two $p_{s_1}^k, p_{s_2}^k$ such that

$$(4.65) \quad \lim_{k \rightarrow \infty} \frac{p_{s_1}^k}{\varepsilon_k M_k} = -\lim_{k \rightarrow \infty} \frac{p_{s_2}^k}{\varepsilon_k M_k} \neq 0, \quad \lim_{k \rightarrow \infty} \frac{p_t^k}{\varepsilon_k M_k} = 0, \quad \forall t \neq s_1, s_2.$$

If we apply the same argument to \tilde{w}_l^k . Then from $\nabla \tilde{w}_l^k(Q_l^k) = 0$ we would get exactly $p_{l_1}^k$ and $p_{l_2}^k$ different from p_l^k and

$$\lim_{k \rightarrow \infty} \frac{p_{l_1}^k - p_l^k}{\varepsilon_k M_k} = -\lim_{k \rightarrow \infty} \frac{p_{l_2}^k - p_l^k}{\varepsilon_k M_k} \neq 0, \quad \lim_{k \rightarrow \infty} \frac{p_t^k - p_l^k}{\varepsilon_k M_k} = 0, \forall t \neq l_1, l_2.$$

Then it is easy to see that this is only possible when we have $N = 2$ because if $N = 1$, we would have just one $a_s \neq 0$, which is not possible based on (4.63) and (4.64). If $N \geq 3$, we have to have p_t^k that satisfies

$$\lim_{k \rightarrow \infty} \frac{|p_t^k|}{\varepsilon_k M_k} = 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{p_t^k - p_{s_1}^k}{\varepsilon_k M_k} = 0$$

which is a contradiction to (4.65).

Finally we rule out the case $N = 2$. In this case we have $p_0^k = 0$,

$$(4.66) \quad \lim_{k \rightarrow \infty} \frac{p_1^k}{\varepsilon_k M_k} = -\lim_{k \rightarrow \infty} \frac{p_2^k}{\varepsilon_k M_k} \neq 0.$$

However from $\tilde{w}_1^k(p_1^k) = 0$ we have

$$\lim_{k \rightarrow \infty} \frac{p_2^k - p_1^k}{\varepsilon_k M_k} = -\lim_{k \rightarrow \infty} \frac{0 - p_1^k}{\varepsilon_k M_k} \neq 0,$$

which is a contradiction to (4.66). Lemma 4.3 is established. \square

Proposition 4.1 is an immediate consequence of Lemma 4.3. \square .

Now we finish the proof of Theorem 4.1.

Let $\hat{w}_k = w_k / \tilde{\delta}_k$. (Recall that $\tilde{\delta}_k = \delta_k |\nabla \eta_k(0)| + \delta_k^2$). If $|\nabla \eta_k(0)| / \delta_k \rightarrow \infty$, we see that in this case $\tilde{\delta}_k \sim \delta_k |\nabla \eta_k(0)|$. The equation of \hat{w}_k is

$$(4.67) \quad \Delta \hat{w}_k + |y|^{2N} e^{\xi_k} \hat{w}_k = a_k \cdot (e_1 - y) |y|^{2N} e^{V_k} + b_k e^{V_k} |y - e_1|^2 |y|^{2N},$$

in Ω_k , where $a_k = \delta_k \nabla \mathfrak{h}_k(0) / \tilde{\delta}_k$, $b_k = o(1)$. By Proposition 4.1, $|\hat{w}_k(y)| \leq C$. Before we carry out the remaining part of the proof we observe that \hat{w}_k converges to a harmonic function in \mathbb{R}^2 minus finite singular points. Since \hat{w}_k is bounded, all these singularities are removable. Thus \hat{w}_k converges to a constant. Based on the information around e_1 , we claim this constant is 0. To do this we use the notation W_k again and use Proposition 4.1 to rewrite the equation for W_k . Let

$$W_k(z) = \hat{w}_k(e_1 + \varepsilon_k z), \quad |z| < \delta_0 \varepsilon_k^{-1}$$

for $\delta_0 > 0$ small. Then from Proposition 4.1 we have

$$(4.68) \quad \mathfrak{h}_k(\delta_k y) = \mathfrak{h}_k(\delta_k e_1) + \delta_k \nabla \mathfrak{h}_k(\delta_k e_1)(y - e_1) + O(\delta_k^2) |y - e_1|^2,$$

$$(4.69) \quad |y|^{2N} = |e_1 + \varepsilon_k z|^{2N} = 1 + O(\varepsilon_k) |z|,$$

$$(4.70) \quad V_k(e_1 + \varepsilon_k z) + 2 \log \varepsilon_k = U_k(z) + O(\varepsilon_k) |z| + O(\varepsilon_k^2) (\log(1 + |z|))^2$$

and

$$(4.71) \quad \xi_k(e_1 + \varepsilon_k z) + 2 \log \varepsilon_k = U_k(z) + O(\varepsilon_k) (1 + |z|).$$

Using (4.68), (4.69), (4.70) and (4.71) in (4.67) we write the equation of W_k as

$$(4.72) \quad \Delta W_k + \mathfrak{h}_k(\delta_k e_1) e^{U_k(z)} W_k = -\varepsilon_k a_k \cdot z e^{U_k(z)} + E_w, \quad 0 < |z| < \delta_0 \varepsilon_k^{-1}$$

where

$$(4.73) \quad E_w(z) = O(\varepsilon_k) (1 + |z|)^{-3}, \quad |z| < \delta_0 \varepsilon_k^{-1}.$$

Since \hat{w}_k obviously converges to a global harmonic function with removable singularity, we have $\hat{w}_k \rightarrow \bar{c}$ for some $\bar{c} \in \mathbb{R}$. Then we claim that

$$(4.74) \quad \bar{c} = 0.$$

The proof of (4.74) is the similar as before: If $\bar{c} \neq 0$, we use $W_k(z) = \bar{c} + o(1)$ on $B(0, \delta_0 \varepsilon_k^{-1}) \setminus B(0, \frac{1}{2} \delta_0 \varepsilon_k^{-1})$ and consider the projection of W_k on 1:

$$g_0(r) = \frac{1}{2\pi} \int_0^{2\pi} W_k(re^{i\theta}) d\theta.$$

If we use F_0 to denote the projection to 1 of the right hand side we have,

$$g_0''(r) + \frac{1}{r} g_0'(r) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(r)} g_0(r) = F_0, \quad 0 < r < \delta_0 \varepsilon_k^{-1}$$

where

$$F_0(r) = O(\varepsilon_k^\varepsilon) (1 + |z|)^{-3}.$$

In addition we also have

$$\lim_{k \rightarrow \infty} g_0(\delta_0 \varepsilon_k^{-1}) = \bar{c} + o(1).$$

For simplicity we omit k in some notations. By the same argument as in Lemma 4.1, we have

$$g_0(r) = O(\varepsilon_k^\varepsilon) \log(2 + r), \quad 0 < r < \delta_0 \varepsilon_k^{-1}.$$

Thus $\bar{c} = 0$.

Based on Lemma 4.74 and standard Harnack inequality for elliptic equations we have

$$(4.75) \quad \tilde{w}_k(x) = o(1), \quad \nabla \tilde{w}_k(x) = o(1), \quad x \in B_3 \setminus (\cup_{l=1}^N (B(e^{i\beta_l}, \delta_0) \setminus B(e^{i\beta_l}, \delta_0/8))).$$

Equation (4.75) is equivalent to $w_k = o(\tilde{\delta}_k)$ and $\nabla w_k = o(\tilde{\delta}_k)$ in the same region.

In the next step we consider the difference between two Pohozaev identities. For $\Omega_{0,k} = B(e_1^k, r)$ ($r > 0$ small) we consider the Pohozaev identity around e_1 . For v_k we have

$$(4.76) \quad \int_{\Omega_{0,k}} \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k y)) e^{v_k} - \int_{\partial\Omega_{0,k}} e^{v_k} |y|^{2N} \mathfrak{h}_k(\delta_k y) (\xi \cdot \nu) \\ = \int_{\partial\Omega_{0,k}} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2} |\nabla v_k|^2 (\xi \cdot \nu)) dS.$$

where ξ is an arbitrary unit vector. Correspondingly the Pohozaev identity for V_k is

$$(4.77) \quad \int_{\Omega_{0,k}} \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k e_1)) e^{V_k} - \int_{\partial\Omega_{0,k}} e^{V_k} |y|^{2N} \mathfrak{h}_k(\delta_k e_1) (\xi \cdot \nu) \\ = \int_{\partial\Omega_{0,k}} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2} |\nabla V_k|^2 (\xi \cdot \nu)) dS.$$

Using $w_k = v_k - V_k$ and $|w_k(y)| = o(\tilde{\delta}_k)$ on $\partial\Omega_{0,k}$ we have

$$\int_{\partial\Omega_{0,k}} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2} |\nabla v_k|^2 (\xi \cdot \nu)) - \int_{\partial\Omega_{0,k}} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2} |\nabla V_k|^2 (\xi \cdot \nu)) = o(\tilde{\delta}_k).$$

The difference between the second term of (4.76) and the second term of (4.77) is minor: If we use the expansion of $v_k = V_k + w_k$ and that of $\mathfrak{h}_k(\delta_k y)$ around e_1 , it is easy to obtain

$$\int_{\partial\Omega_{0,k}} e^{v_k} |y|^{2N} \mathfrak{h}_k(\delta_k y) (\xi \cdot \nu) - \int_{\partial\Omega_{0,k}} e^{V_k} |y|^{2N} \mathfrak{h}_k(\delta_k e_1) (\xi \cdot \nu) = o(\tilde{\delta}_k).$$

To evaluate the first term, we use the following updated estimate of w_k :

$$|w_k(e_1 + \varepsilon_k z)| \leq C \tilde{\delta}_k \varepsilon_k (1 + |z|), \quad |z| \leq r \varepsilon_k^{-1}.$$

Using this expression and

$$(4.78) \quad \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k y)) e^{v_k} \\ = \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k e_1) + |y|^{2N} \delta_k \nabla \mathfrak{h}_k(\delta_k e_1) (y - e_1) + O(\delta_k^2)) e^{V_k} (1 + w_k + O(\delta_k^2)) \\ = \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k e_1)) e^{V_k} + \delta_k \partial_\xi (|y|^{2N} \nabla \mathfrak{h}_k(\delta_k e_1) (y - e_1)) e^{V_k} \\ + \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k e_1)) e^{V_k} w_k + O(\delta_k^2) e^{V_k},$$

we have

$$(4.79) \quad \int_{\Omega_{0,k}} \partial_\xi (|y|^{2N} \mathfrak{h}_k(\delta_k e_1)) e^{V_k} w_k = o(\tilde{\delta}_k).$$

For the second term on the right hand side of (4.78), we have

$$\begin{aligned}
(4.80) \quad & \int_{\Omega_{0,k}} \delta_k \partial_\xi (|y|^{2N} \nabla \mathfrak{h}_k(\delta_k e_1)(y - e_1)) e^{V_k} \\
&= 2N \delta_k \int_{\Omega_{0,k}} y_\xi |y|^{2N-2} \nabla \mathfrak{h}_k(\delta_k e_1)(y - e_1) e^{V_k} + \delta_k \int_{\Omega_{0,k}} |y|^{2N} \partial_\xi \mathfrak{h}_k(\delta_k e_1) e^{V_k} \\
&= \delta_k \partial_\xi (\log \mathfrak{h}_k)(\delta_k e_1) (8\pi + O(\mu_k \varepsilon_k^2)) + o(\tilde{\delta}_k).
\end{aligned}$$

Using (4.79) and (4.80) in the difference between (4.76) and (4.77), we have

$$\delta_k \partial_\xi \mathfrak{h}_k(\delta_k e_1) (1 + O(\mu_k \varepsilon_k^2)) = o(\tilde{\delta}_k).$$

Thus $|\nabla \mathfrak{h}_k(\delta_k e_1)| = O(\delta_k)$ if $\delta_k = o(\varepsilon_k \mu_k^{\frac{1}{2}})$. When $\delta_k \geq C \varepsilon_k \mu_k^{\frac{1}{2}}$ we obtain from [34] $|\nabla \mathfrak{h}_k(0)| = O(\delta_k)$. Theorem 4.1 is established. \square

5. PROOF OF THEOREM 1.1

The key estimate is the following more refined estimate of $\nabla \mathfrak{h}_k(\delta_k Q_s^k)$.

Proposition 5.1.

$$(5.1) \quad |\nabla \mathfrak{h}_k(\delta_k Q_s^k)| = o(\delta_k), \quad s = 0, \dots, N.$$

Proof of Proposition 5.1:

Recall that V_k satisfies

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1) |y|^{2N} e^{V_k} = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N} e^{V_k} < \infty.$$

and

$$w_k = v_k - V_k, \quad w_k(e_1) = |\nabla w_k(e_1)| = 0.$$

The key estimate we establish is

$$(5.2) \quad |w_k(y)| \leq C \delta_k^2, \quad y \in B(0, \tau \varepsilon_k^{-1}).$$

In order to prove (5.2) we shall consider two cases. Either $\varepsilon_k^{-1} |Q_l^k - e^{i\beta_l}| \rightarrow 0$ for all l , or there exists l that satisfies $\varepsilon_k^{-1} |Q_l^k - e^{i\beta_l}| \geq C$. Note that by (3.5) and Theorem 4.1, if $\delta_k = o(\varepsilon_k^{1/2})$, (4.4) holds. If (4.4) is violated, $\delta_k \geq C \varepsilon_k^{1/2}$.

We first prove (5.2) under the assumption

$$(5.3) \quad \varepsilon_k^{-1} |Q_l^k - e^{i\beta_l}| \rightarrow 0 \quad \text{for all } l.$$

If (5.2) does not hold, let $M_k = \max |w_k|$ and suppose $M_k / \delta_k^2 \rightarrow \infty$. Then we set

$$\hat{w}_k = w_k / M_k,$$

the equation of \hat{w}_k is

$$\Delta \hat{w}_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k} \hat{w}_k(y) = a_k \cdot (e_1 - y) |y|^{2N} e^{V_k} + \hat{E}_1$$

where $a_k = \delta_k \nabla \mathfrak{h}_k(0) / M_k \rightarrow 0$ and $\hat{E}_k = o(1) |y - e_1|^2 |y|^{2N} e^{V_k}$.

Then, as before, we prove that $\hat{w}_k = o(1)$ outside bubbling disks. The proof of this part is as before, first we prove $\hat{w}_k = o(1)$ near e_1 . Then we use Harnack inequality to pass through the smallness away from singular sources.

Then we get a contradiction just as the proof of Proposition 4.1, the part before the evaluation of the Pohozaev identity, which essentially says that v_k cannot be very similar to a global solution at $N + 1$ local maximums. (5.2) is established under the assumption (5.3).

Still under (5.3) we use the notation \hat{w}_k and let it be defined as

$$\hat{w}_k(y) = w_k(y)/\delta_k^2.$$

Then the equation of \hat{w}_k is

$$(5.4) \quad \begin{aligned} & \Delta \hat{w}_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k} \hat{w}_k(y) \\ &= a_k \cdot (e_1 - y) |y|^{2N} e^{V_k} + \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k e_1)}{\alpha!} (e_1 - y)^\alpha |y|^{2N} e^{V_k} \\ & \quad + O(\delta_k) |y - e_1|^3 |y|^{2N} e^{V_k}. \end{aligned}$$

where $|a_k| = O(1)$ by (4.1). $\Delta \hat{w}_k \rightarrow 0$ away from finite points and \hat{w}_k is bounded, which means the singularities are removable. Thus $\hat{w}_k \rightarrow c$ in \mathbb{R}^2 minus a few points. By looking at the ODE projected to 1 we see that $c = 0$.

Because of the smallness of w_k ($w_k = O(\delta_k^2)$) we can compare the Pohozaev identities of v_k and V_k and obtain $\nabla \mathfrak{h}_k(\delta_k e_1) = o(\delta_k)$ as before. Similarly $\nabla \mathfrak{h}_k(\delta_k Q_l^k) = o(\delta_k)$ can be obtained similarly. Proposition 5.1 is established under the assumption (5.3).

Now we prove (5.2) if (5.3) does not hold: There exists s such that

$$\varepsilon_k^{-1} |Q_s^k - e^{i\beta_s}| \geq C.$$

In this case necessarily we have $\delta_k > C\varepsilon_k^{1/2}$ and, based on standard result for Liouville equation (say, Theorem 1.1 of [22]), $M_k \geq C$. In this case we set V_k to be the solution that agrees with v_k at e_1 and let $w_k = v_k - V_k$.

Since $M_k \geq C$, we just focus on w_k itself. Using $w_k(e_1) = 0$ we have

$$\begin{aligned} & w_k(y) \\ &= \int_{\Omega_k} (G_k(y, \eta) - G_k(e_1, \eta)) |\eta|^{2N} (\mathfrak{h}_k(\delta_k \eta) e^{v_k} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}) d\eta \\ &= -\frac{1}{2\pi} \int_{\Omega_k} \log \frac{|y - \eta|}{|e_1 - \eta|} |\eta|^{2N} (\mathfrak{h}_k(\delta_k \eta) e^{v_k} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}) d\eta + o(\delta_k^2) \end{aligned}$$

Here we recall

$$\varepsilon_k \leq C \max_l |Q_l^k - e^{i\beta_l}| \leq C\delta_k^2.$$

When we evaluate the integral in the expression of \tilde{w}_k above, we see immediately that we only need to evaluate each $B(Q_l^k, \tau_2)$ for some $\tau_2 > 0$ small. Thus we have

$$(5.5) \quad \begin{aligned} \tilde{w}_k(y) &= \int_{\Omega_k} L(y, \eta) A(\eta) d\eta + o(\delta_k^2). \\ &= \sum_{l=1}^N \int_{B(Q_l^k, \tau_2)} (L(y, \eta) - L(y, Q_l^k)) A(\eta) d\eta + L(y, Q_l^k) A_l^k + o(\delta_k^2). \end{aligned}$$

where

$$L(y, \eta) = -\frac{1}{2\pi} \log \frac{|y - \eta|}{|e_1 - \eta|},$$

$$A(\eta) = |\eta|^{2N} (\mathfrak{h}_k(\delta_k \eta) e^{v_k} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}).$$

and

$$A_l^k = \int_{B(Q_l^k, \tau_3)} A(\eta) d\eta.$$

To evaluate $w_k(y)$ we first observe that

$$(5.6) \quad \int_{B(Q_l^k, \tau)} (L(y, \eta) - L(y, Q_l^k)) A(\eta) = o(\varepsilon_k) = o(\delta_k^2)$$

The reason is $y - Q_l^k$ becomes $\varepsilon_k z$ after the following change of variable: $y = Q_l^k + \varepsilon_k z$. Then we look at (4.33) and use the cancellation in ϕ_1 . Then it is not hard to see that the result is $o(\varepsilon_k)$. For A_l^k we use the standard result for Liouville equation (see [17, 39, 22]) to have

$$(5.7) \quad A_l^k = O(\varepsilon_k^2 \log \frac{1}{\varepsilon_k}) = o(\delta_k^2), \quad l = 1, \dots, N.$$

As a consequence $\tilde{w}_k = o(\delta_k)$ away from the singular sources. Around e_1 ,

$$|\tilde{w}_k(e_1 + \varepsilon_k z)| \leq o(\delta_k \varepsilon_k) (1 + |y|), \quad |y| \leq \tau_1 \varepsilon_k^{-1}.$$

By comparing the Pohozaev identities of V_k and v_k around $B(e_1, \tau_1)$, we have $|\nabla \mathfrak{h}_k(\delta_k e_1)| = o(\delta_k)$. Applying the same argument around each Q_s^k we have

$$|\nabla \mathfrak{h}_k(\delta_k Q_s^k)| = o(\delta_k), \quad s = 0, \dots, N.$$

Proposition 5.1 is established in all cases. \square

Theorem 1.1 immediately follows from Proposition 5.1: For $N = 1$, $\partial_1 \mathfrak{h}_k(\delta_k e_1)$ and $\partial_1 \mathfrak{h}_k(\delta_k(-e_1))$ are both $o(\delta_k)$ implies $\partial_{11} \mathfrak{h}_k(0) = o(1)$. The fact that $\partial_2 \mathfrak{h}_k(\delta_k e_1)$ and $\partial_2 \mathfrak{h}_k(\delta_k(-e_1))$ being both $o(\delta_k)$ implies that $\partial_{12} \mathfrak{h}_k(0) = o(1)$. Finally by $\Delta \mathfrak{h}_k(0) = o(1)$ proved in [36] we have $\partial_{22} \mathfrak{h}_k(0) = o(1)$. When $N \geq 2$ we evaluate

$$\nabla \mathfrak{h}_k(\delta_k Q_l^k) - \nabla \mathfrak{h}_k(\delta_k e_1) = o(\delta_k)$$

for $l \neq 0$. Then it is easy to prove $\partial_{ij} \mathfrak{h}_k(0) = o(1)$ for all $i, j = 1, 2$. Theorem 1.1 is established. \square

6. HIGHER ORDER VANISHING THEOREM.

The key estimate in this section is

Proposition 6.1. *For $N \geq 2$ we have*

$$(6.1) \quad \nabla \mathfrak{h}_k(\delta_k Q_l^k) = o(\delta_k^2), \quad \forall l = 0, 1, \dots, N$$

Proof of Proposition 6.1:

Here we consider two cases:

- (1) $\max_l \varepsilon_k^{-1} |Q_l^k - e^{i\beta_l}| \geq C$. Note that in this case based on (3.5) and (5.1) we have $\delta_k \geq C \varepsilon_k^{\frac{1}{2}}$.

- (2) $\varepsilon_k^{-1}|Q_l^k - e^{i\beta_l}| = o(1)$ for all l . One sufficient condition based on (3.5) is $\delta_k \leq C\varepsilon_k^{1/2}$.

For the first case we prove (6.1). Observe that in this case we have $M_k \geq C$ (so we only work on w_k). Using the argument as in the proof of Theorem 1.1 we have

$$w_k(y) = \sum_{l=1}^N L_1(y, \eta) A_l^k + o(\delta_k^3), \quad y \in B_5 \setminus \{B(Q_k^l, \tau_2) \cup \dots \cup B(Q_N^k, \tau_2)\}.$$

where A_l^k is defined as in (5.6) and

$$L_1(y, \eta) = G_k(y, \eta) - G_k(e_1, \eta),$$

Then using (5.7) we have

$$w_k(y) = O(\varepsilon_k^2 \log \frac{1}{\varepsilon_k}) = o(\delta_k^3), \quad y \in B_5 \setminus (\cup_{l=1}^N B(Q_l^k, \tau_2)).$$

Then the proof of $|\nabla \mathfrak{h}_k(\delta_k e_1)| = o(\delta_k^2)$ is carried out as before. When it comes to $l \neq 0$, we can replace V_k by V_l^k and we still have $v_k - V_l^k = o(\delta_k^3)$ in the neighborhood of $\partial B(Q_l^k, \tau_1)$. Thus by evaluating the Pohozaev identities we obtain (6.1).

Now under the second possibility, we now prove (6.1).

The method of approximating blowup solutions using different global solutions at each local maximum can be used. We set $w_k = v_k - V_k$ with common local maximum at e_1 . Then we set

$$M_k := \max_{x \in \Omega_k} |w_k(x)|$$

Our goal is to show that

$$(6.2) \quad M_k \leq C(\delta_k \sum_{l=0}^N |\nabla \mathfrak{h}_k(\delta_k Q_l^k)| + \delta_k^3)$$

Here we note that we are using a different argument as in the previous section. Based on what we have done for the Hessian vanishing theorem we have already known $M_k = O(\delta_k^2)$.

By way of contradiction we assume that

$$M_k / (\delta_k \sum_{l=0}^N |\nabla \mathfrak{h}_k(\delta_k Q_l^k)| + \delta_k^3) \rightarrow \infty.$$

Then we let $\hat{w}_k = w_k / M_k$ with the obvious implication $|\hat{w}_k| \leq 1$. Then the equation of \hat{w}_k is

$$(6.3) \quad \Delta \hat{w}_k + \mathfrak{h}_k(\delta_k y) |y|^{2N} e^{\xi_k} \hat{w}_k = \frac{\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y)}{M_k} |y|^{2N} e^{V_k},$$

in $\Omega_k := B(0, \tau \delta_k^{-1})$. The estimate of ξ_k is as before

$$e^{\xi_k(x)} = e^{V_k} (1 + \frac{1}{2} w_k + O(w_k^2))$$

Let

$$d_l^k = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k e^{i\beta_l})) / (\mathfrak{h}_k(\delta_k e^{i\beta_l}) M_k).$$

Since

$$\delta_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) / M_k = o(1), \quad \forall i,$$

and $\delta_k^3 / M_k \rightarrow 0$, the proof of Lemma 6.1 below gives $\delta_k^2 \partial_{i_j} \mathfrak{h}_k(0) / M_k = O(1)$ for all $N \geq 2$.

Consequently all a_l are bounded and we can carry out the Fourier analysis around e_1 , we now observe that all the coefficients for \tilde{w}_k in the expansion are $O(1)$. Then (6.2) can be deduced by the same argument as in the proof of Theorem 1.1. Consequently it is easy to prove that $w_k = o(\delta_k^3) + o(\delta_k \sum_l |\nabla \mathfrak{h}_k(\delta_k Q_l^k)|)$ away from the bubbling disks. Since $\mathfrak{h}_k \in C^{N+2}$, obviously the equation above yields

$$(6.4) \quad \partial_{\xi} \mathfrak{h}_k(\delta_k Q_l^k) = O(\delta_k \varepsilon_k \sum_l |\nabla \mathfrak{h}_k(\delta_k Q_l^k)|) + o(\delta_k^2).$$

Thus (6.1) is obtained and Proposition 6.1 is established. \square

Now we complete the proof of Theorem 1.2.

From $|\nabla \mathfrak{h}_k(\delta_k Q_l^k)| = o(\delta_k^2)$ we have

$$|Q_l^k - e^{i\beta_l}| \leq s_k \delta_k^3 + C \mu_k e^{-\mu_k}$$

where $s_k \rightarrow 0+$. So if

$$\delta_k \leq s_k^{-\frac{1}{6}} \varepsilon_k^{1/3},$$

we have

$$|Q_l^k - e^{i\beta_l}| \varepsilon_k^{-1} = o(1).$$

On the other hand if

$$|Q_l^k - e^{i\beta_l}| \geq C \varepsilon_k$$

for at least one l , we have

$$\delta_k \geq C s_k^{-\frac{1}{6}} \varepsilon_k^{1/3}.$$

In this case $\delta_k^5 / (\mu_k e^{-\mu_k}) \rightarrow \infty$. There we can use the same argument to prove

$$(6.5) \quad \max_{x \in \Omega_k} |w_k| \leq C(\delta_k^5 + \delta_k \sum_{l=0}^N |\nabla \mathfrak{h}_k(\delta_k Q_l^k)|).$$

In fact in the proof if we set $M_k = \max_{\Omega_k} |w_k|$, then the function $\hat{w}_k = w_k / M_k$ has coefficient functions satisfying $\delta_k \nabla \mathfrak{h}_k(\delta_k Q_l^k) / M_k = o(1)$ and $\delta_k^5 / M_k = o(1)$. Then by the proof of Lemma 6.1 below, for $N \geq 8$, we have

$$\frac{\nabla^\alpha \mathfrak{h}_k(0) \delta_k^{|\alpha|}}{M_k} = o(1), \quad \text{for } |\alpha| = 2, 3, 4.$$

Thus by looking at the expansion of \hat{w}_k around e_1 we can still obtain the smallness of \hat{w}_k as before. Then using the same argument from evaluating Pohozaev identities we obtain

$$|\nabla \mathfrak{h}_k(\delta_k Q_l^k)| = o(\delta_k^4), \quad l = 0, \dots, N.$$

By Lemma 6.1 below we have, for $N \geq 8$,

$$(6.6) \quad |\nabla^\alpha \mathfrak{h}_k(\delta_k Q_l^k)| = o(\delta_k^{5-|\alpha|}), \quad 1 \leq |\alpha| \leq 5.$$

For $N \geq 8$ with the new rate of $|\nabla \mathfrak{h}_k(\delta_k Q_l^k)|$, we now have

$$|Q_l^k - e^{i\beta_l}| = o(\delta_k^5), \quad l = 0, \dots, N.$$

Using the same reasoning, we put the analysis in two cases: Either $\delta_k \leq S_k \varepsilon_k^{\frac{1}{5}}$ for some $S_k \rightarrow \infty$, or the compliment. As a result, we prove

$$\max_{x \in \Omega_k} |w_k| \leq C(\delta_k^9 + \delta_k \sum_l |\nabla \mathfrak{h}_k(\delta_k Q_l^k)|),$$

which further gives $|\nabla \mathfrak{h}_k(\delta_k e_1)| \leq o(\delta_k^8)$ and

$$|\nabla \mathfrak{h}_k(\delta_k Q_l^k)| = o(\delta_k^8), \quad l = 0, \dots, N.$$

Consequently by Lemma 6.1, for $N \geq 16$, we have

$$(6.7) \quad |\nabla^\alpha \mathfrak{h}_k(0)| = o(\delta_k^{9-|\alpha|}), \quad 1 \leq |\alpha| \leq 9.$$

In general for $N \geq 2^{M+1}$ we have

$$|\nabla^\alpha \mathfrak{h}_k(0)| = O(\delta_k^{2^M+1-|\alpha|}), \quad 1 \leq |\alpha| \leq 2^M + 1.$$

Finally we present this useful lemma:

Lemma 6.1. *Suppose*

$$|\nabla h_k(\delta_k e^{i\beta_l})| = o(\delta_k^{N_1}), \quad l = 0, \dots, N, \quad \beta_l = \frac{2\pi l}{N+1},$$

where N_1 is a positive integer. Then for $N \geq 2N_1$, we have

$$|\nabla^\alpha h_k(0)| = o(\delta_k^{N_1+1-|\alpha|}), \quad \forall 1 \leq |\alpha| \leq N_1 + 1.$$

Proof of Lemma 6.1:

Consider a real valued function $f(x, y)$ being approximated by a polynomial of degree N_1 :

$$f(x, y) = \sum_{i+j \leq N_1} a_{ij} z^i \bar{z}^j + o(r^{N_1}),$$

where we use $z = x + iy$ and $\bar{z} = x - iy$. If we use z and \bar{z} as two variables we have

$$(6.8) \quad f(z, \bar{z}) = \sum_{i=0}^{N_1} a_{ii} r^{2i} + \sum_{i+j \leq N_1, i \neq j} a_{ij} r^{i+j} \left(\frac{z}{r}\right)^i \left(\frac{\bar{z}}{r}\right)^j + o(r^{N_1}),$$

where $r = |z|$. Since f is real valued we have $a_{ij} = \bar{a}_{ji}$. Let $z_0 = e^{\frac{2i\pi}{N+1}}$, suppose

$$f(\delta_k z_0^m) = o(\delta_k^{N_1}), \quad m = 0, 1, \dots, N_1.$$

Then we claim that for $N \geq 2N_1$,

$$f(x, y) = o(\delta_k^{N_1}), \quad \forall |(x, y)| = \delta_k.$$

To prove this we write $f(\delta_k z_0^m)$ as

$$\begin{aligned} f(\delta_k z_0^m) &= \sum_{i+j \leq N_1} a_{ij} \delta_k^{i+j} z_0^{m(i-j)} + o(\delta_k^{N_1}) \\ &= \sum_{i=0}^{N_1} a_{ii} \delta_k^{2i} + \sum_{1 \leq i+j \leq N_1, i \neq j} a_{ij} \delta_k^{i+j} z_0^{m(i-j)} + o(\delta_k^{N_1}) \end{aligned}$$

Let $N \geq 2N_1$. For any $c_m, m = 0, 1, \dots, 2N_1$ we have

$$\begin{aligned} o(\delta_k^{2N_1}) &= \sum_{m=0}^{2N_1} c_m f(\delta_k z_0^m) \\ &= \sum_{i=0}^{2N_1} a_{ii} \delta_k^{2i} \sum_{m=0}^{2N_1} c_m + \sum_{1 \leq i+j \leq N_1, i \neq j} a_{ij} \delta_k^{i+j} \sum_{m=0}^{2N_1} c_m z_0^{(i-j)m} + o(\delta_k^{N_1}) \end{aligned}$$

Then for any given $b_{-N_1}, \dots, b_0, b_1, \dots, b_{N_1}$, we consider the system

$$\begin{aligned} \sum_{m=0}^{2N_1} c_m &= b_0, \\ \sum_{k=0}^{2N_1} c_k (z_0^{i-j})^k &= b_{i-j}, \quad i-j = -N_1, \dots, -1, 1, \dots, N_1. \end{aligned}$$

Then we see that the coefficient matrix for c_0, \dots, c_{2N_1} is a Vandermonde matrix because for $N \geq 2N_1$,

$$z_0^{-N_1}, \dots, z_0^{-1}, 1, z_0, \dots, z_0^{N_1}$$

are all distinct. Thus by choosing one $b_l = 1$ all others equal to 0 we have

$$\sum_{i=0}^{2N_1} a_{ii} \delta_k^{2i} = o(\delta_k^{N_1})$$

and

$$\sum_{j=i+l} a_{ij} \delta_k^{i+j} = o(\delta_k^{N_1}), \quad l = -N_1, \dots, N_1.$$

Putting these two estimates together we obtain from (6.8) that

$$f(z, \bar{z}) = o(\delta_k^{N_1}), \quad |z| = \delta_k.$$

Therefore we have

$$\partial_z^m \partial_{\bar{z}}^n f(0) = o(\delta_k^{N_1 - m - n}), \quad m + n \leq N_1$$

Now we replace f by two functions: $\partial_x \mathfrak{h}_k = (\partial_z + \partial_{\bar{z}}) \mathfrak{h}_k$ and $\partial_y \mathfrak{h}_k = i(\partial_z - \partial_{\bar{z}}) \mathfrak{h}_k$. Using the result above we have

$$\partial_z^m \partial_{\bar{z}}^n (\partial_z + \partial_{\bar{z}}) \mathfrak{h}_k(0) = o(\delta_k^{N_1 - (m+n)}), \quad \forall m + n \leq N_1.$$

and

$$\partial_z^m \partial_{\bar{z}}^n (\partial_z - \partial_{\bar{z}}) \mathfrak{h}_k(0) = o(\delta_k^{N_1 - (m+n)}), \quad \forall m + n \leq N_1.$$

Putting these equations together we have

$$\nabla^\alpha \mathfrak{h}_k(0) = o(\delta_k^{N_1 + 1 - |\alpha|}), \quad \forall 1 \leq |\alpha| \leq N_1 + 1,$$

for $N \geq 2N_1$. Lemma 6.1 is established. \square

Consequently Theorem 1.2 is established. \square

7. DIRICHLET PROBLEM

In this section we address the following Dirichlet problem:

$$(7.1) \quad \begin{cases} -\Delta v_k = \lambda_k |x|^{2N} V(x) e^{v_k} & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

where $N \geq 1 \in \mathbb{N}$. For a sequence of blowup solutions v_k , it is standard to assume a uniform bound on the total integration (see [8]): there exists $C > 0$ independent of k such that

$$(7.2) \quad \lambda_k \int_{\Omega} |x|^{2N} V(x) e^{v_k} dx \leq C.$$

We also suppose that v_k admits the origin as its only blow-up point in B_1 , in other words

$$(7.3) \quad \max_{\Omega} v_k(x) \rightarrow +\infty,$$

and for any compact set $K \subset \Omega \setminus \{0\}$ there exists a constant $C(K)$ (depending on K) such that

$$(7.4) \quad \max_K v_k \leq C(K).$$

The Dirichlet problem when $N = 1$ and Ω is the unit ball $B_1 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ can be described as

$$(7.5) \quad \begin{cases} -\Delta v_k = \lambda_k V(x) |x|^2 e^{v_k} & \text{in } B_1, \\ v_k = 0 & \text{on } \partial B_1, \\ \lambda_k \int_{B_1} |x|^2 e^{v_k} dx \leq C \end{cases}$$

and we suppose that v_k admits the origin as its only blow up point in B_1

$$(7.6) \quad \max_{B_1} v_k \rightarrow +\infty, \quad \sup_{\varepsilon \leq |x| \leq 1} v_k < C(\varepsilon) \quad \forall \varepsilon > 0.$$

In addition, we postulate the usual assumption on V

$$(7.7) \quad V \in C^1(\overline{B_1}), \quad V \in C^2(U), \quad \min_{\overline{B_1}} V > 0, \quad \nabla V(0) = 0$$

where U is a neighborhood of the origin.

We point out that the hypothesis on the vanishing of the first derivatives of the potential V is a necessary condition for the non-simple blowup solutions (see [35]).

The main result in this section is we use two new Pohozaev identities to prove the vanishing property of D^2V :

Theorem 7.1. *Assume that the sequence v_k satisfies (7.5)-(7.6) and has the non-simple blow-up property. Then, if V verifies (7.7), the following holds*

$$D^\alpha V(0) = 0 \quad \forall |\alpha| = 2.$$

The proof relies on the derivation of two new Pohozaev identities. Moreover, the method fails for $N \geq 2$ since in the expansion of the Pohozaev identity we no longer catch information on the Hessian matrix of V (see Remark 7.1 for technical details).

7.1. Pohozaev-type Identities. In this subsection we prove Pohozaev-type identities for solutions of (7.5). We then exploit such integral identities to prove conditions on the potential V for the existence of associated families of blowing up solutions provided by Theorem 7.1.

First we introduce complex notations and identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and we denote by x^N the N -power of the complex number x ; then we fix some algebraic identities in the following lemma. The proof is a direct computation.

Lemma 7.1. *Let $N \in \mathbb{N}$. Then for any $x \in \mathbb{R}^2 \setminus \{0\}$ the following identities hold:*

$$2x_1 \operatorname{Re}(x^N) = \operatorname{Re}(x^{N+1}) + |x|^2 \operatorname{Re}(x^{N-1}), \quad 2x_2 \operatorname{Re}(x^N) = \operatorname{Im}(x^{N+1}) - |x|^2 \operatorname{Im}(x^{N-1}),$$

$$2x_1 \operatorname{Im}(x^N) = \operatorname{Im}(x^{N+1}) + |x|^2 \operatorname{Im}(x^{N-1}), \quad 2x_2 \operatorname{Im}(x^N) = |x|^2 \operatorname{Re}(x^{N-1}) - \operatorname{Re}(x^{N+1}),$$

$$\frac{\partial}{\partial x_1} (\operatorname{Re}(x^N)) = \frac{\partial}{\partial x_2} (\operatorname{Im}(x^N)) = N \operatorname{Re}(x^{N-1}),$$

$$\frac{\partial}{\partial x_1} (\operatorname{Im}(x^N)) = -\frac{\partial}{\partial x_2} (\operatorname{Re}(x^N)) = N \operatorname{Im}(x^{N-1}),$$

$$\frac{\partial}{\partial x_1} \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} \right) = -\frac{\partial}{\partial x_2} \left(\frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) = -N \frac{\operatorname{Re}(x^{N+1})}{|x|^{2(N+1)}},$$

$$\frac{\partial}{\partial x_1} \left(\frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) = \frac{\partial}{\partial x_2} \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} \right) = -N \frac{\operatorname{Im}(x^{N+1})}{|x|^{2(N+1)}}.$$

We proceed to provide the first Pohozaev identity. In the following $G(x, y)$ is the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions and $H(x, y)$ denotes its regular part:

$$H(x, y) := G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

Proposition 7.1. *Let $N \in \mathbb{N}$. Assume that Ω is a smooth and bounded planar domain such that $0 \in \Omega$ and the potential $V > 0$ belongs to the class $C^1(\overline{\Omega})$. Then any solutions $v \in C(\overline{\Omega})$ of the boundary value problem*

$$(7.8) \quad \begin{cases} -\Delta v = \lambda |x|^{2N} V(x) e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the following Pohozaev identity:

$$\begin{aligned} & \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial v}{\partial \nu} \right|^2 v_i dx - \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \frac{\partial v}{\partial x_i} dx - 4N\pi\lambda \int_{\Omega} \frac{\partial H}{\partial x_i}(0, y) |y|^{2N} V(y) e^v dy \\ & \quad - \lambda \int_{\partial\Omega} |x|^{2N} V(x) v_i dx \\ & = -4N\pi \frac{\partial v}{\partial x_i}(0) - \lambda \int_{\Omega} |x|^{2N} \frac{\partial V}{\partial x_i}(x) e^v dx \end{aligned}$$

for $i = 1, 2$.

Proof. Let us multiply on both sides of the equation in (7.8) by $\frac{\partial v}{\partial x_i}$ (for $i = 1, 2$) and integrating on Ω we get

$$(7.9) \quad \int_{\Omega} \nabla v \cdot \nabla \left(\frac{\partial v}{\partial x_i} \right) dx - \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} \lambda |x|^{2N} V(x) e^v \frac{\partial v}{\partial x_i} dx.$$

Taking into account that

$$\nabla v \cdot \nabla \left(\frac{\partial v}{\partial x_i} \right) = \frac{1}{2} \frac{\partial}{\partial x_i} (|\nabla v|^2), \quad e^v \frac{\partial v}{\partial x_i} = \frac{\partial e^v}{\partial x_i},$$

by the divergence theorem (7.9) becomes

$$(7.10) \quad \begin{aligned} & \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 v_i dx - \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \frac{\partial v}{\partial x_i} dx \\ & = -2N\lambda \int_{\Omega} |x|^{2N-2} x_i V e^v dx - \lambda \int_{\Omega} |x|^{2N} \frac{\partial V}{\partial x_i} e^v dx + \lambda \int_{\partial\Omega} |x|^{2N} V(x) v_i dx \end{aligned}$$

where we have used the homogeneous boundary condition $v = 0$ on $\partial\Omega$. By Poisson representation formula we have for $i = 1, 2$ and $x \in \Omega$

$$\begin{aligned} \frac{\partial v}{\partial x_i}(x) & = \lambda \int_{\Omega} \frac{\partial G}{\partial x_i}(x-y) |y|^{2N} V(y) e^{v_k(y)} dy \\ & = \lambda \int_{\Omega} \left(\frac{y_i - x_i}{2\pi |x_i - y_i|^2} + \frac{\partial H}{\partial x_i}(x, y) \right) |y|^{2N} V(y) e^{v_k(y)} dy \end{aligned}$$

where G and H are the Green's function and its regular part as defined above. So we deduce

$$\lambda \int_{\Omega} y_i |y|^{2N-2} V(y) e^v dy = 2\pi \frac{\partial v}{\partial x_i}(0) - 2\pi\lambda \int_{\Omega} \frac{\partial H}{\partial x_i}(0, y) |y|^{2N} V(y) e^{v_k(y)} dy.$$

Combining the last identity with (7.10) we get the thesis. \square

Proposition 7.2. *Let $N \in \mathbb{N}$. Assume that Ω is a smooth and bounded planar domain such that $0 \in \Omega$ and the potential $V > 0$ belongs to the class $C^1(\overline{\Omega})$. Then any solutions $v \in C^1(\overline{\Omega})$ of the boundary value problem (7.8) satisfies the following*

Pohozaev identity:

$$\begin{aligned}
& \lambda \int_{\Omega} e^v \left(\frac{\partial V(x)}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial V(x)}{\partial x_2} \operatorname{Im}(x^N) \right) dx \\
& - \lambda \int_{\partial\Omega} V(x) e^v \left(\operatorname{Re}(x^N) \nu_1 - \operatorname{Im}(x^N) \nu_2 \right) d\sigma \\
& = \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial v}{\partial \nu} \right|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} \nu_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \nu_2 \right) d\sigma \\
& - \frac{1}{2\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \left(\left| \frac{\partial v}{\partial x_1} \right|^2 - \left| \frac{\partial v}{\partial x_2} \right|^2 \right) \operatorname{Re}(x^{N-1}) d\sigma \\
& + \frac{1}{\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} \operatorname{Im}(x^{N-1}) d\sigma + o(1)
\end{aligned}$$

as $\varepsilon \rightarrow 0^+$, where ν stands for the unit outward normal.

Proof. Let us multiply both sides of the equation in (7.8) by

$$\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}};$$

using that v belongs to $C^1(\overline{\Omega})$ by standard regularity theory and integrating on $\Omega \setminus B_\varepsilon$ for $\varepsilon > 0$ sufficiently small we get

$$\begin{aligned}
(7.11) \quad & \int_{\Omega \setminus B_\varepsilon} (-\Delta v) \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) dx \\
& = \lambda \int_{\Omega \setminus B_\varepsilon} V(x) e^v \left(\frac{\partial v}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial v}{\partial x_2} \operatorname{Im}(x^N) \right) dx.
\end{aligned}$$

By applying Gauss Green formula, we have

$$\begin{aligned}
& \int_{\Omega \setminus B_\varepsilon} (-\Delta v) \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) dx \\
& = \int_{\Omega \setminus B_\varepsilon} \nabla v \cdot \nabla \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) dx \\
& - \int_{\partial\Omega \cup \partial B_\varepsilon} \frac{\partial v}{\partial \nu} \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) d\sigma.
\end{aligned}$$

Using Lemma 7.1 we derive

$$\begin{aligned}
& \nabla v \cdot \nabla \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) \\
& = \frac{1}{2} \frac{\partial}{\partial x_1} (|\nabla v|^2) \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{1}{2} \frac{\partial}{\partial x_2} (|\nabla v|^2) \frac{\operatorname{Im}(x^N)}{|x|^{2N}} - N |\nabla v|^2 \frac{\operatorname{Re}(x^{N+1})}{|x|^{2(N+1)}} \\
& = \frac{1}{2} \frac{\partial}{\partial x_1} \left(|\nabla v|^2 \frac{\operatorname{Re}(x^N)}{|x|^{2N}} \right) - \frac{1}{2} \frac{\partial}{\partial x_2} \left(|\nabla v|^2 \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right),
\end{aligned}$$

by which, applying again Gauss-Green formula,

$$\begin{aligned} & \int_{\Omega \setminus B_\varepsilon} \nabla v \cdot \nabla \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) \\ &= \frac{1}{2} \int_{\partial\Omega \cup \partial B_\varepsilon} |\nabla v|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} v_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} v_2 \right) d\sigma. \end{aligned}$$

Hence we deduce

$$\begin{aligned} (7.12) \quad & \int_{\Omega \setminus B_\varepsilon} (-\Delta v) \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) dx \\ &= \frac{1}{2} \int_{\partial\Omega \cup \partial B_\varepsilon} |\nabla v|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} v_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} v_2 \right) d\sigma \\ & \quad - \int_{\partial\Omega \cup \partial B_\varepsilon} \frac{\partial v}{\partial \nu} \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) d\sigma. \end{aligned}$$

Let us examine separately the boundary integrals over $\partial\Omega$ and over ∂B_ε : taking into account of the homogeneous boundary condition we have that $\nabla v = \frac{\partial v}{\partial \nu} \mathbf{v}$ on $\partial\Omega$, which implies

$$\begin{aligned} & \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} v_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} v_2 \right) d\sigma - \\ & \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) d\sigma \\ &= -\frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial v}{\partial \nu} \right|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} v_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} v_2 \right) d\sigma. \end{aligned}$$

On the other hand, $v = -\frac{x}{|x|}$ on ∂B_ε ; consequently

$$\begin{aligned} & \frac{1}{2} \int_{\partial B_\varepsilon} |\nabla v|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} v_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} v_2 \right) d\sigma \\ & \quad - \int_{\partial B_\varepsilon} \frac{\partial v}{\partial \nu} \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) d\sigma \\ &= -\frac{1}{\varepsilon^{2N+1}} \int_{\partial B_\varepsilon} \frac{|\nabla v|^2}{2} \left(\operatorname{Re}(x^N) x_1 - \operatorname{Im}(x^N) x_2 \right) \\ & \quad + \frac{1}{\varepsilon^{2N+1}} \int_{\partial B_\varepsilon} \left(\frac{\partial v}{\partial x_1} x_1 + \frac{\partial v}{\partial x_2} x_2 \right) \left(\frac{\partial v}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial v}{\partial x_2} \operatorname{Im}(x^N) \right) d\sigma \\ &= \frac{1}{2\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \left(\left| \frac{\partial v}{\partial x_1} \right|^2 - \left| \frac{\partial v}{\partial x_2} \right|^2 \right) \operatorname{Re}(x^{N-1}) d\sigma \\ & \quad - \frac{1}{\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} \operatorname{Im}(x^{N-1}) d\sigma. \end{aligned}$$

where in the second identity we have used Lemma 7.1. By inserting the above two boundary estimates into (7.12) we get

$$\begin{aligned}
(7.13) \quad & \int_{\Omega \setminus B_\varepsilon} (-\Delta v) \left(\frac{\partial v}{\partial x_1} \frac{\operatorname{Re}(x^N)}{|x|^{2N}} - \frac{\partial v}{\partial x_2} \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \right) dx \\
&= -\frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial v}{\partial \nu} \right|^2 \left(\frac{\operatorname{Re}(x^N)}{|x|^{2N}} \nu_1 - \frac{\operatorname{Im}(x^N)}{|x|^{2N}} \nu_2 \right) d\sigma \\
&\quad + \frac{1}{2\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \left(\left| \frac{\partial v}{\partial x_1} \right|^2 - \left| \frac{\partial v}{\partial x_2} \right|^2 \right) \operatorname{Re}(x^{N-1}) d\sigma \\
&\quad - \frac{1}{\varepsilon^{2N-1}} \int_{\partial B_\varepsilon} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} \operatorname{Im}(x^{N-1}) d\sigma
\end{aligned}$$

Now let us pass to examine the right hand side of (7.11): by using again Gauss Green formula and Lemma 7.1

$$\begin{aligned}
(7.14) \quad & \int_{\Omega \setminus B_\varepsilon} \lambda V(x) e^v \left(\frac{\partial v}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial v}{\partial x_2} \operatorname{Im}(x^N) \right) dx \\
&= \int_{\Omega \setminus B_\varepsilon} \lambda V(x) \left(\frac{\partial e^v}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial e^v}{\partial x_2} \operatorname{Im}(x^N) \right) dx \\
&= - \int_{\Omega \setminus B_\varepsilon} \lambda e^{v(x)} \left(\frac{\partial V(x)}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial V(x)}{\partial x_2} \operatorname{Im}(x^N) \right) dx \\
&\quad - \int_{\Omega \setminus B_\varepsilon} \lambda e^{v(x)} V(x) \left(\frac{\partial(\operatorname{Re}(x^N))}{\partial x_1} - \frac{\partial(\operatorname{Im}(x^N))}{\partial x_2} \right) dx \\
&\quad + \int_{\partial\Omega \cup \partial B_\varepsilon} \lambda V(x) e^v \left(\operatorname{Re}(x^N) \nu_1 - \operatorname{Im}(x^N) \nu_2 \right) d\sigma \\
&= - \int_{\Omega} \lambda e^{v(x)} \left(\frac{\partial V(x)}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial V(x)}{\partial x_2} \operatorname{Im}(x^N) \right) dx \\
&\quad + \int_{\partial\Omega} \lambda V(x) e^v \left(\operatorname{Re}(x^N) \nu_1 - \operatorname{Im}(x^N) \nu_2 \right) d\sigma + o(1)
\end{aligned}$$

as $\varepsilon \rightarrow 0^+$, where we have used the homogeneous boundary condition $v = 0$ on $\partial\Omega$.

Let us insert (7.13) and (7.14) into (7.11) and letting $\varepsilon \rightarrow 0$ and we get the thesis. \square

7.2. Uniform Behaviour of blowing up solutions. In this section we derive the asymptotic behaviour of blow-up solutions v_k which is a direct consequence of a uniform estimate provided by [4] for bubbling solution to the Liouville equation with no boundary condition; roughly speaking, the analysis reveals that their profiles differs from global solutions of a Liouville type equation only by a uniformly bounded term.

Proposition 7.3. *Let $N \in \mathbb{N}$. Assume that Ω is a smooth and bounded planar domain such that $0 \in \Omega$ and the potential $V \in C^1(\overline{\Omega})$ is bounded from below away*

from zero. If $v_k \in C(\overline{\Omega})$ is a sequence of blow-up solutions for the problem (7.1) satisfying (7.2)-(7.3)-(7.4), then along a subsequence

$$(7.15) \quad \lambda |x|^2 V(x) e^{v_k} \rightharpoonup 8\pi(N+1)\delta_0$$

weakly in the measure sense, where δ_0 denotes the Dirac delta with pole at 0. Moreover, by setting β_k as the maximum point of v_k in Ω :

$$\beta_k \rightarrow 0 : v_k(\beta_k) = \max_{\Omega} v_k(x) \rightarrow +\infty,$$

the following holds¹:

$$(7.16) \quad \frac{1}{\lambda_k} \sim e^{\frac{v_k(\beta_k)}{2}}$$

and

$$(7.17) \quad v_k(x) = \log \frac{e^{v_k(\beta_k)}}{\left(1 + \frac{\lambda_k V(\beta_k)}{8\alpha^2} e^{v_k(\beta_k)} |x|^{N+1} - b_k\right)^2} + O(1) \text{ in } \Omega$$

where $b_k = \beta_k^{N+1}$.

Proof. We need to transform the equation into (7.1) into a Liouville equation with no boundary condition we can use the presence of the free parameter under the transformation

$$\bar{v}_k(x) = v_k(x) + \log \lambda_k$$

so that the sequence \bar{v}_k satisfies

$$\begin{cases} -\Delta \bar{v}_k = V(x) |x|^{2N} e^{\bar{v}_k} & \text{in } \Omega, \\ \int_{\Omega} |x|^{2(N+1)} V(x) e^{\bar{v}_k} dx \leq C \end{cases}$$

and 0 is the only blowup point of \bar{v}_k . Then, by applying the uniform estimate provided in Theorem 1.4 of [4] for a sequence of solutions having a single blow up point:

$$\bar{v}_k(x) = \log \frac{e^{\bar{v}_k(\beta_k)}}{\left(1 + \frac{V(\beta_k)}{8\alpha^2} e^{\bar{v}_k(\beta_k)} |x|^{N+1} - b_k\right)^2} + O(1) \text{ uniformly in } \Omega.$$

and (7.17) follows. By evaluating (7.17) for $x \in \partial\Omega$ (where $v_k(x) = 0$) we get (7.16). □

The Case of the Ball: Necessary Conditions for Blowing Up. In this section we focus on the case when $N = 1$ and Ω is the unit ball B_1 and we establish an integral estimate for bubbling situation at 0.

Proposition 7.4. *Assume that the sequence v_k satisfies (7.5)-(7.6). Then, if V verifies (7.7), the following holds*

$$\frac{\partial v_k}{\partial x_i}(0) = o(1), \quad \int_{B_1} \lambda_k e^{v_k(x)} \left(\frac{\partial V(x)}{\partial x_1} x_1 - \frac{\partial V(x)}{\partial x_2} x_2 \right) dx = o(1).$$

¹We use the notation \sim to denote quantities which in the limit $\lambda \rightarrow 0^+$ are of the same order.

Proof. Consider Propositions 7.1 for $N = 1$ and $\Omega = B_1$: taking into account that $v = (x_1, x_2)$ on ∂B_1 , we get that the homogeneous boundary condition implies

$$\nabla v_k = \frac{\partial v_k}{\partial v} v = \frac{\partial v_k}{\partial v}(x_1, x_2) \text{ on } \partial B_1.$$

On the other hand the regular part $H(x, y)$ of the Green's function takes the form $H(x, y) = \frac{1}{4\pi} \log(1 + |x|^2|y|^2 - 2x \cdot y)$ for $x, y \in B_1$ (where “ \cdot ” denotes the scalar product in \mathbb{R}^2), which gives

$$\frac{\partial H}{\partial x_i}(0, y) = -\frac{y_i}{2\pi}.$$

We deduce that for every k the Pohozaev identity in Proposition 7.1 takes the forms:

$$(7.18) \quad \begin{aligned} \frac{1}{2} \int_{\partial B_1} \left| \frac{\partial v_k}{\partial v} \right|^2 x_i dx - 2N\lambda_k \int_{B_1} y_i |y|^2 V(y) e^{v_k} dy + \lambda_k \int_{\partial B_1} x_i |x|^2 V(x) dx \\ = 4N\pi \frac{\partial v_k}{\partial x_i}(0) + \lambda_k \int_{\Omega} |x|^2 \frac{\partial V}{\partial x_i}(x) e^{v_k} dx. \end{aligned}$$

Next observe that for any fixed k , since v is two-times differentiable by standard regularity theory,

$$\frac{1}{\varepsilon} \int_{\partial B_\varepsilon} \left| \frac{\partial v_k}{\partial x_i} \right|^2 d\sigma = 2\pi \left| \frac{\partial v_k}{\partial x_i}(0) \right|^2 + o(1) \text{ as } \varepsilon \rightarrow 0^+$$

so that the Pohozaev identity in Proposition 7.2 gives

$$(7.19) \quad \begin{aligned} \int_{B_1} \lambda e^{v_k(x)} \left(\frac{\partial V(x)}{\partial x_1} x_1 - \frac{\partial V(x)}{\partial x_2} x_2 \right) dx - \lambda_k \int_{\partial B_1} V(x) (x_1^2 - x_2^2) d\sigma \\ = \frac{1}{2} \int_{\partial B_1} \left| \frac{\partial v_k}{\partial v} \right|^2 (x_1^2 - x_2^2) d\sigma - \pi \left| \frac{\partial v_k}{\partial x_1}(0) \right|^2 + \pi \left| \frac{\partial v_k}{\partial x_2}(0) \right|^2. \end{aligned}$$

We need to estimate $\frac{\partial v_k}{\partial v}$ on ∂B_1 : by Poisson representation formula we have

$$\frac{\partial v_k}{\partial v}(x) = \lambda_k \int_{B_1} \frac{\partial G}{\partial v_x}(x-y) |y|^2 V(y) e^{v_k(y)} dy \quad x \in \partial B_1.$$

Since the Green's function for the unit ball takes the form $G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + \frac{1}{4\pi} \log(1 + |x|^2|y|^2 - 2x \cdot y)$ for $x, y \in B_1$, so we have

$$\frac{\partial G}{\partial v_x}(x-y) = \frac{1}{2\pi} \frac{(y-x) \cdot x}{|x-y|^2} + \frac{1}{2\pi} \frac{(x|y|^2 - y) \cdot x}{1 + |y|^2 - 2x \cdot y} \quad \forall x \in \partial B_1$$

and so

$$\frac{\partial v_k}{\partial v}(x) = \frac{\lambda_k}{2\pi} \int_{B_1} \left(\frac{(y-x) \cdot x}{|x-y|^2} + \frac{(x|y|^2 - y) \cdot x}{1 + |y|^2 - 2x \cdot y} \right) |y|^2 V(y) e^{v_k(y)} dy \quad \forall x \in \partial B_1.$$

By using (7.15) and (7.17), the above integral formula gives

$$\frac{\partial v_k}{\partial v}(x) \rightarrow -8 \text{ unif. for } x \in \partial B_1$$

We have thus proved that

$$(7.20) \quad \int_{\partial B_1} \left| \frac{\partial v_k}{\partial \mathbf{v}} \right|^2 (x_1^2 - x_2^2) d\sigma = 64 \int_{\partial B_1} (x_1^2 - x_2^2) d\sigma + o(1) = o(1)$$

since $\int_{\partial B_1} (x_1^2 - x_2^2) d\sigma = 0$. Similarly

$$(7.21) \quad \int_{\partial B_1} \left| \frac{\partial v_k}{\partial \mathbf{v}} \right|^2 x_i d\sigma = 64 \int_{\partial B_1} x_i d\sigma + o(1) = o(1).$$

Moreover again by (7.15), using that $\nabla V(0) = 0$ by (7.7), we have

$$\lambda_k \int_{B_1} y_i |y|^2 V(y) e^{v_k} dy = o(1), \quad \lambda_k \int_{B_1} |x|^2 \frac{\partial V}{\partial x_i}(x) e^{v_k} dx = o(1),$$

and

$$\lambda_k \int_{\partial B_1} V(x) (x_1^2 - x_2^2) d\sigma = o(1)$$

By inserting the above estimates into (7.18)-(7.19) we deduce the thesis. \square

Proof of Theorem 7.1. In this section we exploit the integral estimates obtained in Proposition 7.4 in order to obtain necessary conditions for the presence of bubbling solutions at 0 for problem (7.5)-(7.6). In the following we assume that v_k is a sequence satisfying (7.5)-(7.6) and the potential V verifies (7.7). Moreover, after suitably rotating the coordinate system, we may assume that in a small neighborhood of 0 the following expansion holds:

$$V(x) = V(0) + \frac{a_{11}x_1^2 + a_{22}x_2^2}{2} + o(|x|^2) \text{ as } x \rightarrow 0,$$

where $a_{ii} = \frac{\partial^2 V}{\partial x_i^2}(0)$, so that

$$(7.22) \quad \frac{\partial V}{\partial x_1}(0)x_1 - \frac{\partial V}{\partial x_2}(0)x_2 = a_{11}x_1^2 - a_{22}x_2^2 + o(|x|^2).$$

Now assume, in addition, that the sequence v_k has the non-simple blow-up property. It has been shown in [36] that Laplacian of coefficient functions must be zero, i.e. $\Delta V(0) = 0$ or, equivalently, $a_{11} + a_{22} = 0$. Then

$$\lambda_k \int_{B_1} e^{v_k(x)} \left(\frac{\partial V}{\partial x_1} x_1 - \frac{\partial V}{\partial x_2} x_2 \right) dx = (a_{11} + o(1)) \lambda_k \int_{B_1} |x|^2 e^{v_k(x)} dx \rightarrow 16\pi \frac{a_{11}}{V(0)}.$$

Combining this convergence with that proved in Proposition 7.4 we deduce

$$(7.23) \quad a_{11} = a_{22} = 0.$$

Then Theorem 7.1 follows.

Remark 7.1. We point out that if we try to apply our technique to $N \geq 2$, then the analogous of Proposition 7.4 would give

$$\lambda_k \int_{B_1} e^{v_k(x)} \left(\frac{\partial V(x)}{\partial x_1} \operatorname{Re}(x^N) - \frac{\partial V(x)}{\partial x_2} \operatorname{Im}(x^N) \right) dx = o(1).$$

Assume that the sequence v_k has the non-simple blow-up property, so that, according to [36] we have $\Delta V(0) = 0$ or, equivalently, $a_{11} + a_{22} = 0$. Then, using Lemma 7.1 the above estimates becomes

$$\lambda_k a_{11} \int_{B_1} |x|^2 e^{v_k(x)} \operatorname{Re}(x^{N-1}) dx + \lambda_k \int_{B_1} o(|x|^{N+1}) e^{v_k} dx = o(1).$$

Observe that only if $N = 1$ the above integrals can be estimated using (7.15), whereas for $N \geq 2$ they cannot be handled by (7.15). This will explain why our method based on the combination of two Pohozaev identities fails to provide the results of Theorem 7.1 for $N \geq 2$.

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